Functional interval observers design for linear systems

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Set-Based Methods for Verified Parameter Identification, State Estimation, and Control, ECC2026, July 7th, 2026

Some related works

- Youdao Ma, Zhenhua Wang, Nacim Meslem, Tarek Raïssi, (2025). Functional interval estimation for continuous-time linear systems with time-invariant uncertainties. Automatica, 172, 112017. https://doi.org/10.1016/j.automatica.2025.112017.
- Youdao Ma, Zhenhua Wang, Nacim Meslem, Tarek Raïssi, Hao Luo, Functional Interval Estimation for Continuous-Time Linear Systems, IFAC-PapersOnLine, Volume 56, Issue 2, Pages 8482-8487, 2023.
- Akremi, R., Lamouchi, R., Amairi, M., Dinh, T. N., Raïssi, T. (2023). Functional interval observer design for multivariable linear parameter-varying systems. European Journal of Control, 71, 100794.

Outline

Dynamical Systems and State Estimation

Dynamical systems are fundamental to modern engineering and play a key role in technologies across multiple domains:

- **Automotive industry:** vehicle dynamics, adaptive control, and stability systems
- **Energy-efficient multi-user systems:** optimal resource allocation and smart grids
- **Communication systems:** antenna arrays, beamforming, and signal processing

State estimation is essential for effective control and monitoring:

- **Controller design:** achieve stability and the desired performance
- **Fault diagnosis:** detect faults early and ensure safety
- **Observer design:** reconstruct system states from limited measurements

Observer design is constantly evolving, providing increasingly efficient and robust solutions to modern engineering challenges.

Functional Observers - motivations

- In many control applications, we do not need to estimate the full system state.
- Controllers, fault detectors, or decision modules often rely on *specific functions* of the state (e.g., linear combinations, performance indicators).
- Full-state observers increase computational cost, complexity, and sensitivity to modeling errors.
- **Functional observers** offer a focused estimation strategy: recover only the quantities that matter.

Functional Observers - motivations

- Reduce observer dimension while preserving relevant information.
- Improve robustness by avoiding unnecessary state reconstruction.
- Enable real-time implementation on resource-limited platforms.
- Provide a flexible framework adaptable to nonlinearities, disturbances, or partial sensor availability.
- Key idea: **estimate functions, not states**.

Functional Observers - motivations

- Goal: Estimate a linear function of system states using an auxiliary dynamical system.
- Advantage: Reduced order and complexity compared to general observers.
- Existing works:
    - Linear descriptor systems, Zhang et al., (2021)
    - Nonlinear systems, Kravaris and Venkateswaran (2021)
    - ...

Interval Observers

- Estimate upper and lower bounds of system states under uncertainties.
- Key results:
  - Guaranteed interval estimation if lower and upper errors are cooperative/positive, Moisan et al., (2009); Mazenc and Bernard, (2011); Raïssi et al., (2010)
  - Relaxation of cooperativity assumption via changes of coordinate, Mazenc and Bernard, (2010); Raïssi et al., (2012)
- Extended to LTI, LPV, and fault-tolerant systems, Chebotarev et al., (2015); Lamouchi et al., (2021); Li et al., (2019); Zhu et al., (2021), ...

Functional Interval Observers

- Combine functional and interval observers to handle uncertainties, Che et al., (2020); Gu et al., (2018); Huong, (2019); Huang et al., (2021); Huong et al., (2020); Huong and Yen, (2020); Yen and Huong, (2020).
- Motivation: Functional observers alone may not handle effects of uncertainties.
- Robust design approaches:
  - H_∞: energy-bounded uncertainties
  - L_∞: peak-bounded uncertainties (more suitable for interval observers)
- Gap: L_∞ and H_∞ performance indexes are not fully addressed for functional interval observers.

Outline

Notations

- Element-wise inequalities: $>, \geqslant, <, \leqslant$ for vectors/matrices
- Norms: $||x||$ Euclidean or induced matrices norm
- Spaces: $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$
- Matrix operators:
  - $A^+ = \max\{0, A\}$, $A^- = A^+ - A$, $|A| = A^+ + A^-$
  - $A^\dagger$ pseudo-inverse
  - I_n identity, $P \succ 0$ positive definite

Functional Observer & Interval Estimation

Functional observer:

- A special form of reduced-order observer
- Estimates the desired functional directly
- Relaxed design conditions, low computational complexity

Interval estimation:

- Point estimation may not converge to the real linear functional
- Computes interval enclosing admissible values
- Methods include ellipsoids, zonotopes, polytopes, interval observers, etc.

Set-Based Methods

Set-based methods:

1. Use geometrical sets to represent and propagate uncertainties
2. Achieve interval estimation based on outer-box approximation

Advantages: high accuracy, relaxed design conditions

Limitation: only suitable for discrete-time systems

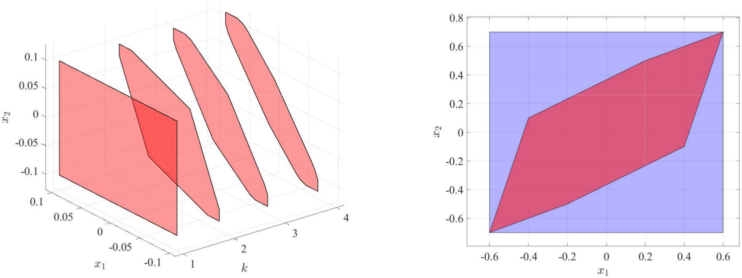


Figure: Zonotopic approximation

Interval Observers

Based on monotone systems theory, interval observers:

- construct two coupled sub-observers estimating upper/lower state bounds
- are easy to implement, high computational efficiency
- applicable to both discrete- and continuous-time systems
- have restrictive design conditions $\rightarrow$ less accurate estimation

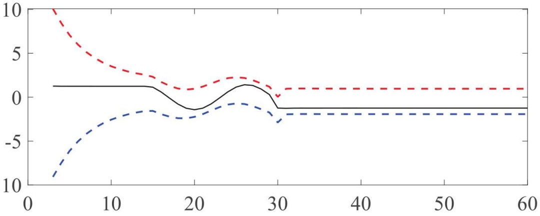


Figure: Interval estimation

Set-Integration Method (Meslem et al., (2020))

- Based on explicit expression + interval arithmetic
- Easy implementation, high computational efficiency
- High estimation accuracy, relaxed design conditions
- Provides point estimation with performance index (e.g. H_∞)
- Used for both discrete- and continuous-time linear systems
- Still has some conservatism

Functional Interval Estimation

- Aims to achieve interval estimation of the linear functional
- Limited works for continuous-time systems
- Existing: strict cooperativity condition (functional interval observers)
- Often neglects noise / requires strong conditions on $\dot{v}(t)$
- Based on classical observability assumptions $\rightarrow$ meaningful to extend

Method proposed in Ma et al., (2023)

Consider a continuous-time linear time-invariant system described by:

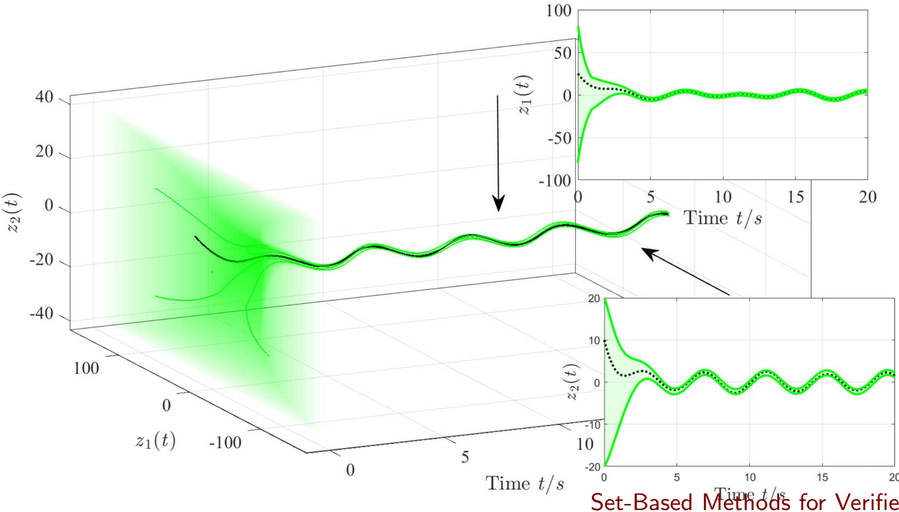
$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + D_w w(t)\\ y(t) &= Cx(t) + D_v v(t)\\ z(t) &= Lx(t)\end{aligned}$$

Assumptions:

- $|x(0) - c_0| \leq x_0$
- $|w(t)| \leq \overline{w}$, $|v(t)| \leq \overline{v}$ (known bounds)
- $|\dot{v}(t)| \leq \nu$ with unknown ν but finite

Goal: compute $\underline{z}(t), \overline{z}(t)$ such that $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$.

Functional interval estimation illustration



Problems to be Addressed

1. How to design the functional observer? $\mathcal{H}_\infty$ functional observer design
2. How to handle the derivative of measurement noise? Auxiliary error dynamics
3. How to estimate tight intervals of $z(t)$? Interval-based functional interval estimation

Point functional observer structure

$$\begin{cases} \dot{\eta}(t) = N\eta(t) + Hu(t) + Jy(t) \\ \hat{z}(t) = \eta(t) + Ey(t) \end{cases}$$

$$T = L - EC, \qquad TA - NT - JC = 0, \qquad H = TB$$

Observer design parameters:

$$E = \Xi\alpha_1 + Z\Psi\alpha_1, \qquad K = \Xi\alpha_2 + Z\Psi\alpha_2, \qquad N = F - ZG, \qquad J = K + NE$$

where Z is an arbitrary matrix of appropriate dimensions.

$$\alpha_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \Gamma = \begin{bmatrix} CA \\ C \end{bmatrix} \quad \Theta = I - LL^\dagger,$$

$$\Sigma = \Gamma\Theta, \quad \Xi = LA\Theta\Sigma^\dagger, \quad \Psi = I - \Sigma\Sigma^\dagger \quad F = LAL^\dagger - \Xi\Gamma L^\dagger, \qquad G = \Psi\Gamma L^\dagger$$

How to determine the optimal Z with an $\mathcal{H}_\infty$ performance?

$\mathcal{H}_\infty$ Functional Observer Design

Estimation error: $e(t) = z(t) - \hat{z}(t)$
Error dynamics:

$$\dot{e}(t) = Ne(t) + TD_w w(t) - KD_v v(t) - ED_v \dot{v}(t) = Ne(t) + Dd(t)$$

$\mathcal{H}_\infty$ **performance:** $\|e(t)\| \leq \gamma \|d(t)\|$
Linear Matrix Inequality (with $W = PZ$**):**

$$P \succ 0$$

$$\begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \\ \star & -\gamma^2 I & 0 & 0 \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \prec 0 \tag{1}$$

Where:
$\Pi_1 = \mathrm{He}(PF - WG) + I$, $\Pi_2 = PLD_w - P\Xi\alpha_1 CD_w - W\Psi\alpha_1 CD_w$,
$\Pi_3 = -P\Xi\alpha_2 D_v - W\Psi\alpha_1 CD_v$, $\Pi_4 = -P\Xi\alpha_1 D_v - W\Psi\alpha_1 CD_v$.

$\mathcal{H}_\infty$ Functional Observer Design

Constraint on the observer gain matrix:

$$\|Z\| \leq \delta \qquad \text{(to avoid numerical issues)}$$

LMI (for a given δ, with $W = PZ$):

$$\begin{bmatrix} -\delta^2 I & W^T \\ \star & -2P + I \end{bmatrix} \prec 0 \tag{3}$$

Optimization problem:

$$\min\ \gamma^2, \qquad \text{s.t. LMIs } (1) - (3)$$

Optimal solution:

$$Z = P^{-1} W$$

Auxiliary Error Dynamics

Error dynamics:

$$\dot{e}(t) = Ne(t) + TD_w w(t) - KD_v\, v(t) - ED_v\, \dot{v}(t)$$

Issue: the bounds of $\dot{v}(t)$ are unknown.

Auxiliary estimation error:

$$\varepsilon(t) = e(t) + E_{D_v} v(t)$$

Auxiliary error dynamics:

$$\begin{cases} \dot{\varepsilon}(t) = N\varepsilon(t) + TD_w w(t) - JD_v v(t) \\ e(t) = \varepsilon(t) - ED_v v(t) \end{cases}$$

Interval-Based Functional Interval Estimation

From the definition of the estimation error:

$$e(t) = z(t) - \hat{z}(t) \quad \Longrightarrow \quad z(t) = \hat{z}(t) + e(t)$$

Problem: $\hat{z}(t)$ is known, but how to compute the bounds of $z(t)$?

If we can obtain upper and lower bounds on the estimation error,

$$\underline{e}(t) \leq e(t) \leq \overline{e}(t),$$

then, an interval estimation $[\underline{z}, \overline{z}]$ for $z(t)$ is defined by:

$$\underline{z}(t) = \hat{z}(t) + \underline{e}(t), \qquad \overline{z}(t) = \hat{z}(t) + \overline{e}(t).$$

Remaining question:

How to compute $\underline{e}(t)$ and $\overline{e}(t)$?

Interval-Based Functional Interval Estimation

Explicit expression of the auxiliary error $\varepsilon(t)$:

$$\varepsilon(t) = \Phi(t)\,\varepsilon(0) + \zeta_w(t) + \zeta_v(t)$$

$$\zeta_w(t) = \int_0^t \Phi(t-\tau)\,T D_w\, w(\tau)\, d\tau, \qquad \zeta_v(t) = -\int_0^t \Phi(t-\tau) J D_v\, v(\tau)\, d\tau$$

Explicit expression of the estimation error $e(t)$:

$$e(t) = \Phi(t)L(x(0) - c_0) + \Phi(t)ED_v v(0) - ED_v v(t) + \zeta_w(t) + \zeta_v(t)$$

$$\Phi(t) = e^{Nt}$$

Initial condition choice:

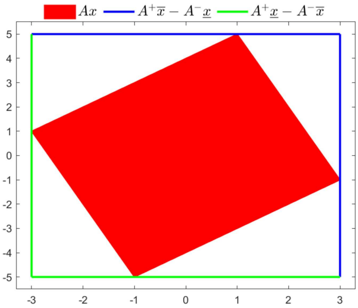
$$\hat{x}(0) = c_0 \quad \Rightarrow \quad \varepsilon(0) = Lx(0) - Lc_0 + E_{D_v} v(0)$$

Interval-Based Functional Interval Estimation

[redacted]

Efimov et al., (2013) For any $x \in [\underline{x}, \overline{x}]$:

$$A^{+}\underline{x} - A^{-}\overline{x} \leq Ax \leq A^{+}\overline{x} - A^{-}\underline{x}$$



Interval-Based Functional Interval Estimation

Efimov et al., (2013) For any $x \in [\underline{x}, \overline{x}]$*:*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}$$

Used to bound terms like $\Phi(t)L(x(0) - c_0)$:

$$-\left|\Phi(t)L\right| x_0 \;\leq\; \Phi(t)L\left(x(0) - c_0\right) \;\leq\; \left|\Phi(t)L\right| x_0$$

$$-\left|\Phi(t)ED_v\right| \overline{v} \;\leq\; \Phi(t)ED_v v(0) \;\leq\; \left|\Phi(t)E_{D_v}\right| \overline{v}$$

$$-\left|ED_v\right| \overline{v} \;\leq\; -ED_v v(t) \;\leq\; \left|ED_v\right| \overline{v}$$

Interval-Based Functional Interval Estimation: $\zeta_w(t)$, $\zeta_v(t)$

Computation of $\zeta_w(t)$:

$$\zeta_w(t) = \int_0^t \Phi(t-\tau)TD_w\, w(\tau)\, d\tau \quad \Rightarrow \quad \zeta_w(t) \in \int_0^t \Phi(\beta)TD_w[-\overline{w}, \overline{w}]\, d\beta$$

$$\dot{\zeta}_w(t) \in \Phi(t)TD_w[-\overline{w}, \overline{w}] \quad \Rightarrow$$

$$-|\Phi(t)TD_w|\overline{w} = \dot{\underline{\zeta}}_w(t) \leq \dot{\zeta}_w(t) \leq \dot{\overline{\zeta}}_w(t) = |\Phi(t)TD_w|\overline{w}$$

$$\zeta_w(0) = 0 \quad \Rightarrow \quad \underline{\zeta}_w(0) = \overline{\zeta}_w(0) = 0$$

Computation of $\zeta_v(t)$:

$$\begin{aligned} \zeta_v(t) \quad &= -\int_0^t \Phi(t-\tau)JD_v\, v(\tau)\, d\tau \quad \Rightarrow \\ &-|\Phi(t)JD_v|\overline{v} = \dot{\underline{\zeta}}_v \leq \dot{\zeta}_v(t) \leq \dot{\overline{\zeta}}_v = |\Phi(t)JD_v|\overline{v} \end{aligned}$$

$$\underline{\zeta}_v(0) = \overline{\zeta}_v(0) = 0$$

Change of variable: $\beta = t - \tau$

Interval-Based Functional Interval Estimation

Interval estimation of the error $e(t)$:

$$\begin{cases} \overline{e}(t) = |\Phi(t)L|\, x_0 + |\Phi(t)ED_v|\, v + |ED_v|\,\overline{v} + \overline{\zeta}(t) \\ \underline{e}(t) = -|\Phi(t)L|\, x_0 - |\Phi(t)ED_v|\,\overline{v} - |ED_v|\,\overline{v} + \underline{\zeta}(t) \end{cases}$$

where $\underline{\zeta}(t)$ and $\overline{\zeta}(t)$ satisfy

$$\begin{cases} \dot{\overline{\zeta}}(t) = |\Phi(t)TD_w|\,\overline{w} + |\Phi(t)JD_v|\,\overline{v} \\ \dot{\underline{\zeta}}(t) = -|\Phi(t)TD_w|\,\overline{w} - |\Phi(t)JD_v|\,\overline{v} \end{cases}$$

$$\overline{\zeta}(0) = 0, \qquad \underline{\zeta}(0) = 0$$

Functional interval estimation of $z(t)$:

$$\begin{cases} \overline{z}(t) = \hat{z}(t) + |\Phi(t)L|\, x_0 + |\Phi(t)ED_v|\,\overline{v} + |ED_v|\,\overline{v} + \overline{\zeta}(t) \\ \underline{z}(t) = \hat{z}(t) - |\Phi(t)L|\, x_0 - |\Phi(t)ED_v|\,\overline{v} - |ED_v|\,\overline{v} + \underline{\zeta}(t) \end{cases}$$

Summary of the Proposed Method

1. **$\mathcal{H}_\infty$ functional observer design**

$$\begin{cases} \dot{\eta}(t) = N\,\eta(t) + H\,u(t) + J\,y(t) \\ \hat{z}(t) = \eta(t) + E\,y(t) \end{cases}$$

2. **Auxiliary error dynamics**

$$\begin{cases} \dot{\varepsilon}(t) = N\,\varepsilon(t) + TD_w w(t) - JD_v v(t) \\ e(t) = \varepsilon(t) - ED_v v(t) \end{cases}$$

3. **Interval-based functional interval estimation**

$$\begin{cases} \overline{z}(t) = \hat{z}(t) + |\Phi(t)L|\,\overline{x}_0 + |\Phi(t)ED_v|\,\overline{v} + |ED_v|\,\overline{v} + \overline{\zeta}(t) \\ \underline{z}(t) = \hat{z}(t) - |\Phi(t)L|\,\overline{x}_0 - |\Phi(t)ED_v|\,\overline{v} - |ED_v|\,\overline{v} + \underline{\zeta}(t) \end{cases}$$

Simulation

Matrices from Meyer (2019)

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, D_w = I_3, D_v = I_2$$

Bounds:

$$|x(0)| \leq [20, 20, 20]^T, \quad |w| \leq [0.5, 0.5, 0.5]^T, \quad |v| \leq [0.1, 0.1]^T$$

Simulation

Functional interval estimation compared with Meslem et al., (2020).

$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \delta = 5$$

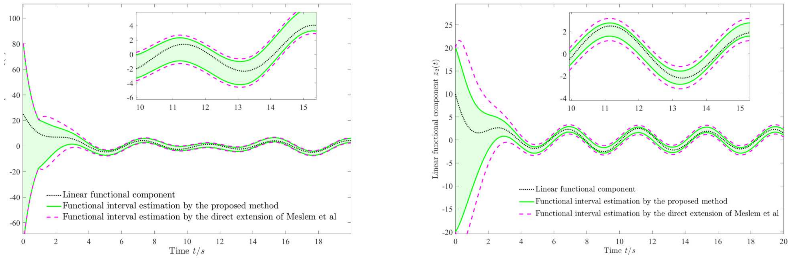


Figure: Functional interval estimation results provided by the proposed method and the direct extension of the method in Meslem et al., (2020).

Simulation

Comparison with Meyer (2019), including $|\dot{v}|$ constraints.

$$\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \delta = 5, \quad |\dot{v}(t)| \leq \begin{bmatrix} 0.7 \\ 0.4 \end{bmatrix}$$

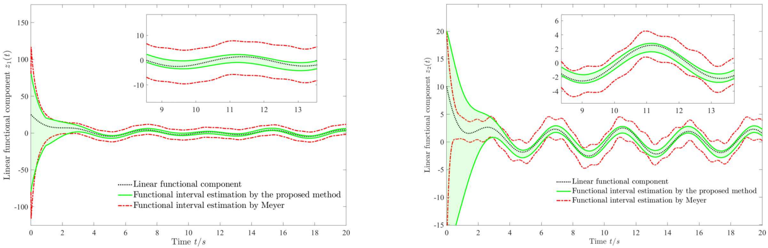


Figure: Functional interval estimation results provided by the proposed method and the method in Meyer (2019)

Outline

Objectives and Contributions

- Aim: Design a functional interval observer for LPV systems subject to bounded disturbances.
- Assumption: Scheduling vector described by a convex combination (time-varying parameters belonging to a polytope)
- Goals:
  1. A functional interval observer for LPV systems estimating linear functions of states.
  2. Novel L_∞-based design to attenuate uncertainties and improve interval estimation.

Section organization

1. Preliminaries
2. Problem statement
3. Main results: Functional interval observer design for linear LPV systems
4. Simulation results

Interval Analysis

Let $A \in [\underline{A}, \overline{A}]$ and $x \in [\underline{x}, \overline{x}]$, what about Ax?

Chebotarev et al., (2013) For $x \in \mathbb{R}^n$ with $\underline{x} \leqslant x \leqslant \overline{x}$:

1. *If A is constant: $A^+\underline{x} - A^-\overline{x} \leqslant Ax \leqslant A^+\overline{x} - A^-\underline{x}$*
2. *If $\underline{A} \leqslant A \leqslant \overline{A}$:*

$$\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leqslant Ax$$
$$\leqslant \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-$$

Definitions

[redacted]

- A is Schur stable if all eigenvalues have norm < 1.
- A is nonnegative if all elements ≥ 0.
- For $x(k+1) = Ax(k) + w(k)$ with $A \geq 0$, $w \geq 0$, $x(0) \geq 0$ implies $x(k) \geq 0$ for all k.

Input-to-State Stability (ISS)

[redacted]

- $\mathcal{K}$ function: continuous, strictly increasing, $\alpha(0) = 0$
- $\mathcal{K}_\infty$: $\mathcal{K}$ and radially unbounded
- $\mathcal{KL}$ function: $\beta(s, t)$ decreasing in t, $\beta(s, 0)$ is $\mathcal{K}$

ISS Lyapunov Function

[redacted]

Jiang and Wang, (2001); Sontag, (2008) A system $x(k+1) = f(x(k), w(k))$ is ISS if $\exists \beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}$ s.t.

$$|x(k,w)| \leq \beta(|x(0)|, k) + \alpha(\|w\|), \forall k \geq 0$$

[redacted]

Jiang and Wang, (2001); Sontag, (2008) $V : \mathbb{R}^n \to \mathbb{R}_+$ is an ISS Lyapunov function if $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad V(f(x,w)) - V(x) \leq \alpha_3(|x|) + \sigma(|w|)$$

[redacted]

If a system admits an ISS Lyapunov function, then it is ISS.

Problem Statement

Consider the multivariable discrete-time LPV system:

$$\begin{cases} x(k+1) = A(\rho(k))x(k) + B(\rho(k))u(k) + Fw(k) \\ y(k) = Cx(k) \\ z(k) = Lx(k) \end{cases} \tag{1}$$

where:

- $x \in \mathbb{R}^{n_x}$: state
- $u \in \mathbb{R}^{n_u}$: input
- $y \in \mathbb{R}^{n_y}$: output
- $z \in \mathbb{R}^{n_z}$: functional state
- $w \in \mathbb{R}^{n_w}$: state disturbance

C, F, L are known constant matrices. $A(\rho(k))$ and $B(\rho(k))$ are parameter-varying matrices scheduled by $\rho(k)$.

Polytopic Representation

$A(\rho(k))$ and $B(\rho(k))$ can be expressed as convex combinations of vertex matrices:

$$A(\rho(k)) = \sum_{i=1}^{s} h_i(\rho(k))A_i, \quad B(\rho(k)) = \sum_{i=1}^{s} h_i(\rho(k))B_i$$

- $\rho(k) \in \mathcal{M}$: unmeasurable scheduling vector
- s: number of vertices of polytopic set $\mathcal{M}$
- $h_i(\rho(k)) \geq 0$, $\sum_{i=1}^{s} h_i(\rho(k)) = 1$
- A_i, B_i: known bounded matrices

Assumptions (1/3)

[redacted]

The states of the system are bounded:

$$x(k) \in \mathcal{X} \subset \mathcal{L}_\infty^n$$

where $\mathcal{X}$ is a given compact set.

Assumptions (2/3)

[redacted]

Uncertainties in system parameters and input are unknown but bounded:

$$\underline{w}(k) \leq w(k) \leq \overline{w}(k)$$

$$\underline{A}(\rho(k)) \leq A(\rho(k)) \leq \overline{A}(\rho(k))$$

$$\underline{B}(\rho(k)) \leq B(\rho(k)) \leq \overline{B}(\rho(k))$$

where $\underline{w}, \overline{w}$, $\underline{A}(\rho(k)), \overline{A}(\rho(k))$, $\underline{B}(\rho(k)), \overline{B}(\rho(k))$ *are known.*

Assumptions (3/3)

[redacted]

The matrices C and L satisfy (full rank assumption):

$$rank(C) = n_y, \quad rank(L) = n_z$$

Objective

The aim is to design a functional interval observer for multivariable LPV systems (1) that estimates a linear function of the states:

- Observer gains can be computed using:
  - $\mathcal{H}_\infty$ formalism
  - $\mathcal{L}_\infty$ criterion
- Time index k may be omitted for brevity when no ambiguity arises

Functional Interval Observer Design

For system (1), consider the functional interval observer:

$$\begin{cases} \overline{g}(k+1) = D(\rho)\overline{g}(k) + J(\rho)y(k) + H(\rho)u(k) + \overline{\delta}(k) \\ \underline{g}(k+1) = D(\rho)\underline{g}(k) + J(\rho)y(k) + H(\rho)u(k) + \underline{\delta}(k) \\ \qquad \overline{z}(k) = S^{+}\overline{g}(k) + S^{-}\underline{g}(k) + Ny(k) \\ \qquad \underline{z}(k) = S^{+}\underline{g}(k) + S^{-}\overline{g}(k) + Ny(k) \end{cases} \tag{2}$$

with $\overline{g}(k), \underline{g}(k) \in \mathbb{R}^{n_x}$, $\overline{z}(k), \underline{z}(k) \in \mathbb{R}^{n_z}$ and $\overline{\delta}(k), \underline{\delta}(k)$ are defined in the next slide.

Expression of $\overline{\delta}(k), \underline{\delta}(k)$

$$
\begin{aligned}
\overline{\delta}(k) = {} & (RF)^{+}\overline{w}(k) - (RF)^{-}\underline{w}(k) + \overline{A}^{+}\overline{x}^{+} - \underline{A}^{+}\overline{x}^{-} \\
& - \overline{A}^{-}\underline{x}^{+} + \underline{A}^{-}\underline{x}^{-} + \overline{B}^{+}\overline{u}^{+} - \underline{B}^{+}\overline{u}^{-} - \overline{B}^{-}\underline{u}^{+} + \underline{B}^{-}\underline{u}^{-} \\
\underline{\delta}(k) = {} & (RF)^{+}\underline{w}(k) - (RF)^{-}\overline{w}(k) + \underline{A}^{+}\underline{x}^{+} - \overline{A}^{+}\underline{x}^{-} \\
& - \underline{A}^{-}\overline{x}^{+} + \overline{A}^{-}\overline{x}^{-} + \underline{B}^{+}\underline{u}^{+} - \overline{B}^{+}\underline{u}^{-} - \underline{B}^{-}\overline{u}^{+} + \overline{B}^{-}\overline{u}^{-}
\end{aligned}
$$

Matrix Conditions

- $D(\rho) \in \mathbb{R}^{n_x \times n_x}$ is non-negative
- $J(\rho) \in \mathbb{R}^{n_x \times n_y}$ is non-singular
- $H(\rho) \in \mathbb{R}^{n_x \times n_u}$ depends on $\rho(k)$
- Design constants: S, N, R

LPV representation:

$$D(\rho) = \sum_{i=1}^{s} h_i(\rho) D_i, \quad J(\rho) = \sum_{i=1}^{s} h_i(\rho) J_i, \quad H(\rho) = \sum_{i=1}^{s} h_i(\rho) H_i$$

Theorem 1: Guaranteed estimation

[redacted]

Under Assumptions 1-3, the observer (2) ensures:

$$\underline{z}(k) \leq z(k) \leq \overline{z}(k)$$

if

$$D(\rho)R + J(\rho)C - RA(\rho) = 0, \quad H(\rho) - RB(\rho) = 0, \quad SR + NC - L = 0$$

with $D(\rho)$ nonnegative and $\underline{z}(0) \leq z(0) \leq \overline{z}(0)$.

Proof of Theorem 1

Define errors:

$$\overline{e}(k) = \overline{g}(k) - Rx(k), \quad \underline{e}(k) = \underline{g}(k) - Rx(k)$$

$$\overline{e}_z(k) = \overline{z}(k) - z(k), \quad \underline{e}_z(k) = z(k) - \underline{z}(k)$$

Error dynamics:

$$\overline{e}(k+1) = D(\rho)\overline{e}(k) + \overline{\Phi}(k), \quad \underline{e}(k+1) = D(\rho)\underline{e}(k) + \underline{\Phi}(k)$$

Output reconstruction:

$$\overline{e}_z(k) = S^+\overline{e}(k) + S^-\underline{e}(k), \quad \underline{e}_z(k) = S^+\underline{e}(k) + S^-\overline{e}(k)$$

Initialization of Bounds

$$\overline{g}(0) = \overline{R}\overline{x}(0) - \underline{Rx}(0), \quad \underline{g}(0) = \overline{R}\underline{x}(0) - \underline{R}\overline{x}(0)$$

$$\overline{z}(0) = \overline{L}\overline{x}(0) - \underline{Lx}(0), \quad \underline{z}(0) = \overline{L}\underline{x}(0) - \underline{L}\overline{x}(0)$$

$$\implies \overline{e}(0) \geq 0, \underline{e}(0) \geq 0, \overline{e}_z(0) \geq 0, \underline{e}_z(0) \geq 0$$

Bound on Perturbations

$$(RF)^{+}\underline{w}(k)-(RF)^{-}\overline{w}(k)\leq(RF)w(k)\leq(RF)^{+}\overline{w}(k)-(RF)^{-}\underline{w}(k)$$

$$\underline{\delta}(k)\leq A(\rho(k))x(k)+B(\rho(k))u(k)\leq\overline{\delta}(k)$$

$$\Longrightarrow\ \overline{\Phi}(k)\geq 0,\underline{\Phi}(k)\geq 0$$

Final Result

$$\underline{z}(k) \leq z(k) = Lx(k) \leq \overline{z}(k), \quad \forall k \geq 0$$

Measurable LPV Observer

If $\rho(k)$ is measurable:

$$\overline{\delta}(k) = (RF)^{+}\overline{w}(k) - (RF)^{-}\underline{w}(k), \quad \underline{\delta}(k) = (RF)^{+}\underline{w}(k) - (RF)^{-}\overline{w}(k)$$

State Augmentation

$$e(k) = \begin{bmatrix} \overline{e}(k) \\ \underline{e}(k) \end{bmatrix}, \quad e(k+1) = \mathcal{D}(\rho)e(k) + \mathcal{G}\Sigma(k)$$

$\mathcal{D}(\rho) = \text{diag}(D(\rho), D(\rho))$

Remark 2: Robustness

Classical observers do not minimize the effect of uncertainties.

[redacted]

Use H_∞ design to:

- Choose gains $D(\rho), H(\rho), S, N, R$
- Attenuate the effect of disturbances

H_∞ Performance

Objective: $\|e(k)\| < \gamma^2 \|\Sigma(k)\|$

[redacted]

If matrices $P > 0$, $Q_i > 0$ exist and

$$\begin{bmatrix} -\xi P + I & * & * \\ 0 & -\gamma^2 I & * \\ Q_i & P\mathcal{G} & -P \end{bmatrix} < 0$$

$$D_i L + J_i C - L A_i = 0, \quad H_i - L B_i = 0, \quad SR + NC - L = 0$$

Then, the error system is stable and satisfies $\|e(k)\| < \gamma^2 \|\Sigma(k)\|$

L_∞ Performance

Objective:

$$\|e(k)\| < \sqrt{\gamma\left[(1-\lambda)^k V(0) + \gamma \Sigma_\infty^2\right]}$$

[redacted]

If matrices $P > 0$, $Q_i > 0$ exist and LMIs hold:

$$\begin{bmatrix} (\lambda-1)P & * & * \\ 0 & -\alpha I & * \\ Q_i & P\mathcal{G} & -P \end{bmatrix} < 0, \quad \begin{bmatrix} \lambda P & 0 & I \\ 0 & (\gamma-\alpha)I & 0 \\ I & 0 & \gamma I \end{bmatrix} > 0$$

Then, the error system is stable and satisfies $\|e(k)\|_{L_\infty}$

Two-tank system example

In this section, a two-tank system described in Figure 8 is considered to demonstrate the feasibility of the proposed interval estimation method.

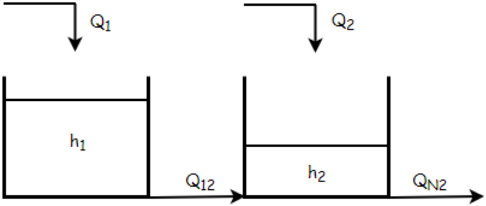


Figure: Two-tank system.

Inspired by Rotondo et al., (2014), this system consists of two liquid tanks that can be filled with two identical, independent pumps delivering the liquid flows Q_1 and Q_2. The tanks are interconnected through the pipe Q_{12}, while the outflow Q_{N2} is at the right tank. The liquid levels h_1 and h_2 are the state variables. The tank area is $A = 1.54 \times 10^{-2}\, m^2$, with flow

Nonlinear and LPV models

The nonlinear model is obtained from mass balance and Torricelli's law:

$$\begin{cases} \dfrac{dh_1(t)}{dt} = \dfrac{1}{A}(Q_1(t) - Q_{12}(t)) \\ \dfrac{dh_2(t)}{dt} = \dfrac{1}{A}(Q_2(t) + Q_{12}(t) - Q_{N2}(t)) \\ Q_{12}(t) = c_{12}\sqrt{h_1(t) - h_2(t)} \\ Q_{N2}(t) = c_2\sqrt{h_2(t)} \end{cases} \tag{3}$$

Assuming $h_1(t) > h_2(t)$, the system is reformulated in quasi-LPV form:

$$\begin{bmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{bmatrix} = \begin{bmatrix} -\rho_1(v(t)) & 0 \\ \rho_1(v(t)) & \rho_2(v(t)) \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{4}$$

Time-varying parameters

Where

$$\begin{cases} \rho_1(v(t)) = \dfrac{c_{12}}{A}\dfrac{\sqrt{h_1(t) - h_2(t)}}{h_1(t)} \\ \rho_2(v(t)) = \dfrac{c_2}{A}\dfrac{\sqrt{h_2(t)}}{h_2(t)} \\ b_{11} = b_{22} = \dfrac{1}{A} \end{cases} \tag{5}$$

Discrete-time Euler approximation with sampling T_s:

$$\begin{bmatrix} h_1(k+1) \\ h_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 - T_s\rho_1(v(k)) & 0 \\ T_s\rho_1(v(k)) & 1 - T_s\rho_2(v(k)) \end{bmatrix} \begin{bmatrix} h_1(k) \\ h_2(k) \end{bmatrix} + \begin{bmatrix} T_s b_{11} & 0 \\ 0 & T_s b_{22} \end{bmatrix} \tag{6}$$

Parameter bounds

Assume $h_1 \in [0.6, 1.8]m$, $h_2 \in [0.1, 0.3]m$, $T_s = 0.1s$. The parameter bounds:

$$0.6804\, m^{-1/2} < \rho_1(k) < 1.1785\, m^{-1/2} \tag{7}$$

$$1.8257\, m^{-1/2} < \rho_2(k) < 3.1623\, m^{-1/2} \tag{8}$$

Gain scheduling functions:

$$h_1(\rho(k)) = \frac{\rho_1(k) - \underline{\rho_1}}{\overline{\rho_1} - \underline{\rho_1}} \frac{\rho_2(k) - \underline{\rho_2}}{\overline{\rho_2} - \underline{\rho_2}} \tag{9}$$

$$h_2(\rho(k)) = \frac{\rho_1(k) - \underline{\rho_1}}{\overline{\rho_1} - \underline{\rho_1}} \frac{\overline{\rho_2} - \rho_2(k)}{\overline{\rho_2} - \underline{\rho_2}} \tag{10}$$

$$h_3(\rho(k)) = \frac{\overline{\rho_1} - \rho_1(k)}{\overline{\rho_1} - \underline{\rho_1}} \frac{\rho_2(k) - \underline{\rho_2}}{\overline{\rho_2} - \underline{\rho_2}} \tag{11}$$

$$h_4(\rho(k)) = \frac{\overline{\rho_1} - \rho_1(k)}{\overline{\rho_1} - \underline{\rho_1}} \frac{\overline{\rho_2} - \rho_2(k)}{\overline{\rho_2} - \underline{\rho_2}} \tag{12}$$

Polytopic form

System matrices in polytopic form:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.0002 & 0 \\ 0.0002 & -0.0014 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.0002 & 0 \\ 0.0002 & -0.0072 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -0.0013 & 0 \\ 0.0013 & -0.0014 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -0.0013 & 0 \\ 0.0013 & -0.0072 \end{bmatrix}, \quad B = \begin{bmatrix} 6.4935 & 0 \\ 0 & 6.4935 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}, \qquad L = \begin{bmatrix} 0.0031 & 0 \\ 0.1 & 0 \end{bmatrix}$$

Initial conditions:

$$x(0) = \begin{bmatrix} 1.7 \\ 0.3 \end{bmatrix}, \quad \overline{x}(0) = \begin{bmatrix} 2.7 \\ 0.6 \end{bmatrix}, \quad \underline{x}(0) = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}$$

Observer gains

Based on H_∞ performance: $\xi = 0.63$, $\gamma = 33.7242$

$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0.2153 & 0 \\ 0 & 0.2173 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ldots$$

$$R = \begin{bmatrix} -8.6377 & 1.7550 \\ -10.0589 & 1.2769 \end{bmatrix}, \quad H = \begin{bmatrix} -56.0890 & 11.3962 \\ -65.3174 & 8.2917 \end{bmatrix}$$

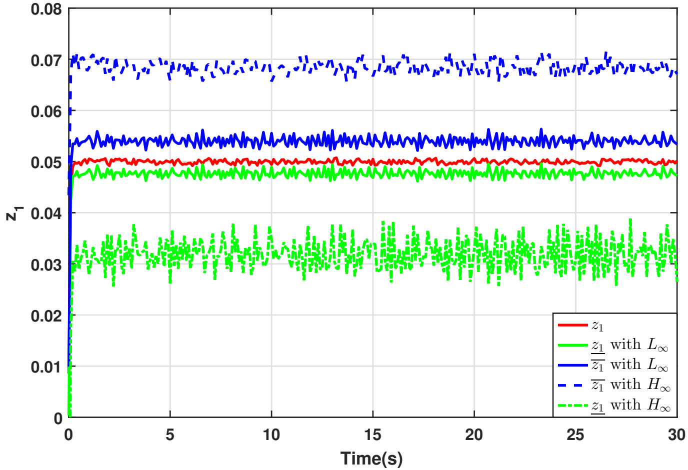
$$S = \begin{bmatrix} 0.0033 & -0.0030 \\ 0.509 & -0.0513 \end{bmatrix}, \quad N = \begin{bmatrix} 0.0014 & 0.0090 \\ 0.0234 & 0.1442 \end{bmatrix}$$

Lyapunov matrix:

$$P = \begin{bmatrix} 132.1777 & 0 & 0 & 0 \\ 0 & 128.5187 & 0 & 0 \\ 0 & 0 & 132.1777 & 0 \\ 0 & 0 & 0 & 132.5187 \end{bmatrix}$$

Simulation results

Figures show interval estimation for $z_1(k)$ and $z_2(k)$:



State evolution

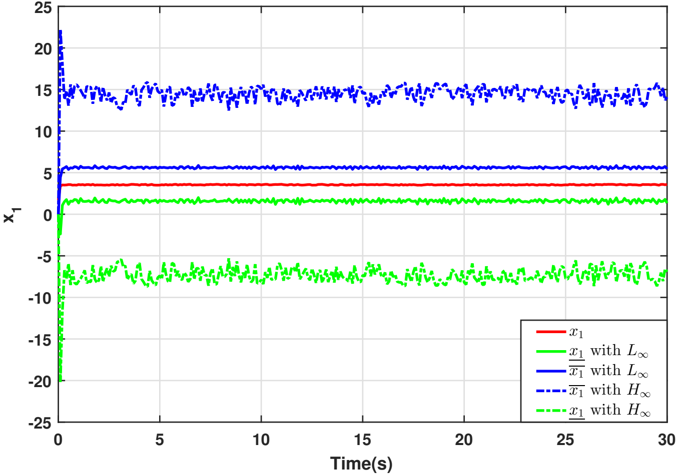


Figure: Evolution of $x_1(k)$.

Conclusions

- Functional interval estimation methods are proposed for LTI and LPV systems subject to disturbances and measurement noise.
- The reachability analysis method apploied to LTI systems improves the estimation performance **without requiring the stringent cooperativity condition** traditionally imposed in interval observer design.
- The L_∞ norm gives better estimation accuracy than H_∞.

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