Using Taylor Models in the Reachability Analysis of Non-linear Hybrid Systems

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We propose a novel approach to use Taylor models, independently developed by Berz and Makino, in the reachability analysis of non-linear hybrid systems. A *Taylor model* $\mathbb{T} = (p, I)$ *over domain* D *is specified by an* n*-dimensional polynomial* p *and interval domains (boxes)* $D, I \subseteq \mathbb{R}^n$ and represents the set

$$\mathbb{T} = \{\boldsymbol{x} \in \mathbb{R}^m | \exists \boldsymbol{x}_0 \in D.\, \exists \boldsymbol{i} \in I.\, \boldsymbol{x} = p(\boldsymbol{x}_0) + \boldsymbol{i}\}\,. \quad (1)$$

A Taylor model (p, I) over domain D is a k*-order over-approximation* for a $(k+1)$ times differentiable function $f : D \to \mathbb{R}^m$, if the polynomial p is the order k Taylor expansion of f around the center point of D and the interval I encloses the remainder. *Taylor model arithmetic* can be used to apply basic arithmetic operations to Taylor models as well as, e.g., anti-derivation and Lie derivation w.r.t. a non-constant vector field.

Our goal is to integrate Taylor models into the reachability analysis of hybrid systems. *Hybrid systems*, often modeled as hybrid automata, exhibit both *continuous dynamics* (*flows* modeled by ordinary differential equations and restricted by invariants) and *discrete behaviour* (guarded *jumps*). Besides the higher-order approximations of the flows by Taylor models, the main challenge in our work is the computation of flow/invariant and flow/guard intersections. Note that, given a Taylor model representing a flowpipe, in general the above intersections cannot be exactly represented by Taylor models.

Assume a Taylor model $\mathbb{T}$ and a convex polytope S (representing an invariant or a guard). We introduce three techniques to over-approximate $\mathbb{T} \cap S$ by a Taylor model $\mathbb{T}^*$. Firstly, our *domain contraction* technique of polynomial complexity computes a contraction D^* of D such that $\mathbb{T}^*$ resulting from $\mathbb{T}$ when replacing D by D^* over-approximates $\mathbb{T} \cap S$. We use a combination of the *branch-and-prune* algorithm and *interval constraint propagation* to compute D^*. Secondly, we present the *template* method which computes $\mathbb{T}^*$ by SMT solving as an instance of a template $(p^*, [0, 0]^m)$ over D^*, where D^* is fixed and p^* is a polynomial with parametric coefficients, whose parameters are fixed by the computation. Thirdly, we offer some *geometric* methods to wrap the intersection by some geometric objects, e.g., *zonotopes* or *support functions*.

We have implemented our ideas in a tool, which we use to show results for several interesting and challenging benchmarks.