Capture Basin Characterization for Frugal Extremum Seeking Cyclic Navigation using Set Methods

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Introduction

Context: Navigation in GNSS-denied environments

- In many scenarios, such as underwater or indoor environments, GNSS signals are not available.
- This requires the development of new control strategies for autonomous robots.

Objective: Navigation without localization

- The goal is to design control laws that enable a robot to navigate without knowing its precise location.
- This approach focuses on cyclic behaviors and a stabilization of the robot' based on sensor measurements.

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- Set-based methods work with sets of values guaranteed to contain the true state, rather than single point estimates.
- Uncertainties from models and sensors are represented as bounded sets, like intervals or boxes.

Advantages

- **Guaranteed results**: Provides robust and verified enclosures.
- **Weak noise assumptions**: Only requires noise to be bounded, no need for statistical distribution.

Drawbacks

- **Conservatism**: The wrapping effect can lead to pessimistic (large) sets over time.
- **Complexity**: Can be computationally expensive for high-dimensional systems.

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Formalism

Consider a **dynamical system** described by the following equations:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ y = g(\mathbf{x}) \end{cases} \tag{1}$$

Objective

- Converge towards the **extremum** of the output function $g(\mathbf{x})$.
- Achieve this with **minimal sensing** and **actuation** capabilities.

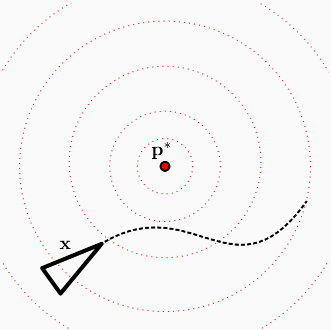

Figure 1: Navigating following a cyclic path

Unicycle Model

The robot of state $\mathbf{x} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$ is following the equations of a unicycle model:

$$\dot{\mathbf{x}} = \begin{cases} \dot{x} = u_1 \cdot \cos(\theta) \\ \dot{y} = u_1 \cdot \sin(\theta) \\ \dot{\theta} = u_2 \end{cases} \tag{2}$$

Navigating through a cyclic path

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_0 \\ \omega_0 \end{bmatrix} \tag{3}$$

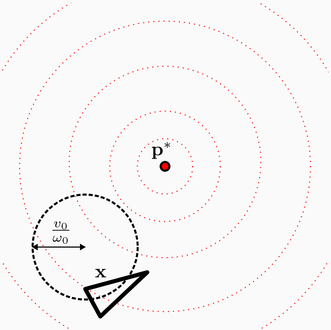

Figure 2: Illustration of the unicycle model

Discretization

A measurement is taken every T_s seconds

$$y_k = g(\mathbf{x}(t_0 + k \cdot T_s)) \quad \text{with} \quad k \in \mathbb{N} \quad (4)$$

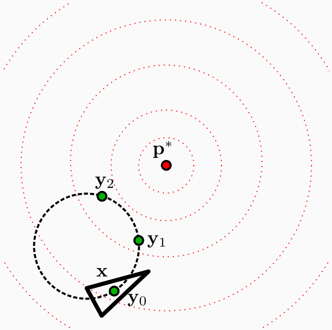

Figure 3

Zero Order Hold Control

$$\mathbf{u}_k = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_0 \\ \omega_0 + K_p \cdot (y_k - y_{k-1}) \end{bmatrix} \quad (5)$$

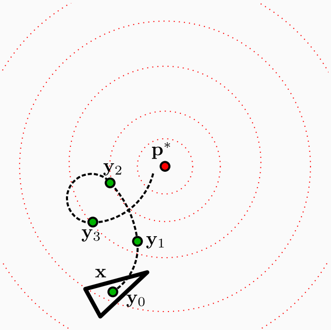

Figure 4: Taking measurements along the cycle

Figure 5: Experimental platform used for the navigation method

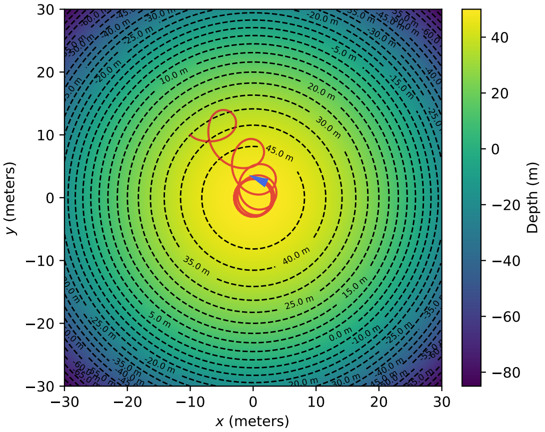

Figure 6: Simulation of the navigation method

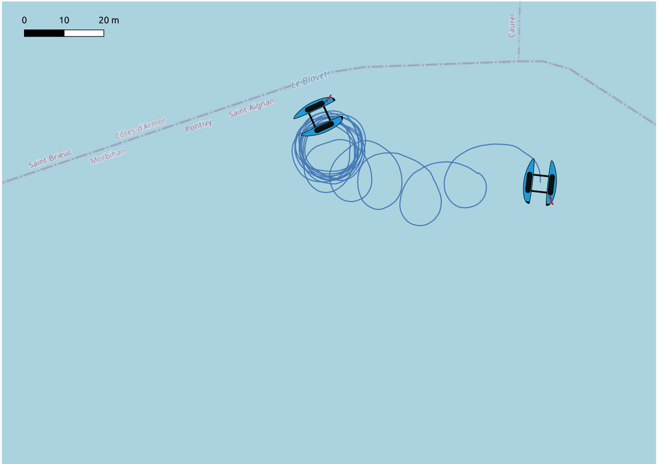

Figure 7: First Trial of the navigation method

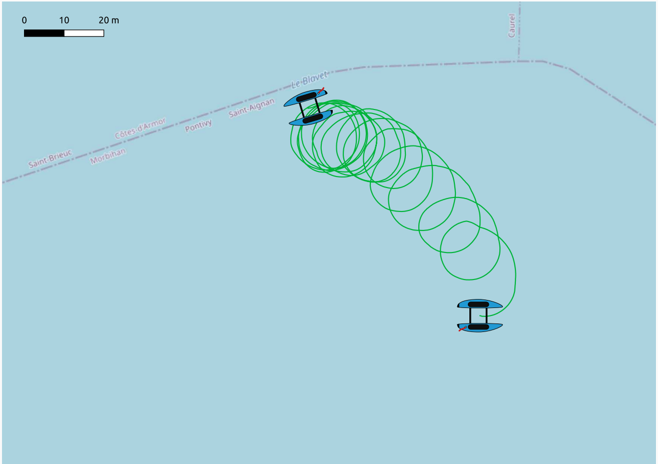

Figure 7: Second Trial of the navigation method

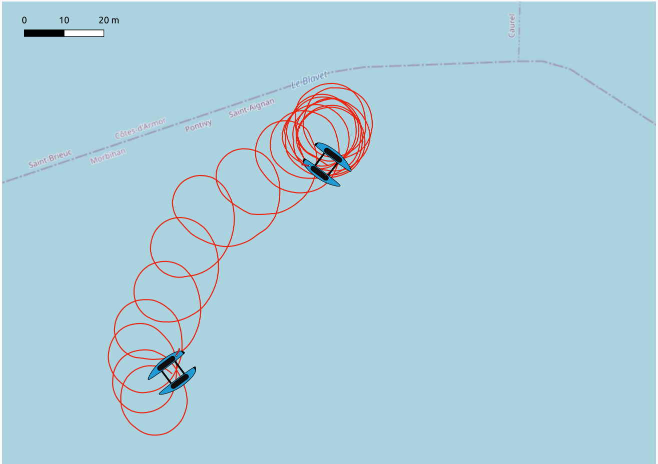

Figure 7: Third Trial of the navigation method

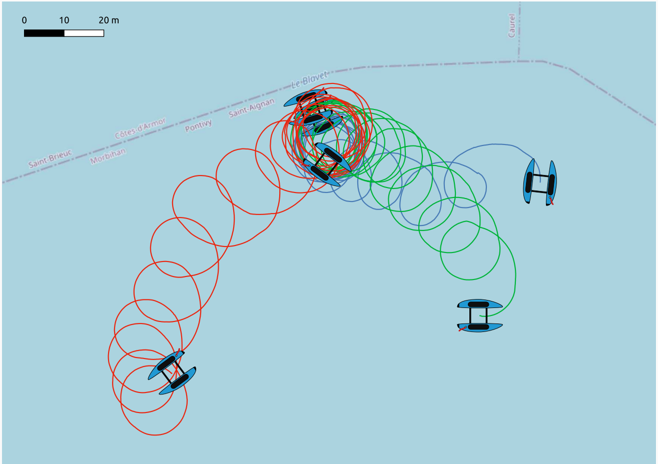

Figure 8: Experimental trials of the navigation method

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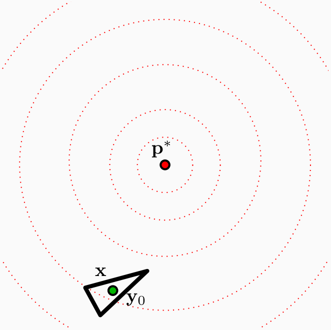
Figure 9: Experimental results of the navigation method

Stability Analysis

Center Coordinates

The center coordinates of the cyclic path are given by:

$$\mathbf{c}_k = \begin{bmatrix} x_k + \frac{v_k}{\omega_k} \cdot \cos(\theta_k + \omega_k \cdot T_s) \\ y_k + \frac{v_k}{\omega_k} \cdot \sin(\theta_k + \omega_k \cdot T_s) \end{bmatrix} \quad (6)$$



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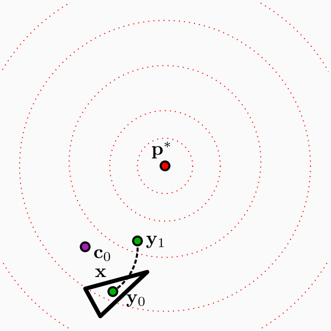

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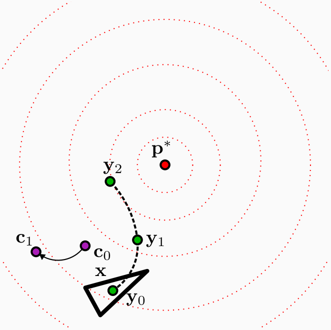

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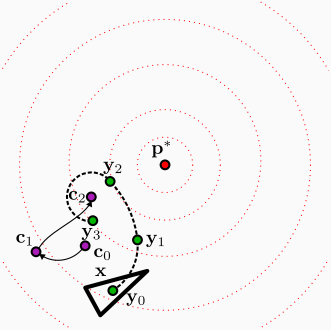

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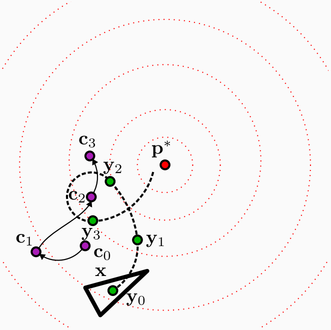

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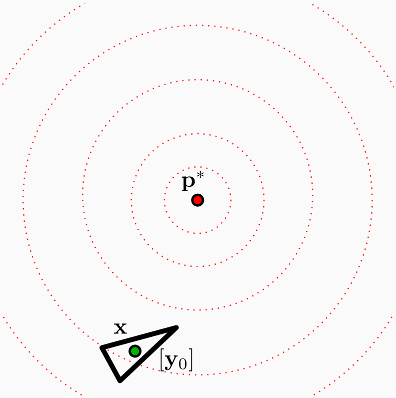


Figure 11: Propagation of uncertainties through the navigation method

Adding uncertainties

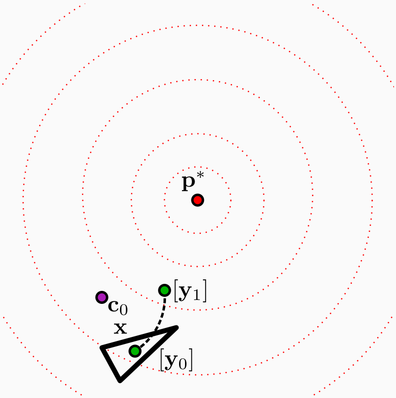
The system can be modeled by the **differential inclusion**:

$$\mathcal{S}: \begin{cases} \mathbf{c}_{k+1} \in \boldsymbol{\phi}(\mathbf{c}_k, \mathbf{u}_k) + [\mathbf{w}] \\ y_k \in \mu(\mathbf{c}_k) + [v], \end{cases} \tag{7}$$

with $[\mathbf{w}] \in \mathbb{W}$, and $[v] \in \mathbb{V}$ representing the uncertainties.

Autonomous System Equation

$$\mathbf{c}_{k+1} \in [\varphi](\mathbf{c}_k) \tag{8}$$



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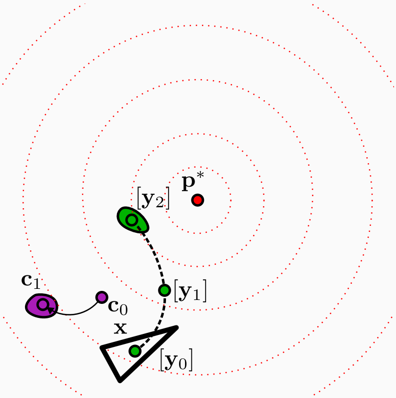

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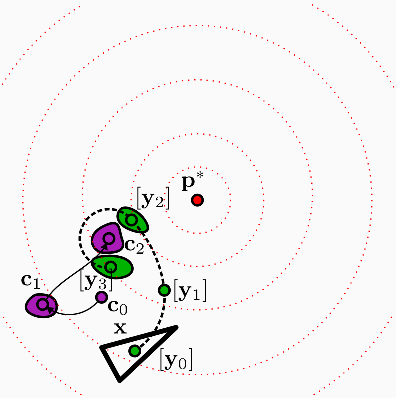

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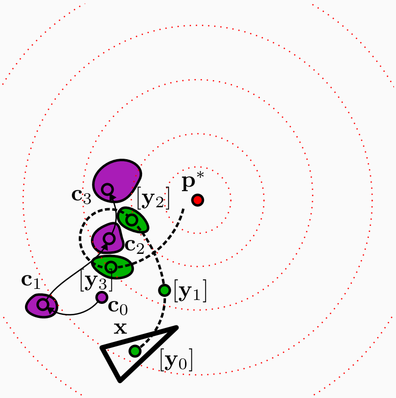

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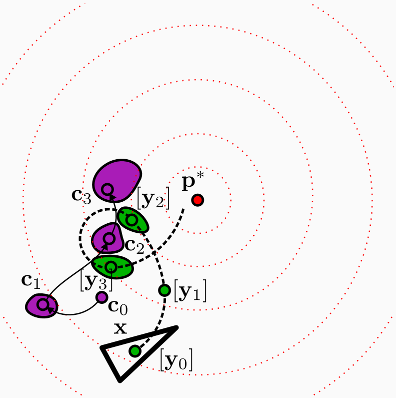

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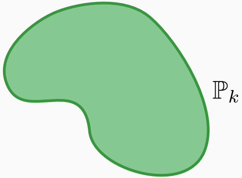
Positive Invariant Set

A **Positively Invariant Set** for a **dynamical system** is a set $\mathbb{P}$ such that:

$$\boldsymbol{\varphi}(\mathbb{P}) \subseteq \mathbb{P} \tag{9}$$

Construction of a Positively Invariant Set

$$\begin{cases} \mathbb{P}_0 = \text{initial set} \\ \mathbb{P}_{k+1} = \mathbb{P}_k \setminus \boldsymbol{\varphi}^{-1}(\overline{\mathbb{P}_k}) \end{cases} \tag{10}$$



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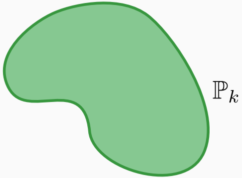

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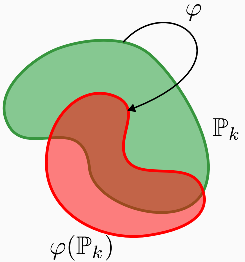

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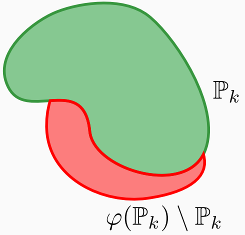

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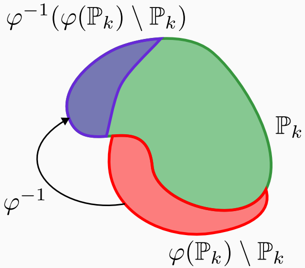

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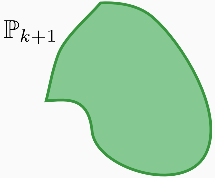

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Initial set

$$\mathbb{P}_0 = [-\pi, \pi] \times [0, 5] \quad (11)$$

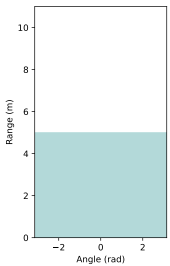

Figure 13: Cartesian Plot

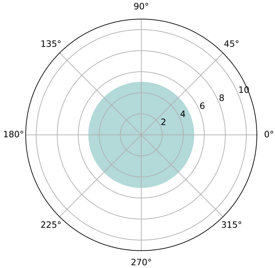

Figure 14: Polar Plot

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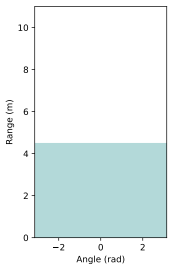

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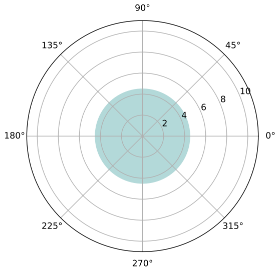

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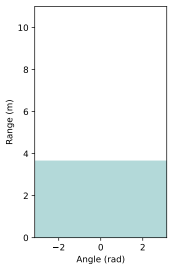

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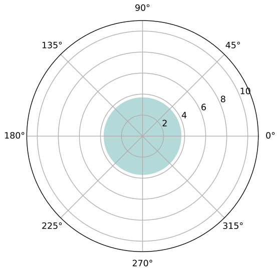

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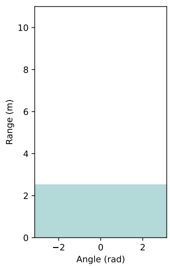

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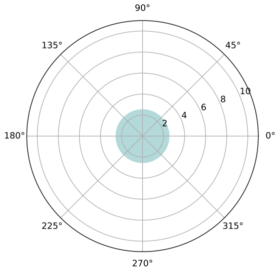

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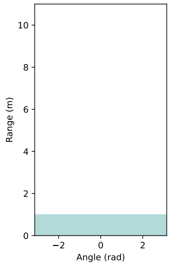

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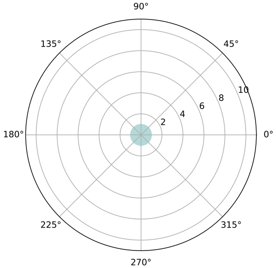

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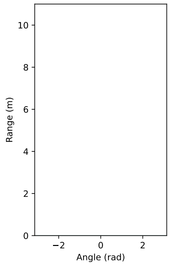

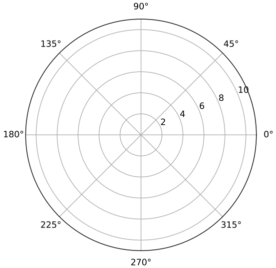

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Positive Invariant set

The largest positive invariant set contained in $\mathbb{P}_0 = [-\pi, \pi] \times [0, 5]$ for our dynamical system is:

$$\mathbb{P} = \emptyset \tag{12}$$

Initial set

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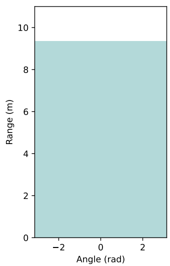


Figure 15: Cartesian Plot

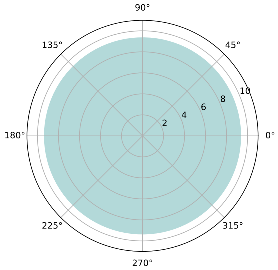
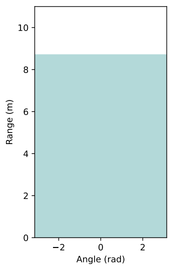


Figure 16: Polar Plot

Initial set

$$\mathbb{P}_0 = [-\pi, \pi] \times [0, 10] \quad (13)$$



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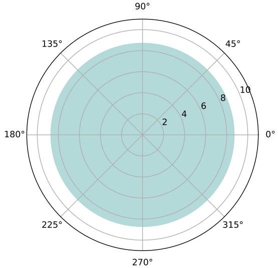

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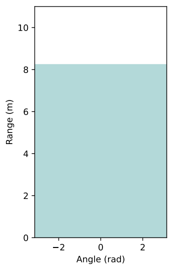

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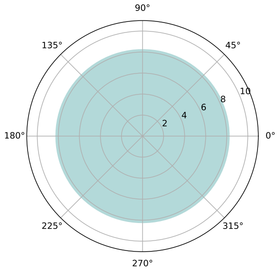

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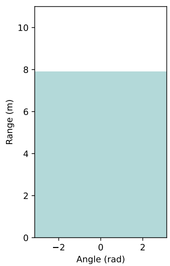

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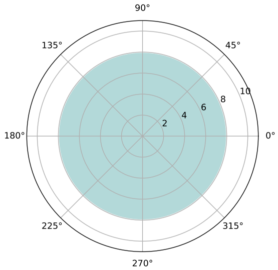

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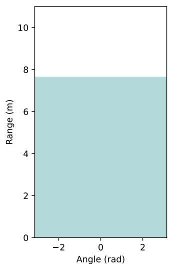

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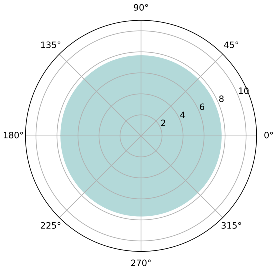

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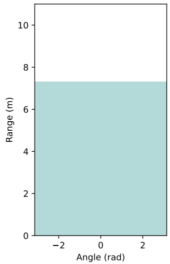

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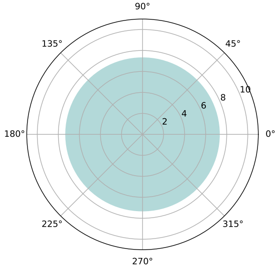

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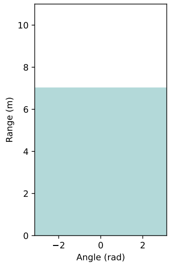

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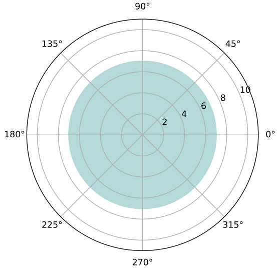

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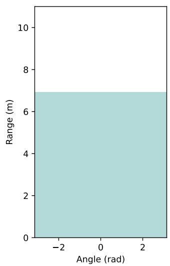

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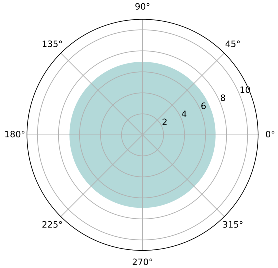

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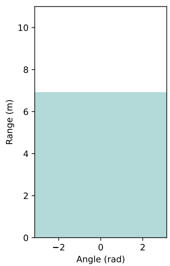

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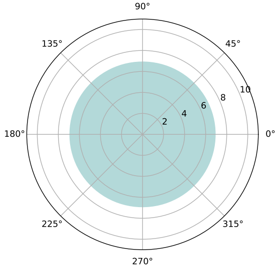

Figure 16: Polar Plot

Positive Invariant set

The largest positive invariant set contained in $\mathbb{P}_0 = [-\pi, \pi] \times [0, 10]$ for our dynamical system is:

$$\mathbb{P} = [-\pi, \pi] \times [0, 6.8944] \tag{14}$$

Capture Basin

A **Capture Basin** $\mathbb{B}$ of order k associated with a positively invariant set $\mathbb{P}$ is a set such that:

$$\exists k \in \mathbb{N}, \quad \mathbb{B}_k = \underbrace{\mathbf{f}^{-1} \circ \ldots \circ \mathbf{f}^{-1}}_{k}(\mathbb{P}) \tag{15}$$

Construction of a Capture Basin

$$\begin{cases} \mathbb{B}_0 = \mathbb{P} \\ \mathbb{B}_{k+1} = \mathbb{B}_k \cup \mathbf{f}^{-1}(\mathbb{B}_k) \end{cases} \tag{16}$$

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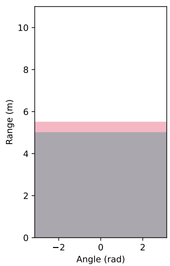
Initial set

$$\mathbb{B}_0 = [-\pi, \pi] \times [0, 6.8944] \quad (17)$$



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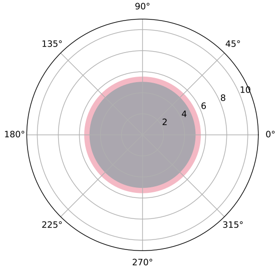

Figure 18: Polar Plot

Initial set

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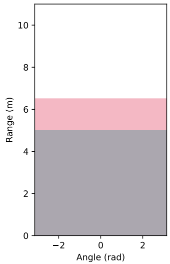

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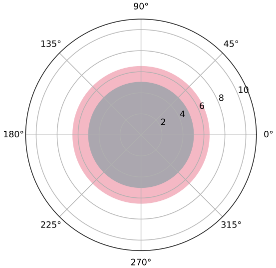

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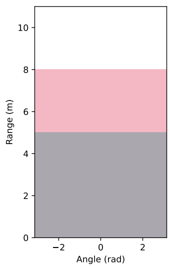

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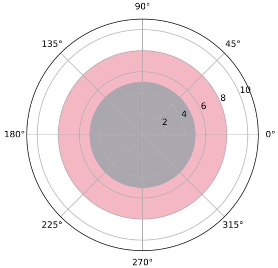

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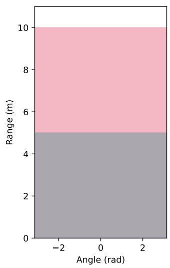

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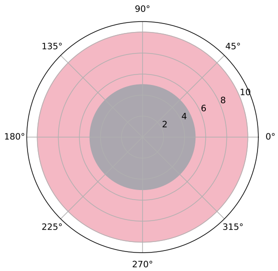

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Conclusion

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- Development of **frugal extremum-seeking controller** for **navigation without localization**.
- Validated method in both **simulation** and **experimental trials**.
- **Stability** of such navigation is proven using set methods.

Outlook

- Try to compute the capture basin for more complex scenarios and on real bathymetric data.
- Find strategies to explore the area and find the global extremum

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Questions?

Capture Basin Characterization for Frugal Extremum Seeking Cyclic Navigation using Set Methods

Quentin Brateau

ECC2026, Reykjavik, Iceland
July 07, 2026

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