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Inequality reducing taxation reconsidered

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Abstract

The paper investigates inequality reducing taxation for various inequality views. Using the general definition of an inequality concept (Ebert (2004)) corresponding definitions of Lorenz dominance, inequality reduction and measures of tax progression are provided. The framework allows us to simplify and clarify the different approaches found in the literature, to extend this analysis, and to present brief and transparent proofs.

1. Introduction

Nowadays one of the objectives of income taxation is inequality reduction. Public economists seem to agree that Lorenz dominance is the appropriate criterion for ranking income distributions. If the distribution of post-tax income dominates the distribution of pre-tax income, inequality is reduced. When this is the case for any income distribution, the tax function is progressive. Progression can also be determined by the properties of the tax schedule itself: The average tax rate has to be increasing, or, equivalently, residual progression – a measure of progression – has to be less than unity for all levels of income.

All this is well known and well established. Furthermore, the Lorenz criterion can be justified convincingly. When net incomes dominate gross incomes, the post-tax income distribution is also unanimously preferred to the pre-tax income distribution by all inequality measures possessing few important properties. The measures have to be relative (equal proportional changes of all incomes do not change inequality) and symmetric (the individuals' identity does not play a role). Moreover they have to satisfy the principle of progressive transfers (a rank-preserving redistribution of income from a richer to a poorer individual reduces inequality). These properties seem to be indispensable for inequality measurement.

On the other hand these characterizations are based on the relative inequality view. Here inequality is not altered if all incomes are changed in the same proportion. But this concept of inequality is not the only one considered by economists. Kolm (1976) has already proposed the concept of absolute inequality (inequality is not changed if incomes are changed by the same amount); Bossert and Pfingsten (1990) have introduced intermediate inequality (which can be interpreted as a compromise between relative and absolute inequality). This concept is also mentioned and, respectively, investigated in Ebert and Moyes (2000, 2002) and Chakravarty (2009). A number of other concepts have been proposed recently (see e.g. Pfingsten and Seidl (1997), Zoli (1998), del Rio and Ruiz-Castillo (2000), Ebert (2004), and Yoshida (2005)). Furthermore, empirical studies demonstrate that the relative inequality view is not unanimously accepted (see e.g. Amiel and Cowell (1992)). On the contrary, further attitudes towards inequality can be observed. Thus the question arises whether inequality reducing taxation can be defined with respect to further concepts and how it can be characterized.

The present paper deals with this problem. In order to not restrict the analysis too much a priori, we will start by using the general definition of an inequality concept provided in Ebert (2004). Then in a first step we confine ourselves to the subclass of coherent inequality concepts, namely to those for which we are able to describe the operations leaving inequality unchanged by linear transformations. We also need a corresponding concept of Lorenz dominance. It is required for checking the reduction of inequality. The dominance concept will be proposed in the second step. Given this device, finally the set of inequality reducing tax functions is described and corresponding concepts and measures of progression are defined and investigated.

The paper is organized as follows: The notation is defined in section 2. Section 3 provides a general definition of an inequality concept (originally proposed in Ebert (2004)). Its basic idea is to characterize the equivalence relation of having equal inequality by a one-parameter family of transformations which leave inequality *unchanged*. It then turns out that every coherent concept can be uniquely described by its domain and a characteristic function. The characteristic function maps the domain of incomes onto the set of strictly positive incomes (the domain of the relative inequality concept) and the characteristic function defines an isomorphism between both concepts. It therefore allows us to describe the original concept by means of the concept of relative inequality. As a consequence this one-to-one relationship can be exploited later on; we borrow many definitions and ideas from the relative inequality concept. Furthermore, the proof of many results makes use of the isomorphism. The class of coherent concepts then consists of the relative, absolute, intermediate and reference-point inequality views.

In section 4 Lorenz dominance is investigated. It forms an important ingredient when inequality reducing taxation is to be considered. Then it is required that a progressive tax schedule decreases inequality in every case, i.e., for every income distribution. Given the characteristic function of a coherent inequality concept it is obvious how to define the corresponding Lorenz curve. All incomes are transformed by means of the characteristic function and then the usual Lorenz curve is used.

Following the discussion of Lorenz dominance it is clear how inequality reducing taxation has to be introduced. In section 5 the corresponding definition is given. We call a tax schedule progressive if and only if it is inequality reducing. For a given inequality concept a tax schedule reduces inequality if and only if the corresponding tax schedule which can be derived by means of the characteristic function is progressive in the standard sense.

Analogous results are presented in section 6 for measures of tax progression. A measure of residual income progression can be defined by transforming income by means of the characteristic function and employing the standard measure. Finally section 7 concludes.

The paper makes several contributions to the literature: First, it leads to a clarification of ideas and definitions. Using the close relationship between the coherent inequality concepts considered and the relative inequality view we are able to present simple definitions of Lorenz dominance, progression and measures of progression. These definitions are obvious – given the above isomorphism – , but often replace and always simplify the definitions proposed in the literature for the intermediate inequality view. Above that the isomorphism facilitates the application of the nonstandard inequality concepts to practical and empirical analysis. Up to now the standard approach is to use the relative inequality view. Second, the basic results in this area can now be derived directly by making use of those for relative inequality. Thus a number of proofs which can be found in the literature now reduce to few lines. Third, some results already proven in the past are here collected and presented by a unifying approach. Fourth, all definitions and results for the concepts of reference point inequality have not been discussed up to now.

2. Notation

We consider a population consisting of $n \geq 3$ individuals. They are supposed to be identical with respect to all attributes but possibly income. Let Ω_d be the set of feasible incomes. We confine the analysis to $\Omega_d := (d, \infty)$ for $d \in \mathbb{R}$ and $\Omega_{-\infty} := \mathbb{R}$ for $d = -\infty$. Incomes are either bounded from below by income d or may be arbitrary. Sometimes also negative incomes are permitted. Income can be negative in a given period (as long as an individual is able to survive by getting credit or by using savings). The income d can be interpreted as reference or minimum income. Individual i's income is denoted by $X_i \in \Omega_d$, $i = 1,...,n$. An income distribution is represented by a vector $\boldsymbol{X} = (X_1,...,X_n) \in \Omega_d^n$. The vector $\boldsymbol{X}_{()} = (X_{(1)},...,X_{(n)})$ is generated by permuting the components of $\boldsymbol{X}$ in such a way that incomes are nondecreasing. $\mu(\boldsymbol{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the average income of $\boldsymbol{X}$ and $\mathbf{1}$ a vector containing n ones, $\mathbf{1} = (1,...,1)$.

$\mathcal{F}(\Omega_d)$ denotes the set of all continuous and nondecreasing functions $F : \Omega_d \to \Omega_d$. For every function $F \in \mathcal{F}(\Omega_d)$ we define a (corresponding) transformation $\boldsymbol{F} : \Omega_d^n \to \Omega_d^n$ by $\boldsymbol{F}(\mathbf{X}) := (F(X_1),...,F(X_n))$. It is individualistic (i.e., $F(X_i)$ depends only on individual i's income) and symmetric (all individuals are treated identically). Sometimes we confine ourselves to the subset $\mathcal{F}_c(\Omega_d) \subset \mathcal{F}(\Omega_d)$ containing all functions which are once continuously differentiable. $\mathcal{T}(\Omega_d) \subset \mathcal{F}(\Omega_d)$ denotes the set of all functions being strictly increasing and one-to-one. For $F \in \mathcal{F}_c(\Omega_d)$ we define the elasticity of $F(X)$ with respect to X by $\eta(F,X) := \frac{dF}{F} \Big/ \frac{dX}{X} = \frac{dF}{dX} \frac{X}{F(X)}$.

3. Inequality concepts

Whenever we are talking about 'inequality reducing taxation', the meaning of the term 'inequality' has to be made precise. This section is based on Ebert (2004) and provides a general definition of an inequality concept which represents an inequality view.

3.1 Definition

The standard example of an inequality concept is given by the relative inequality view. Here we assume that incomes are strictly positive, i.e., $\Omega_0 = \mathbb{R}_{++}$. Furthermore, according to this view equiproportional changes of all incomes leave inequality unchanged. In other words, every income distribution $\boldsymbol{X} \in \mathbb{R}^n_{++}$ and its transform $\boldsymbol{Y} = \lambda \boldsymbol{X} := (\lambda X_1, \ldots, \lambda X_n)$ (a priori) possess the same degree of inequality for $\lambda > 0$; i.e., in this case both income distributions having different means are related as far as inequality is concerned.

This relationship defines a (mathematical) relation on the set of feasible income distributions:

$$\boldsymbol{X} \sim_{rel} \boldsymbol{Y} :\Leftrightarrow \text{There is } \lambda > 0 \text{ such that } \boldsymbol{Y} = \lambda \boldsymbol{X}.$$

It is easily seen that $\sim_{rel}$ is an equivalence relation, i.e., $\boldsymbol{X} \sim_{rel} \boldsymbol{Y}$ and $\boldsymbol{Y} \sim_{rel} \boldsymbol{Z}$ imply that $\boldsymbol{X} \sim_{rel} \boldsymbol{Z}$ (transitivity), and that $\boldsymbol{Y} \sim_{rel} \boldsymbol{X}$ (symmetry) for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{R}^n_{++}$. Moreover we have $\boldsymbol{X} \sim_{rel} \boldsymbol{X}$ (reflexivity) for all $\boldsymbol{X} \in \mathbb{R}^n_{++}$.

More formally, the relative inequality view can be characterized by the equivalence relation $\sim_{rel}$ which in turn is based on a set of transformations or, more simply, on a set of functions defining the admissible transformations.

Let $S_\lambda : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be given by $S_\lambda(X) = \lambda X$ for $X \in \mathbb{R}_{++}, \lambda > 0$. Then we introduce the set of functions $\mathcal{T}^{rel} := \{S_\lambda \mid \lambda \in \mathbb{R}_{++}\}$. Since the individuals considered can differ only in income we treat them identically. S_λ defines the transformation $\boldsymbol{S}_\lambda(\boldsymbol{X}) := (S_\lambda(X_1), \ldots, S_\lambda(X_n))$ for all $X \in \mathbb{R}^n_{++}$, $S_\lambda \in \mathcal{T}^{rel}$. Then the relative inequality view is uniquely described by the relation $\sim_{rel}$:

$$X \sim_{rel} Y \Leftrightarrow \text{There is } S_\lambda \in \mathcal{T}^{rel} \text{ such that } Y = S_\lambda(X),$$

i.e., X and Y possess the same degree of inequality according to this view if Y can be generated from X by means of a(n admissible) transformation S_λ, where $S_\lambda \in \mathcal{T}^{rel}$.

This idea of characterizing an inequality view can be generalized to any domain Ω_d by an appropriate definition of an equivalence relation based on a (one-parameter) family of functions $\mathcal{T}$. We therefore introduce

Definition 1

Given a domain Ω_d and a set $\mathcal{T} = \{T_\lambda \mid \lambda \in \mathbb{R}_{++}\} \subset \mathcal{T}(\Omega_d)$ we call a relation $\sim^d_{\mathcal{T}}$ defined on Ω^n_d an inequality concept[1] *if*

(a) *for all $X, Y \in \Omega^n_d$:*

$$X \sim^d_{\mathcal{T}} Y \Leftrightarrow \textit{There is } T_\lambda \in \mathcal{T} \text{ s.t. } Y = T_\lambda(X)$$

(b) (i) $T_1(X) = X$ *for all* $X \in \Omega_d$.

(ii) $T_\lambda^{-1} = T_{1/\lambda}$ *for all* $\lambda > 0$ *where* T_λ^{-1} *denotes the inverse of* T_λ.

(iii) $T_\lambda \circ T_\kappa = T_{\lambda\kappa}$ *for all* $\lambda, \kappa \in \mathbb{R}_{++}$ *where the operation (composition)* $\circ$ *is defined by* $T_\lambda \circ T_\kappa(X) = T_\lambda(T_\kappa(X))$ *for all* $X \in \Omega_d$ *and* $\lambda, \kappa \in \mathbb{R}_{++}$.

[$\mathcal{T}$ is a(n algebraic) group with the group operation $\circ$ (see Lang (1968))].

(c) $T_\lambda(X)$ *is continuous and strictly increasing in* $\lambda \in \mathbb{R}_{++}$ *for all* $X \in \Omega_d$.

Thus an inequality view is described by a binary relation $\sim^d_{\mathcal{T}}$ satisfying definition 1. It is completely determined by the domain Ω_d and the set of admissible transformations characterized by $\mathcal{T}$. Indifference between X and Y according to $\sim^d_{\mathcal{T}}$ means that these distributions have the same degree of inequality.

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1 In Ebert (2004) the relation $\sim^d_{\mathcal{T}}$ is called a path-independent inequality concept. Since in this paper only this type of concept will be considered, the attribute will be dropped.

Given the properties of $\mathcal{T}$ the relation $\sim_{\mathcal{T}}^{d}$ is an equivalence relation: b(iii) implies transitivity, b(ii) symmetry, and reflexivity is an implication of b(i). The set $\mathcal{T}$ is a one-parameter family. Parameters are restricted to be strictly positive. Condition (c) is a regularity condition. Monotonicity in λ guarantees that equivalence classes $\{T_\lambda(X) | \lambda \in \mathbb{R}_{++}\}$ are 'thin'; i.e., for every income distribution there is always an arbitrarily close distribution being less or more unequal.

It is obvious that $\sim_{rel}$ is the inequality concept representing the relative inequality view. Another concept $\sim_{abs}$ can be defined by setting $\Omega_d = \Omega_{-\infty} = \mathbb{R}$ and $\mathcal{T}^{-\infty} := \{T_\lambda : \mathbb{R} \to \mathbb{R} \,|\, T_\lambda(X) = X + \ln\lambda \text{ for } \lambda \in \mathbb{R}_{++}\}$. It corresponds to the absolute inequality view. Inequality is not changed by equal additions to all incomes. The functions involved may also be nonlinear: Consider e.g. the inequality concept $\sim_{\exp}$ given by $\Omega_d = \Omega_1$ and $\mathcal{T}^{exp} = \{T_\lambda : \Omega_1 \to \Omega_1 \,|\, T_\lambda(X) = X^\lambda \text{ for } \lambda \in \mathbb{R}_{++}\}$. Further examples are discussed below.

3.2 Properties

Though the idea of defining inequality concepts is a general one and the variety of concepts is great, inequality concepts can be described simply. We have

Theorem 1 (Ebert (2004))

Let $\sim_{\mathcal{T}}^{d}$ be an inequality concept. Then there exists a continuous, strictly increasing, and surjective function $G : \Omega_d \to \mathbb{R}_{++}$ such that

$$\mathcal{T} = \{T_\lambda \,|\, T_\lambda(X) = G^{-1} \circ S_\lambda \circ G(X) \text{ for all } X \in \Omega_d, \quad S_\lambda \in \mathcal{T}^{rel}\},$$

given an appropriate parameterization of the functions T_λ.

The function G is unique up to a nonzero multiplicative factor.

The set of inequality concepts can be completely characterized. The admissible transformations T_λ possess a simple and clear structure (when written down appropriately). Therefore an inequality concept $\sim_{\mathcal{T}}^{d}$ can also be described by its domain Ω_d and the characteristic function $G : \Omega_d \to \mathbb{R}_{++}$ which is unique up to a multiplicative constant and which connects the domain Ω_d with $\Omega_0 = \mathbb{R}_{++}$. The relative inequality concept $\sim_{rel}$ is given by $\Omega_0 = \mathbb{R}_{++}$, and $G(X) = X$, the absolute concept $\sim_{abs}$ by $\Omega_{-\infty} = \mathbb{R}$, and $G(X) = e^X$ and the nonlinear concept $\sim_{exp}$ by Ω_1, and $G(X) = \ln X$. The function G allows the definition of a one-to-one correspondence between $\mathcal{T}$ and $\mathcal{T}^{rel}$ and is consistent with the composition $\circ$:

$$T_\lambda \circ T_\kappa = \left(G^{-1} \circ S_\lambda \circ G\right) \circ \left(G^{-1} \circ S_\kappa \circ G\right) = G^{-1} \circ S_\lambda \circ S_\kappa \circ G = G^{-1} \circ S_{\lambda\kappa} \circ G = T_{\lambda\kappa}$$

Furthermore the equivalence relations are linked in a simple manner:

$$X \sim_T^d Y \Leftrightarrow \text{There is } T_\lambda \in \mathcal{T} \text{ such that } Y = T_\lambda(X)$$

$$\Leftrightarrow \text{There is } S_\lambda \in \mathcal{T}^{rel} \text{ such that } Y = G^{-1} \circ S_\lambda \circ G(X)$$

$$\Leftrightarrow \text{There is } S_\lambda \in \mathcal{T}^{rel} \text{ such that } G(Y) = S_\lambda \circ G(X)$$

$$\Leftrightarrow G(X) \sim_{rel} G(Y)$$

X and Y possess the same degree of inequality according to $\sim_T^d$ if and only if the transformed income distributions $G(X)$ and $G(Y)$ are equivalent according to the relative inequality view. This relationship will be used below intensively. Theorem 1 says that all inequality concepts are essentially isomorphic to the relative inequality concept. The relationship is characterized by the characteristic function G. I.e., we obtain

Corollary 2

Let $\sim_T^d$ be an inequality concept. Then

$$X \sim_T^d Y \Leftrightarrow G(X) \sim_{rel} G(Y).$$

In this paper we will confine ourselves to inequality concepts for which the admissible transformations are linear.

3.3 Linear concepts

In order to characterize linear concepts we introduce progressive transfers and the property of transfer-consistency:

Definition 2

An income distribution $X' \in \Omega_d^n$ is derived from $X \in \Omega_d^n$ by a progressive transfer if there are i and j, $i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$, and $\varepsilon > 0$ such that

$$X_i < X_i' = X_i + \varepsilon \leq X_j - \varepsilon = X_j' < X_j \text{ and } X_k' = X_k \text{ for } k = 1,\ldots,n, \quad k \neq i, \quad k \neq j.$$

Progressive transfers redistribute a small amount of income from a richer to a poorer individual.

Definition 3

An inequality concept $\sim_T^d$ *is called transfer-consistent if it satisfies the following condition for all* $X, X' \in \Omega_d^n$ *and* $T_\lambda \in \mathcal{T}$*: If* X' *is derived from* X *by a progressive transfer then* $Y' = T_\lambda(X)$ *can be derived by a(n appropriate) transfer from* $Y = T_\lambda(X)$.

In this case the sequence of the transformation T_λ and the redistribution of income does not play a role. We then get

Theorem 3 (Ebert (2004))

Let the inequality concept $\sim_T^d$ *be given. Then* $\sim_T^d$ *is transfer-consistent if and only if* $\mathcal{T}_d = \{T_\lambda^d | \lambda \in \mathbb{R}_{++}\}$ *where*

$$T_\lambda^d(X) = \lambda(X - d) + d \quad \text{for } d \in \mathbb{R} \text{ and}$$

$$T_\lambda^d(X) = X + \ln\lambda \qquad \text{for } d = -\infty$$

for all $X \in \Omega_d$ *and* $\lambda \in \mathbb{R}_{++}$.

Thus the admissible transformations have to be linear for transfer-consistent inequality concepts. These concepts are called *coherent*.

Now some comments can be made.

1) If we have $T_\lambda(X) = \lambda(X - d) + d$ for $d \in \mathbb{R}$ the characteristic function G is also linear: $G(X) = a(X - d)$. Obviously there exists exactly one coherent inequality concept defined on every Ω_d. It is denoted by $\sim_d \equiv \sim_{\mathcal{T}^d}^d$ where $\mathcal{T}^d = \{T_\lambda^d | T_\lambda^d(X) = \lambda(X - d) + d, \lambda \in \mathbb{R}_{++}\}$. Then $T_\lambda(X) = \lambda(X - d\mathbf{1}) + d\mathbf{1}$ for all $X \in \Omega_d^n$, i.e., $d\mathbf{1}$ is the point in the 'lower left corner' of Ω_d^n, just outside the domain. T_λ translates X at first, then applies a relative transformation (equiproportional scaling), and then translates back. The distribution $d\mathbf{1}$ represents a reference point.

We denote the characteristic function[2] simply by

$$D(X) := X - d \text{ for } X \in \Omega_d \text{ and } D^{-1}(X) := X + d \text{ for } X \in \Omega_0 = \mathbb{R}_{++}.$$

The functions T_λ^d are indexed explicitly by *d* if necessary.

For $d = -\infty$ we obtain $T_\lambda(X) = X + \ln \lambda$ and define $D(X) := \exp(X)$.

2) If $d < 0$ we obtain Bossert and Pfingsten's (1990) intermediate inequality concepts. According to their notation the domain is equal to $\Omega_d = \left(-\frac{1-\theta}{\theta}, \infty\right)$ where $\theta \in (0,1]$ and $d = -\frac{1-\theta}{\theta}$. They consider the transformations $\boldsymbol{T}_\kappa^\theta(\boldsymbol{X}) = \boldsymbol{X} + \kappa(\theta \boldsymbol{X} + (1-\theta)\mathbf{1})$ for $\kappa > -1/\theta$. The concept is identical to $\sim_{rel}$ for $\theta = 1$ and to $\sim_{abs}$ for $\theta = 0$. If $\theta \in (0,1)$ we obtain $\sim_d$ for $d = -(1-\theta)/\theta$. It turns out that T_κ^θ coincides with $T_\lambda^d \in \mathcal{T}^d$ if $\lambda = 1 + \theta\kappa$.

3) For $d > 0$ we get a class of inequality concepts for which the reference point $d\mathbf{1} > 0$ is relevant. *d* can be interpreted as some basic income which is necessary for surviving. Therefore only the surplus income $X - d$ is taken into account (cf. Ebert (2004)).

In the following sections we restrict the analysis to the coherent inequality concepts $\sim_d$ for $d \in \mathbb{R} \cup \{-\infty\}$, i.e., to the relative, absolute, intermediate, and reference-point inequality view.

The definitions presented and the results derived will be related to the literature if possible. Furthermore, if not mentioned explicitly, the citations refer to intermediate inequality. To the best knowledge of the author, there are no papers – apart from Ebert (2004) – dealing with reference-point inequality. Therefore the corresponding definitions and results are new.

4. Lorenz dominance

Having clarified the concept of inequality we will use, we have to deal with a dominance criterion in the next step. Therefore we now introduce the corresponding notions of Lorenz dominance for $d \in \mathbb{R} \cup \{-\infty\}$. Since the characteristic function *D* allows us to define an isomorphism from $(\mathcal{T}^d, \circ)$ to $(\mathcal{T}^{rel}, \circ)$ the following definitions seem to be obvious in view of Corollary 2. It will turn out that they are appropriate. At first we define a *d*-Lorenz curve by

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2 Since the factor *a* is not relevant, we set $a = 1$. The notation *D* should remind the reader of *d*.

$$LC^d(0,X):=0,$$

$$LC^d\left(\frac{i}{n},X\right):=\frac{1}{n}\sum_{k=1}^{i}\frac{D\left(X_{(k)}\right)}{D(\mu(X))}.$$

$LC^d(p,X)$ is linear on $\left[\frac{i}{n},\frac{i+1}{n}\right]$ for $i=1,\ldots,n-1$, $X\in\Omega_d^n$ and $d\in\mathbb{R}\cup\{-\infty\}$.

For $d=0$ this definition coincides with the usual one; we get the ordinary Lorenz curve $LC:=LC^0$. Whenever $d\in\mathbb{R}$ incomes are translated, i.e. $X\to D(X)=X-d$ and the ordinary concept of Lorenz curve is computed for the vector of translated incomes $\boldsymbol{D}(\boldsymbol{X})$; i.e., we obtain

$$LC^d(p,X)=LC(p,\boldsymbol{D}(\boldsymbol{X}))\text{ for }X\in\Omega_d^n\text{ and }p\in[0,1].$$

For $d<0$ this definition has already been proposed in a remark by Besley and Preston (1988), p. 162. But in the literature (cf. e.g. Moyes (1992)) a different concept has been used for $d\le 0$, namely

$$LC^\theta\left(\frac{i}{n},X\right)=\frac{1}{n}\sum_{k=1}^{i}\left(\frac{X_{(k)}-\mu(X)}{\theta\,\mu(X)+(1-\theta)}+\theta\right)\text{ for }X\in\Omega_d^n\text{ and }d=-(1-\theta)/\theta$$

where $\theta\in(0,1]$ and

$$LC^\theta\left(\frac{i}{n},X\right)=\frac{1}{n}\sum_{k=1}^{i}\left(X_{(k)}-\mu(X)\right)\text{ for }X\in\Omega_d^n\text{ and }\theta=0.$$

It is also identical to LC for $\theta=1$ or $d=0$, but differs from LC^d for $\theta\in(0,1)$ and $d<0$. But there is a definite relationship between both concepts:

$$LC^\theta\left(\frac{i}{n},X\right)=(1-d)LC^d\left(\frac{i}{n},X\right)+\frac{i}{n}\frac{1}{1-d}=LC^d\left(\frac{i}{n},X\right)\Big/\theta+\frac{i}{n}\theta. \qquad (1)$$

In other words, LC^θ is a transform of LC^d. The concept of LC^d is much simpler than the corresponding LC^θ since it directly reveals the underlying idea. On the other hand the transform LC^θ is attractive if the concept of *absolute* inequality is investigated. The curves LC^θ converge pointwise to the absolute Lorenz curve as defined by Moyes (1987) if θ tends to zero. This is no longer true for the curves LC^d. They do not converge to $LC^{-\infty}$ for $d\to-\infty$

since there is no direct functional relationship between $LC^{-\infty}(i/n, X)$ and Moyes' Lorenz curve for absolute inequality. We have $LC^{-\infty}(i/n, X) = (1/n)\sum_{k=1}^{i}\exp\left(X_{(k)} - \mu(X)\right)$.

Now we are in the position to introduce

Definition 4

Let $X, Y \in \Omega_d^n$. Then X d-Lorenz dominates Y, i.e., $X \succsim_L^d Y$, if and only if

$$LC^d\left(\frac{i}{n}, X\right) \geq LC^d\left(\frac{i}{n}, Y\right) \quad \textit{for } i = 1,\ldots,n-1.$$

The symmetric part of this relation is denoted by $\sim_L^d$, its asymmetric one by $\succ_L^d$. For $d = 0$ $\succsim_L^d$ is abbreviated by $\succsim_L$. It should be clear that further concepts like generalized Lorenz dominance or ratio dominance (see e.g. Moyes (1992)) can be extended in the same way. As they are not discussed below, here no definition is presented. Furthermore, the relationship (1) demonstrates that for $d < 0$ d-Lorenz dominance can equivalently be defined by means of the Lorenz curves LC^θ. Moreover, Lorenz dominance for $d = -\infty$ is equivalent to the dominance criterion for absolute inequality suggested in Moyes (1987).

Given this definition we can establish a number of results. At first we investigate whether the definition of d-Lorenz dominance is well chosen. We establish[3]

Proposition 4

Let $d \in \mathbb{R} \cup \{-\infty\}$. Then

(a) *For all $X, Y \in \Omega_d^n$: $X \succsim_L^d Y \Leftrightarrow D(X) \succsim_L D(Y)$*

(b) *For all $X, Y \in \Omega_d^n$: $X \sim_d Y \Leftrightarrow X \sim_L^d Y$*

Part (a) of Proposition 4 reflects the definition of the d-Lorenz curve and of d-Lorenz dominance. It demonstrates again the simple relationship between $\sim_d$ and $\sim_{rel}$.

A minimal requirement for the ordering $\succsim_L^d$ is its compatibility with the inequality concept $\sim_d$ and the idea that two income distributions possess the same degree of inequality if one is a transform by T_λ^d of the other one. Thus we expect that two income distributions having the

3 The proofs of all propositions and of equations (1)-(3) have been collected in the Appendix.

same degree of inequality should be equivalent as far as d-Lorenz dominance is concerned. (b) shows even more: The criterion of d-Lorenz dominance is very sensitive. Whenever two income distributions are not equivalent, as far as inequality is concerned, they cannot be equivalent with respect to the Lorenz criterion.

Next we consider some implications of the definition of d-Lorenz dominance. They will be used below when taxation is examined. We obtain:

Proposition 5

(a) *Let* $H:\Omega_d \to \Omega_d$. *Then*

$$X \sim_L^d H(X) \textit{ for all } X \in \Omega_d^n \Leftrightarrow H \in \mathcal{T}^d.$$

(b) *Let* $H_i:\Omega_d \to \Omega_d$ *and let* H_i *be nonconstant for* $i=1,2$. *Then*

$$H_1(X) \sim_L^d H_2(X) \textit{ for all } X \in \Omega_d^n.$$

$$\Leftrightarrow \textit{ There is } T_\lambda^d \in \mathcal{T}^d \textit{ such that } H_1(X) = T_\lambda^d(H_2(X)) \textit{ for } X \in \Omega_d.$$

(c) *Let* $H:\Omega_d \to \Omega_d$. *Then*

$$\textit{[For all } X,Y \in \Omega_d^n : X \succsim_L^d Y \Rightarrow H(X) \succsim_L^d H(Y)] \Leftrightarrow H \in \mathcal{T}^d \textit{ or } H \textit{ is constant.}$$

(a) and (b) consider the relationship between $\sim_L^d$ and symmetric transformations. (a) is related to part (b) of Proposition 4. Equivalence with respect to d-Lorenz dominance requires that the transformation is admissible. Part (b) derives an analogous relationship for two transformations. The last part (c) is important for the discussion of tax schedules. d-Lorenz dominance is only preserved if the transformation applied is admissible $\left(H \in \mathcal{T}^d\right)$ or always maps an income vector into the same distribution. It is a result already proven in Theorem 3.3 of Moyes (1992) and Proposition 4.6 of Ebert and Moyes (2002) for $d \leq 0$. But it holds for all $d \in \mathbb{R} \cup \{-\infty\}$.

The analysis of this section can be summarized simply. The relation $\succsim_L^d$, i.e., d-Lorenz dominance, possesses the same properties as $\succsim_L$ when interpreted appropriately.

5. Taxation

Next we will consider the taxation of income. Given an inequality concept $\sim_d$ for $d \in \mathbb{R} \cup \{-\infty\}$ we examine a net income function $F^d \in \mathcal{F}(\Omega_d)$. It assigns to any pre-tax income $X \in \Omega_d$ the post-tax income $F^d(X)$. Since $F^d(X) \in \Omega_d$, negative net incomes are admitted if $d < 0$. Monotonicity implies that there are no rank reversals, i.e.,

$$X \leq X' \Rightarrow F^d(X) \leq F^d(X')$$

for all $X, X' \in \Omega_d$. This condition can be interpreted as the property of incentive preservation. For F^d the corresponding tax-liability can be derived by defining $t^d(X) = X - F^d(X)$ for $X \in \Omega_d$. Since there is a duality between F^d and t^d (cf. Moyes and Shorrocks (1998)) it is sufficient to investigate net income functions. Corresponding results for tax schedules can then be derived by duality.

The sets of net income functions for different $d \in \mathbb{R} \cup \{-\infty\}$ are isomorphic, i.e., they are essentially 'identical' with $\mathcal{F}(\Omega_0)$ since we have

$$F^d \in \mathcal{F}(\Omega_d) \Leftrightarrow D \circ F^d \circ D^{-1} \in \mathcal{F}(\Omega_0) \text{ and}$$
$$F^0 \in \mathcal{F}(\Omega_0) \Leftrightarrow D^{-1} \circ F^0 \circ D \in \mathcal{F}(\Omega_d).$$

There is also a one-to-one correspondence between the respective tax functions for $d \in \mathbb{R}$:

$$t^d(X) = t^0(D(X)) \Leftrightarrow F^d = D^{-1} \circ F^0 \circ D.$$

For the evaluation of taxation we introduce the usual concepts (cf. for example Pfingsten (1986)):

Definition 5

$F^d \in \mathcal{F}(\Omega_d)$ is called d-inequality preserving [reducing] if and only if

$$F^d(X) \sim_L^d X \quad \left[F^d(X) \succsim_L^d X\right] \textit{ for all } X \in \Omega_d^n.$$

A net income function is d-inequality preserving (reducing) if the distribution of net incomes possesses the same (less) inequality than the distribution of gross incomes (for *all* income distributions). The implications of taxation are evaluated by means of the criterion of d-Lorenz dominance.

Now we are able to establish

Proposition 6

Let $F^d, F_1^d, F_2^d \in \mathcal{F}(\Omega_d)$ *and* $d \in \mathbb{R} \cup \{-\infty\}$. *Then*

(a) F^d *is d-inequality preserving* $\Leftrightarrow F^d \in \mathcal{T}^d$.

(b) F^d *is d-inequality reducing* $\Leftrightarrow \dfrac{D(F^d(X))}{D(X)}$ *is nonincreasing in* $X \in \Omega_d$.

(c) *Let* F_1^d *and* F_2^d *be not constant.* $F_1^d(X) \sim_L^d F_2^d(X)$ *for all* $X \in \Omega_d^n$ $\Leftrightarrow$ *There is* $T_\lambda^d \in \mathcal{T}^d$ *such that* $F_1^d = T_\lambda^d \circ F_2^d$.

Part (a) is not surprising given Proposition 5(a). If F^d is *d*-inequality preserving there exists $\lambda > 0$ such that $F^d(X) = \lambda(X-d)+d$. Then the corresponding tax function has a particular simple form $t^d(X) = (1-\lambda)D(X)$. Our approach simplifies Pfingsten's (1986) Theorem 4.2 considerably. But here the factor $1-\lambda$ can be positive or negative in contrast to Pfingsten's result. This is a consequence of the way the domains are chosen. Pfingsten considers only strictly positive pre- and post-tax incomes.

Part (b) demonstrates for $d \in \mathbb{R}$ that *d*-inequality reduction is equivalent to the fact that the average retention rate – measured for the 'appropriately translated net and gross income' $D(F^d(X))$ and $D(X) = (X-d)$ – is nonincreasing in income. This condition is reasonable since *d* represents the basic income which is the point of reference. For $d = 0$ a net income function F^d reduces inequality if and only if F^d is progressive (in the standard sense) (cf. Jakobsson (1976), Eichhorn, Funke, and Richter (1984)); for $d = -\infty$ the condition requires that $F^d(X) - X$ is nonincreasing. The condition for inequality reduction presented here is much simpler than the definition in Pfingsten (1988) for $d < 0$. It requires

$$\frac{F^d(X)-X}{\theta X - 1 + \theta} \quad \text{to be nonincreasing in } X.$$

It is easy to see that this condition is equivalent to (b):

$$\frac{F^d(X)-X}{\theta X - 1+\theta} = \frac{1}{\theta}\left(\frac{D(F^d(X))}{D(X)} - 1\right) \quad \text{for } d = -(1-\theta)/\theta \text{ and } \theta \in (0,1]. \tag{2}$$

Therefore the following definition suggests itself:

Definition 6

$F^d \in \mathcal{F}(\Omega_d)$ is called d-progressive if and only if $D\left(F^d(X)\right)/D(X)$ is nonincreasing in $X \in \Omega_d$.

Accordingly we define $\hat{\mathcal{F}}(\Omega_d) := \left\{F^d \in \mathcal{F}(\Omega_d) \middle| D\left(F^d(X)\right)/D(X) \text{ is } \textit{non}\text{increasing in } X \in \Omega_d\right\}$. The set $\hat{\mathcal{F}}(\Omega_d)$ consists of all d-progressive net income functions. Furthermore we have a direct implication:

$$F^d \in \mathcal{F}(\Omega_d) \text{ is d-progressive} \Leftrightarrow D \circ F^d \circ D^{-1} \in \mathcal{F}(\Omega_0) \text{ is 0-progressive}$$

$$F^0 \in \mathcal{F}(\Omega_0) \text{ is 0-progressive} \Leftrightarrow D^{-1} \circ F^0 \circ D \in \mathcal{F}(\Omega_d) \text{ is d-progressive}$$

Part (c) of Proposition 6 demonstrates that two net income functions are equivalent as far as inequality reduction is concerned if and only if they satisfy a simple relationship. One must be a T_λ^d-transform of the other. Again (c) is more general than a corresponding result by Pfingsten (1986) (Theorem 3.4) for $d \leq 0$. Moreover, in our framework the connection between the corresponding tax functions is more transparent. It is described by $t_1^d(X) = (1-\lambda)D(X) + \lambda t_2^d(X)$ for $X \in \Omega_d$ for $d \in \mathbb{R}$. The factor $(1-\lambda)$ can again be negative or positive. t_1^d is a combination of a proportional tax/transfer and the tax schedule t_2^d.

Summing up, the characterization of a d-inequality reducing net income function F^d is straightforward. Since d is the basic income, the translated (and corresponding) net income function which assigns $D\left(F^d(X)\right)$ to $D(X)$ has to be inequality reducing and therefore progressive in the standard sense. Moreover, the usual results can be extended directly.

6. Measures of tax progression

Up to now it has been examined under what condition a net income function is d-progressive. The degree of progression for a given income tax has not been investigated. In the following measures of progression will be introduced for the inequality concepts $\sim_d$ where $d \in \mathbb{R} \cup \{-\infty\}$.

For the rest of the paper we assume that all net income functions considered are once continuously differentiable; i.e., we confine ourselves to $\mathcal{F}_c(\Omega_d)$ and to $\hat{\mathcal{F}}_c(\Omega_d) := \hat{\mathcal{F}}(\Omega_d) \cap \mathcal{F}_c(\Omega_d)$ since we want to employ elasticities.

We propose a measure of tax progression.

Definition 7

Let $F^d \in \mathcal{F}_c(\Omega_d)$ and $d \in \mathbb{R} \cup \{-\infty\}$. The elasticity $\eta^d(F^d, X) := \eta(D \circ F^d, D(X))$ is called d-residual progression.

The way d-residual progression is defined is suggested by earlier definitions. Since $F^d = D^{-1} \circ F^0 \circ D$ for $F^d \in \mathcal{F}_c(\Omega_d)$ where $F^0 \in \mathcal{F}_c(\Omega_0)$ we obtain

$$\eta^d(F^d, X) = \eta\left(D \circ \left(D^{-1} \circ F^0 \circ D\right), D(X)\right) = \eta^0\left(F^0, Y\right)\Big|_{Y=D(X)}.$$

In other words d-residual progression is measured by the corresponding standard residual progression for the translated net income function and income.

The definition of $\eta^d(F^d, X)$ is simpler than the measure of progression introduced by Pfingsten (1987). He proposed for $\theta \in (0,1)$ and $d = -(1-\theta)/\theta < 0$

$$P^\theta(t^d, X) = \frac{\theta X\left[t^{d\prime}(X) - t^d(X)/X\right] + (1-\theta)t^{d\prime}(X)}{\theta X\left[1 - t^d(X)/X\right] + (1-\theta)}$$

when t^d corresponds to F^d. It turns out that

$$P^\theta(t^d, X) = 1 - \eta^d(F^d, X) \tag{3}$$

(see the Appendix; cf. also Besley and Preston's (1988) remark on this point and Lambert's (2001) presentation).

Now it is not surprising that the following results hold:

Proposition 7

Let $F^d, F_1^d, F_2^d \in \mathcal{F}_c(\Omega_d)$ and $d \in \mathbb{R} \cup \{-\infty\}$.

(a) $F^d \in \hat{\mathcal{F}}_c(\Omega_d) \Leftrightarrow \eta^d(F^d, X) \leq 1$ *for all* $X \in \Omega_d$.

(b) $\eta^d\left(F^d,X\right)=\varepsilon$ *for all* $X\in\Omega_d$

$$\Leftrightarrow \textit{There is } \alpha>0 \textit{ such that } F^d\left(X\right)=D^{-1}\left(\alpha D\left(X\right)^{\varepsilon}\right) \textit{ for all } X\in\Omega_d.$$

(c) $F_1^d\left(\mathbf{X}\right)\succsim_L^d F_2^d\left(\mathbf{X}\right)$ *for all* $\mathbf{X}\in\Omega_d^n \Leftrightarrow \eta^d\left(F_1^d,X\right)\le\eta^d\left(F_2^d,X\right)$ *for all* $X\in\Omega_d$.

(a) demonstrates that the definition of the *d*-residual progression η^d is appropriate. F^d is *d*-progressive if and only if $\eta^d\left(F^d,X\right)$ is less than or equal to unity for all incomes.

Part (b) considers net income functions possessing constant *d*-residual progression. For $d\in\mathbb{R}$ they are given by $F^d\left(X\right)=\alpha\left(X-d\right)^{\varepsilon}+d$ for an $\alpha>0$. For $d=-\infty$ we obtain $F^{-\infty}\left(X\right)=\gamma+\varepsilon X$. The result is related to Theorem 3.2 in Moyes (1992).

Finally, part (c) is a generalization of a result derived by Jakobsson (1976) for $d=0$. A net income function is more *d*-inequality reducing than another one if and only if its *d*-residual progression is not greater than the *d*-residual progression of the other one (cf. also Theorem 2.2. in Moyes (1992)).

In summary, we find that the criterion of *d*-residual progression defined above is a reasonable analogue to the standard residual progression.

7. Conclusion

The paper has investigated *d*-Lorenz dominance and *d*-inequality reduction for the coherent inequality concepts $\sim_d$ where $d\in\mathbb{R}\cup\{-\infty\}$. The characteristics of these concepts are again summarized in Table 1. It has turned out that these inequality concepts are essentially equivalent to the relative inequality concept. The only difference is that the reference point *d* is different from 0 (whenever $d\neq 0$). The definitions introduced and proposed are identical with the usual ones – apart from the fact that the reference point differs. Therefore it is easy to describe the set of *d*-inequality reducing or *d*-progressive tax functions. There is a one-to-one mapping onto the respective set of relative inequality measures and, respectively, the set of (in the standard sense) progressive tax functions. Given these relationships it is not surprising that the results we obtain in Proposition 5-7 are generalizations of the results we know from the usual framework.

Nevertheless, the paper has clarified these relationships and presented simple and transparent definitions and brief and elegant proofs of the (general) results. Furthermore an investigation of reference point inequality has not yet been performed before. Finally, a number of results for the intermediate inequality view, which can be found in the literature, have been collected and extended. They are presented in a unifying framework. Moreover the results of this paper demonstrate that the nonstandard inequality concepts considered are closely related and isomorphic to the relative inequality view. Therefore there should be no difficulties to apply them in practice. The examination has proven that these concepts are not all exotic and also present a feasible and consistent framework for empirical work.

The analysis has been performed under the assumption that the individuals considered differ only with respect to income and are identical otherwise. In principle the framework can be extended. Then things are more complicated since the differences in attributes have to be taken into account. One possibility is to use equivalence scales which reflect the type of household. Ebert and Moyes (2003) investigate the implications of reasonable conditions for the definition of Lorenz dominance in a heterogeneous framework. In Ebert and Moyes (2002) welfare and inequality are examined when households are heterogeneous. The corresponding problem of inequality reducing taxation is examined in Ebert and Moyes (2000). Finally Ebert and Lambert (2004) define measures of progression in the extended model. In all these cases various concepts of inequality are admitted.

It is well known that criteria like Lorenz dominance are incomplete. They do not allow us to compare two income distributions if the corresponding Lorenz curves intersect. In this case it is helpful to employ summary measures of inequality. In the literature many inequality measures are considered. An inequality measure is in general only invariant with respect to exactly one type of transformation. Relative measures are not changed if all incomes are changed in the same proportion. Absolute measures are not altered if each income is changed by the same amount. One can similarly derive measures which are invariant with respect to the other coherent inequality concepts. Nevertheless, in some cases it is possible to define inequality measures, which are invariant with respect to various inequality concepts, by a slight variation. For example the Gini coefficient is a relative measure:

$$G(X) = \left(1/n^2\right)\sum_{i=1}^{n}(n+1-2i)X_{[i]}\Big/\mu(X)$$

where $X_{[\,]}$ is generated from X by permuting the components such that incomes are non-increasing. The measure can be interpreted as the relative welfare loss evaluated by means of the Gini welfare function. If we renormalize this welfare loss we get an analogue to the Gini coefficient by

$$G(X,d)=\left(1/n^2\right)\sum_{i=1}^{n}(n+1-2i)X_{[i]}/\left(\mu(X)-d\right)$$

and, respectively,

$$G(X,-\infty)=\left(1/n^2\right)\sum_{i=1}^{n}(n+1-2i)X_{[i]}$$

for $d\in\mathbb{R}\cup\{-\infty\}$. Thus in this situation the renormalization guarantees that the measure is invariant with respect to the inequality concept $\sim_d$ (cf. Ebert (1997)).

The paper has confined itself to an investigation of inequality reducing taxation. But it should be emphasized that everything shown and proven for *d*-inequality reducing tax functions can also be derived for *d*-inequality increasing tax functions. Then in the definition 5 the inequality sign has to be reversed ($F^d(X)\precsim_L^d X$), i.e., the post-tax income distribution has to possess more inequality than the pre-tax income distribution. Similarly, a net income schedule F^d is called *d*-regressive if and only if $D\left(F^d(X)\right)/D(X)$ is nondecreasing in income X (definition 6). In this case the elasticity representing *d*-residual progression has to be (weakly) greater than unity. Thus we observe the same kind of symmetry we know from the standard framework (cf. Lambert (2001)).

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	Ω_d	$D(X)$	$T_\lambda(X)$
$d = 0$	$\mathbb{R}_{++}$	X	λX
$d \gtrless 0$	(d,∞)	$X - d$	$\lambda(X-d)+d$
$d = -\infty$	$\mathbb{R}$	$\exp(X)$	$X + \ln\lambda$

Table 1: Summary of characteristics of the coherent inequality concepts

References

Amiel, Y. and F.A. Cowell (1992), Measurement of income inequality. Experimental test by questionnaire, Journal of Public Economics 47, 3-26.

Arnold, B.C. (1990), The Lorenz order and the effects of taxation policies, Bulletin of Economic Research 42, 249-265.

Besley, T.J. and I.P. Preston (1988), Invariance and the axiomatics of income tax progression: a comment, Bulletin of Economic Research 40, 159-163.

Bossert, W. and A. Pfingsten (1990), Intermediate inequality: concepts, indices, and welfare implications, Mathematical Social Sciences 19, 117-134.

Chakravarty, S.R. (2009), Deprivation, inequality and welfare, Japanese Economic Review 60, 172-190.

del Rio, C. and J. Ruiz-Castillo (2000), Intermediate inequality and welfare, Social Choice and Welfare 17, 223-239.

Ebert, U. (1997), Linear inequality concepts and social welfare, DARP Discussion Paper 33, London School of Economics.

Ebert, U. (2004), Coherent inequality views: Linear invariant measures reconsidered, Mathematical Social Sciences 47, 1-20.

Ebert U. and P. Moyes (2000), Consistent Income Tax Structures When Households Are Heterogeneous, Journal of Economic Theory 90, 116-150.

Ebert U. and P. Moyes (2002), Welfare, inequality and the transformation of incomes. The case of weighted income distributions, in P. Moyes, C. Seidl and A.F. Shorrocks (eds.), Inequalities: Theory, Measurement and Applications, Journal of Economics, Supplementum # 9, 9-50.

Ebert, U. and P. Moyes (2003), Equivalence scales reconsidered, Econometrica 71, 319-343.

Ebert U. and P.J. Lambert (2004), Horizontal equity when equivalence scales are not constant, Public Finance Review 32, 426-440.

Eichhorn, W., H. Funke, and W.F. Richter (1984), Tax progression and inequality of income distribution, Journal of Mathematical Economics 13, 127-131.

Jakobsson, U. (1976), On the measurement of the degree of progression, Journal of Public Economics 5, 161-168.

Kolm, S.-C. (1976), Unequal inequalities I, Journal of Economic Theory 12, 416-442.

Lambert, P.J. (2001), The Distribution and Redistribution of Income, third edition, Manchester University Press, Manchester.

Lang, S. (1968), Algebra, Addison-Wesley, Reading, Mass.

Moyes, P. (1987), A new concept of Lorenz domination, Economics Letters 23, 203-207.

Moyes, P. (1992), The through-time redistributive effect of income taxation: The intermediate inequality view, Mathematical Social Sciences 24, 59-71.

Moyes, P. and A.F. Shorrocks (1998), The impossibility of a progressive tax structure, Journal of Public Economics 69, 49-65.

Pfingsten, A. (1986), Distributionally-neutral tax changes for different inequality concepts, Journal of Public Economics 30, 385-393.

Pfingsten, A. and C. Seidl (1997), Ray invariant inequality measures, in: S. Zandvikili (Ed.), Taxation and Inequality, Research on Economic Inequality, Vol. 7, 107-129, JAI Press.

Yoshida, T. (2005), Social welfare rankings of income distributions – A new parametric concept of intermediate inequality, Social Choice and Welfare 24, 557-574.

Zoli, C. (1998), A surplus sharing approach to the measurement of inequality, mimeo, University of York and University of Pavia.

Appendix

Proof of equation (1)

Observe that $\theta = 1/(1-d) > 0$. Then

$$\frac{X_{(k)} - \mu(X)}{\theta\mu(X) + (1-\theta)} + \theta = \frac{X_{(k)} - \mu(X)}{\frac{\mu(X)}{1-d} + 1 - \frac{1}{1-d}} + \frac{1}{1-d}$$

$$= (1-d)\frac{X_{(k)} - d}{\mu(X) - d} - (1-d)\frac{\mu(X) - d}{\mu(X) - d} + \frac{1}{1-d} = (1-d)\frac{D(X_{(k)})}{D(\mu(X))} + \frac{1}{1-d} \qquad \square$$

Proof of Proposition 4

(a) Obvious

(b) $X \sim_{\mathcal{T}^d} Y \Leftrightarrow$ There is $T_\lambda^d \in \mathcal{T}^d$ such that $Y = T_\lambda^d(X)$

$\Leftrightarrow$ There is $S_\lambda \in \mathcal{T}^{rel}$ such that $Y = D^{-1} \circ S_\lambda \circ D(X)$

$\Leftrightarrow$ There is $S_\lambda \in \mathcal{T}^{rel}$ such that $D(Y) = S_\lambda(D(X))$

$\Leftrightarrow D(X) \sim_L D(Y)$ $\square$

Proof of Proposition 5

(a) see (b) for $H_1(X) = X$ and $H_2(X) = H(X)$

(b) $H_1(X) \sim_L^d H_2(X)$ for all $X \in \Omega_d^n$

$\Leftrightarrow D(H_1(X)) \sim_L D(H_2(X))$ for all $X \in \Omega_d^n$

$\Leftrightarrow$ There is $\lambda \in \mathrm{R}_{++}$ such that $D(H_1(X)) = S_\lambda(D(H_2(X)))$ for all $X \in \Omega_d^n$

$\Leftrightarrow H_1(X) = D^{-1} \circ S_\lambda \circ D(H_2(X))$ for all $X \in \Omega_d^n$

$\Leftrightarrow H_1(X) = T_\lambda^d(H_2(X))$ for all $X \in \Omega_d$.

(c) $\left[X \succsim_L^d Y \Rightarrow H(X) \succsim_L^d H(Y) \text{ for all } X, Y \in \Omega_d^n\right]$

$\Leftrightarrow \left[D(X) \succsim_L D(Y) \Rightarrow D(H(X)) \succsim_L D(H(Y)) \text{ for all } X, Y \in \Omega_d^n\right]$

$$\Leftrightarrow \left[\boldsymbol{D}(\boldsymbol{X}) \succsim_L \boldsymbol{D}(\boldsymbol{Y}) \Rightarrow \boldsymbol{D} \circ \boldsymbol{H} \circ \boldsymbol{D}^{-1}(\boldsymbol{D}(\boldsymbol{X})) \succsim_L \boldsymbol{D} \circ \boldsymbol{H} \circ \boldsymbol{D}^{-1}(\boldsymbol{D}(\boldsymbol{Y})) \text{ for all } \boldsymbol{X}, \boldsymbol{Y} \in \Omega_d^n \right]$$

$$\Leftrightarrow \left[\boldsymbol{X} \succsim_L \boldsymbol{Y} \Rightarrow (\boldsymbol{D} \circ \boldsymbol{H} \circ \boldsymbol{D}^{-1})(\boldsymbol{X}) \succsim_L (\boldsymbol{D} \circ \boldsymbol{H} \circ \boldsymbol{D}^{-1})(\boldsymbol{Y}) \text{ for all } \boldsymbol{X}, \boldsymbol{Y} \in \Omega_0^n \right]$$

$\Leftrightarrow$ There is $\lambda \in \mathbb{R}_{++}$ such that $D \circ H \circ D^{-1} = S_\lambda$ or H is constant (cf. Theorem 3.1 in Arnold (1990)).

$\Leftrightarrow H = D^{-1} \circ S_\lambda \circ D = T_\lambda^d \in \mathcal{T}^d$ or H is constant. □

Proof of equation (2)

$$\frac{F^d(X) - X}{\theta X - 1 + \theta} = \frac{1}{\theta} \frac{F^d(X) - X}{X - \frac{1-\theta}{\theta}} = \frac{1}{\theta} \left(\frac{F^d(X) - d}{X - d} - \frac{X - d}{X - d} \right) = \frac{1}{\theta} \left(\frac{D(F^d(X))}{D(X)} - 1 \right)$$

Proof of Proposition 6

(a) See Proposition 5(a) above.

(b) $\boldsymbol{F}^d(\boldsymbol{X}) \succsim_L^d \boldsymbol{X}$ for all $\boldsymbol{X} \in \Omega_d^n$

$\Leftrightarrow \boldsymbol{D}(\boldsymbol{F}^d(\boldsymbol{X})) \succsim_L \boldsymbol{D}(\boldsymbol{X})$ for all $\boldsymbol{X} \in \Omega_d^n$

$\Leftrightarrow \dfrac{D(F^d(X))}{D(X)}$ is nonincreasing in X (see Jakobsson (1976)). □

(c) See Proposition 5(b) above.

Proof of Proposition 7

Let $F^d \in \mathcal{F}_c(\Omega_d)$: There is $F^0 \in \mathcal{F}_c(\Omega_0)$ such that $F^d = D^{-1} \circ F^0 \circ D$. Then

(a) $F^d \in \hat{\mathcal{F}}_c(\Omega_d) \Leftrightarrow F^0 \in \hat{\mathcal{F}}_c^0(\Omega_d) \Leftrightarrow \eta^0(F^0, Y) \leq 1$ for all $Y \in \Omega_0$.

$\Leftrightarrow \eta^d(F^d, X) \leq 1$ for all $X \in \Omega_d$.

(b) $\eta^d(F^d, X) = \varepsilon$ for all $X \in \Omega_d \Leftrightarrow \eta^0(F^0, Y) = \varepsilon$ for all $Y \in \Omega_0$.

$\Leftrightarrow$ There is $\alpha > 0$ such that $F^0(Y) = \alpha Y^\varepsilon$ for all $Y \in \Omega_0$.

$\Leftrightarrow$ There is $\alpha > 0$ such that $F^d(X) = D^{-1}\left(\alpha D(X)^\varepsilon\right)$ for all $X \in \Omega_d$

(see Jakobsson (1976)).

(c) Analogous. Cf. Jakobsson (1976). □

Proof of equation (3)

$$P^{0}\left(t^{d},X\right)=\frac{\theta X\left[t^{d'}(X)-t^{d}(X)/X\right]+(1-\theta)t^{d'}(X)}{\theta X\left[1-t^{d}(X)/X\right]+(1-\theta)}$$

$$=\frac{t^{d'}(X)X-t^{d}(X)+\frac{(1-\theta)}{\theta}t^{d'}(X)}{X-t^{d}(X)+\frac{(1-\theta)}{\theta}}=\frac{X-t^{d}(X)-X+t^{d'}(X)(X-d)}{X-t^{d}(X)+d}$$

$$=\frac{F^{d}(X)-d+(d-X)+t^{d'}(X)(X-d)}{F^{d}(X)-d}=1-\frac{\left(1-t^{d'}(X)\right)(X-d)}{F^{d}(X)-d}$$

$$=1-F^{d'}(X)\frac{D(X)}{D\left(F^{d}(X)\right)}=1-D'\left(F^{d}(X)\right)F^{d'}(X)\frac{D(X)}{D\left(F^{d}(X)\right)}=1-\eta\left(F^{d},X\right)$$

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