An interval approach for stability analysis of nonlinear systems

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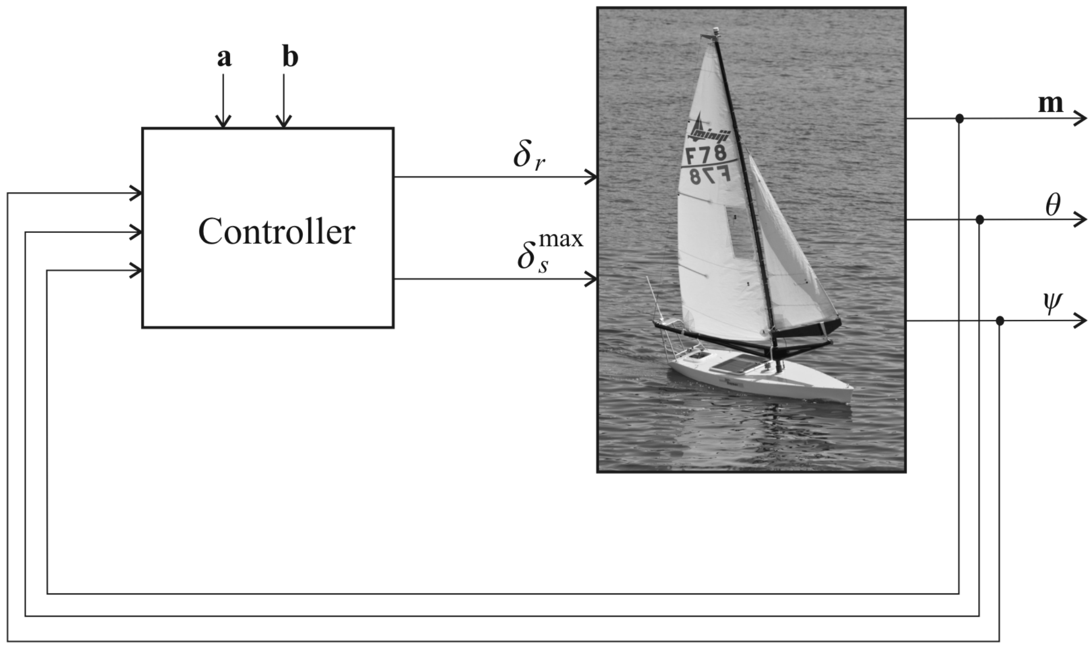
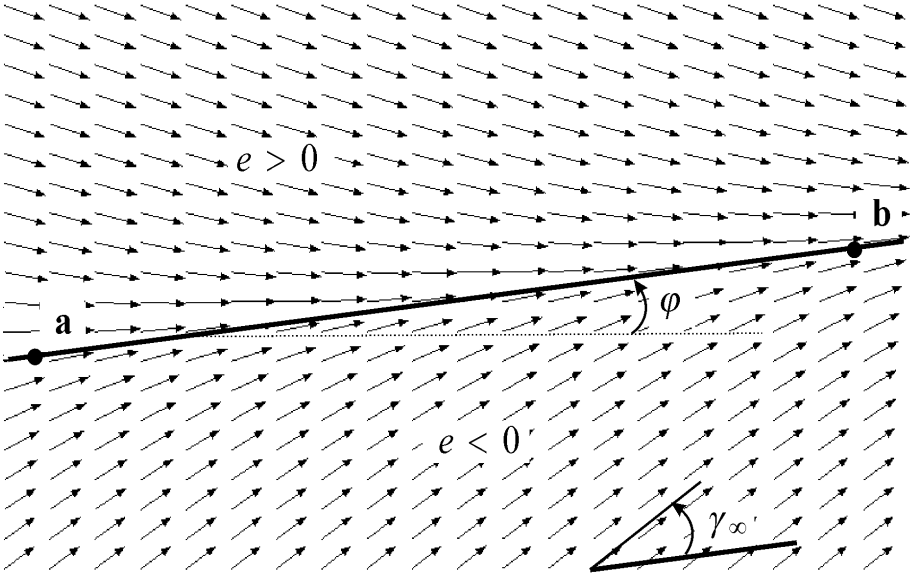
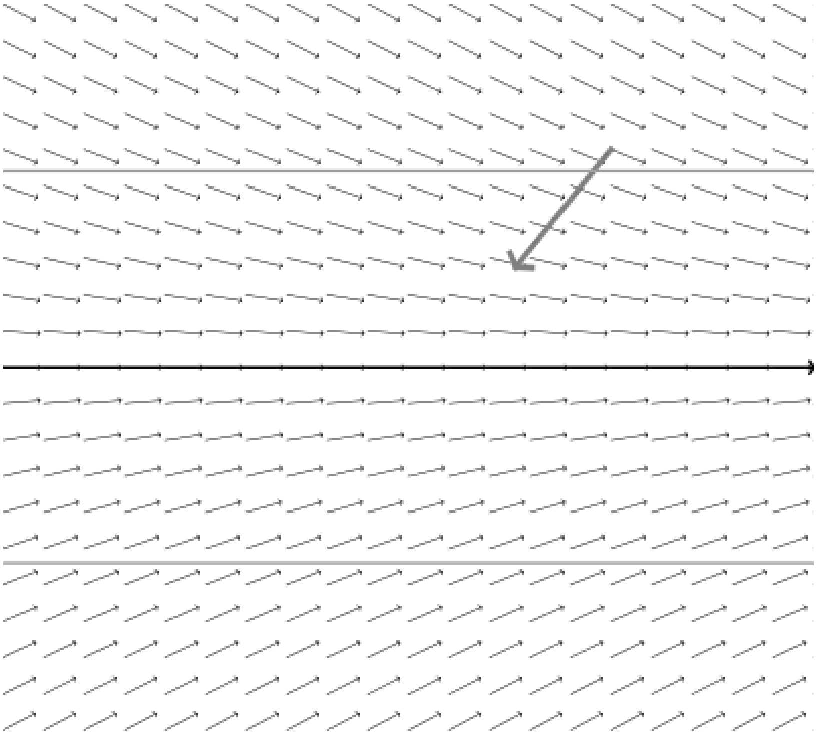
1 Line following

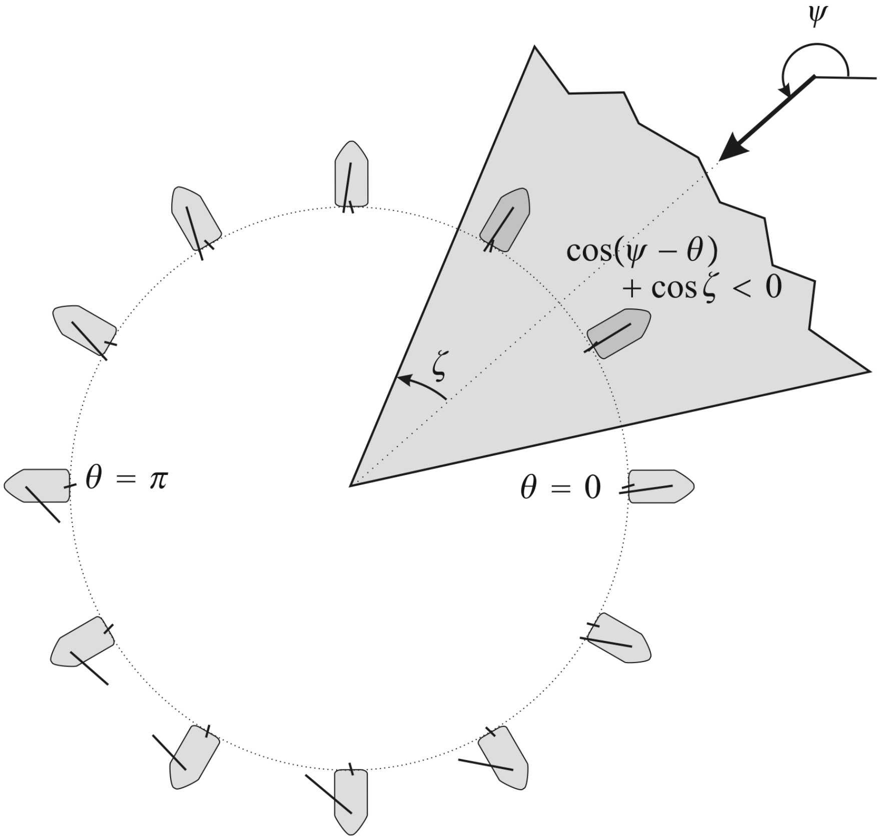

a
b
Controller
δ_r
$\delta_s^{\max}$
m
θ
ψ



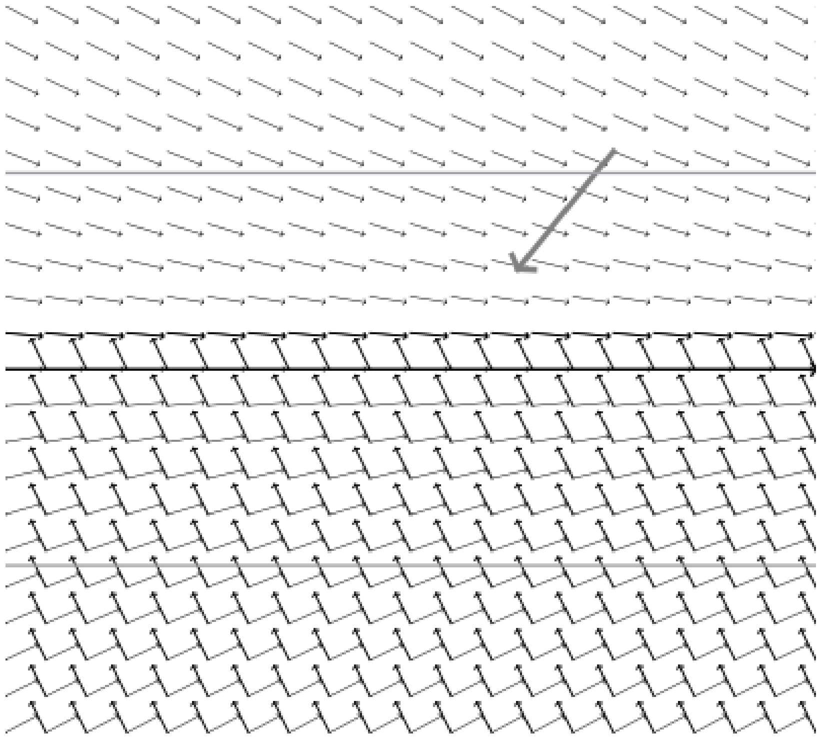
Nominal vector field θ^*



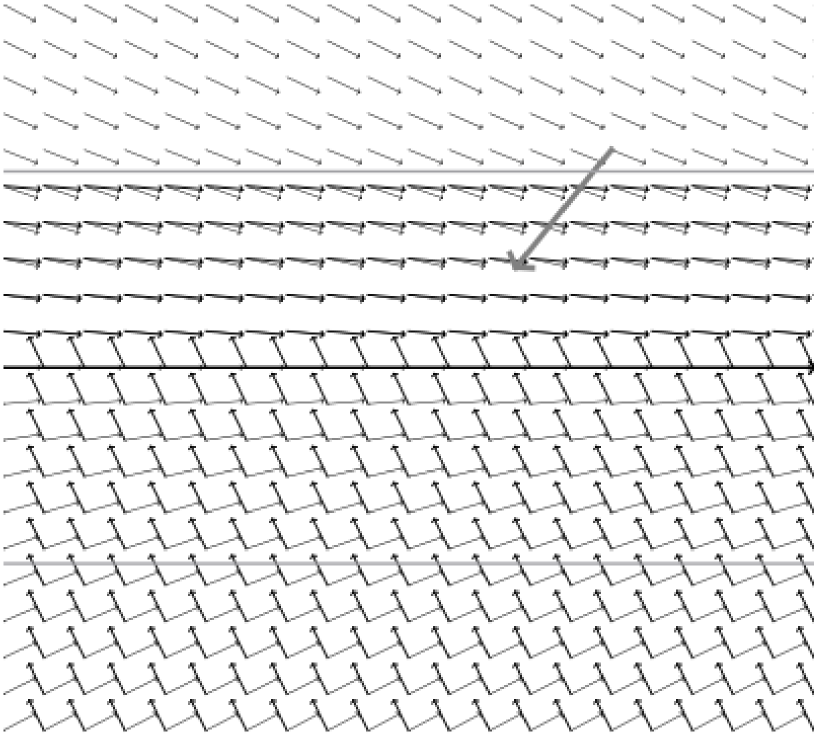
$$\theta^* = -\frac{2.\gamma_\infty}{\pi}.atan\left(\frac{e}{r}\right)$$



$$\cos\left(\psi - \bar{\theta}\right) + \cos\zeta \;<\; 0 \Rightarrow \;\bar{\theta} \text{ is unfeasible}$$
$$\text{In this case, take } \bar{\theta} \;=\; \pi + \psi - \zeta.sign\left(e\right)$$



Polar projection strategy



Keep tack strategy: even if the route $\bar{\theta}$ is feasible, we keep the tack mode

Keep tack strategy:

$$\left\{ \begin{array}{c} |e| < r \text{ and} \\ \cos(\psi - \varphi) + \cos\zeta < 0 \end{array} \right. \Rightarrow \bar{\theta} = \pi + \psi - \zeta.\text{sign}\,(e)$$

.

Function $\bar{\theta}\,(e, \psi, \gamma_{\infty}, r, \zeta)$

$\theta^{*} = -\frac{2.\gamma_{\infty}}{\pi}.\text{atan}\left(\frac{e}{r}\right)$ // nominal route

if $\cos\,(\psi - \theta^{*}) + \cos\zeta < 0$ // θ^{*} unfeasible

  or $(|e| < r$ and $\cos\psi + \cos\zeta < 0)$ // line unfeasible

    then $\bar{\theta} = \pi + \psi - \zeta.\text{sign}(e)$; // tack mode

    else $\bar{\theta} = \theta^{*}$; // nominal route

end

Choose a frame $\mathcal{R}(\mathbf{a}, \mathbf{i}, \mathbf{j})$ based on the line.

Function $\bar{\theta}(\mathbf{x}, \psi, \gamma_\infty, r, \zeta)$

$\theta^* = -\frac{2.\gamma_\infty}{\pi}.\text{atan}\left(\frac{x_2}{r}\right)$

if $\cos(\psi - \theta^*) + \cos\zeta < 0$

    or $(|e| < r$ and $(\cos(\psi - \varphi) + \cos(\zeta) < 0))$

      then $\bar{\theta} = \pi + \psi - \zeta.\text{sign}(x_2)$;

      else $\bar{\theta} = \theta^*$;

end

The motion of the sailboat robot satisfies

$$\dot{\mathbf{x}} = \begin{pmatrix} \cos\left(\bar{\theta}\left(\mathbf{x},\psi\right)+w\right) \\ \sin\left(\bar{\theta}\left(\mathbf{x},\psi\right)+w\right) \end{pmatrix}, \text{ with } w \in \left[w^{-}, w^{+}\right],$$

i.e.

$$\dot{\mathbf{x}} \in \mathbf{F}\left(\mathbf{x}\right)$$

which is a differential inclusion.

2 V-stability

The system

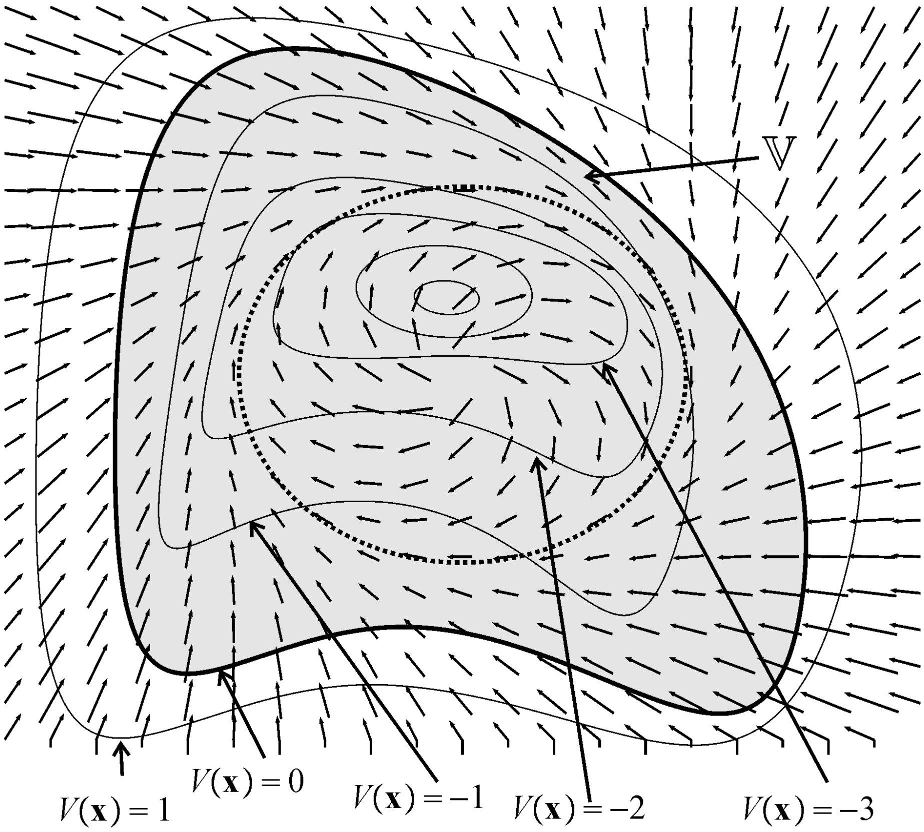
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) is there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned} \dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}. \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0} \end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x})$. The system is V-stable if

$$\left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right).$$


𝕍
$V(\mathbf{x}) = 1$
$V(\mathbf{x}) = 0$
$V(\mathbf{x}) = -1$
$V(\mathbf{x}) = -2$
$V(\mathbf{x}) = -3$

Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V-stable then

(i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
(ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0,\ V(\mathbf{x}(t+\tau)) < 0$.

Now,

$$
\begin{aligned}
& \left(V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\; & \left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0\right) \\
\Leftrightarrow\; & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\
\Leftrightarrow\; & \forall \mathbf{x}, \min\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}), V(\mathbf{x})\right) < 0 \\
\Leftrightarrow\; & \forall \mathbf{x}, g(\mathbf{x}) < 0 \\
\Leftrightarrow\; & g^{-1}([0, \infty]) = \emptyset.
\end{aligned}
$$

Theorem. If

$$g(\mathbf{x}) = \min\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) . \mathbf{f}(\mathbf{x}), V(\mathbf{x})\right),$$

we have

$$g^{-1}([0, \infty[) = \emptyset \quad \Leftrightarrow \quad \text{the system is } V\text{-stable.}$$

3 Robust case

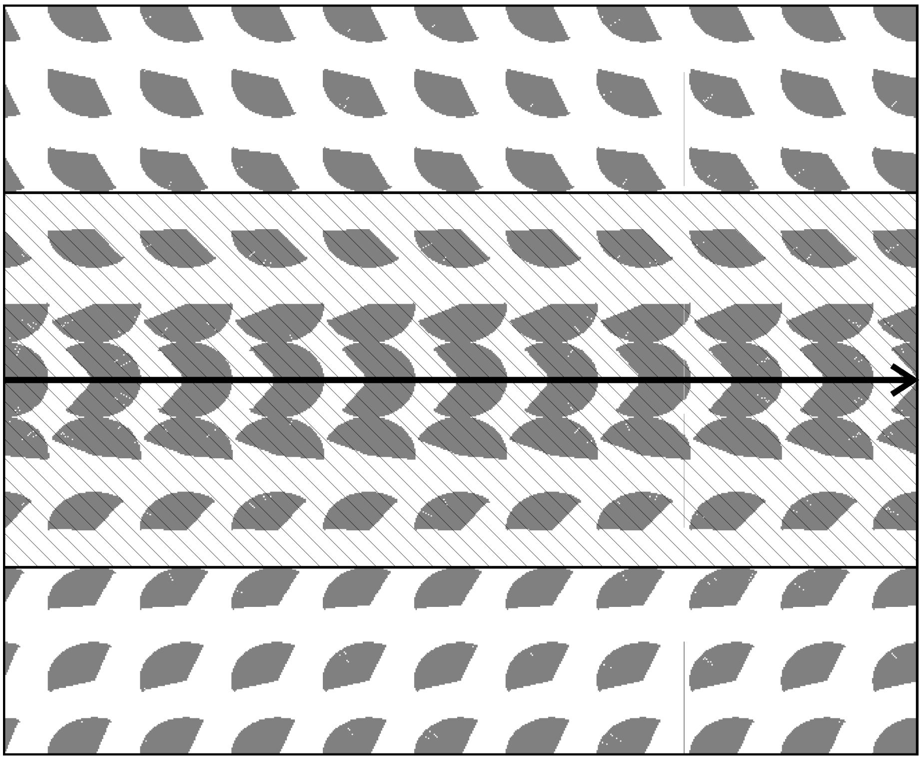
The state equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

becomes a differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}).$$

where $\mathbf{F}$ is a thick function.

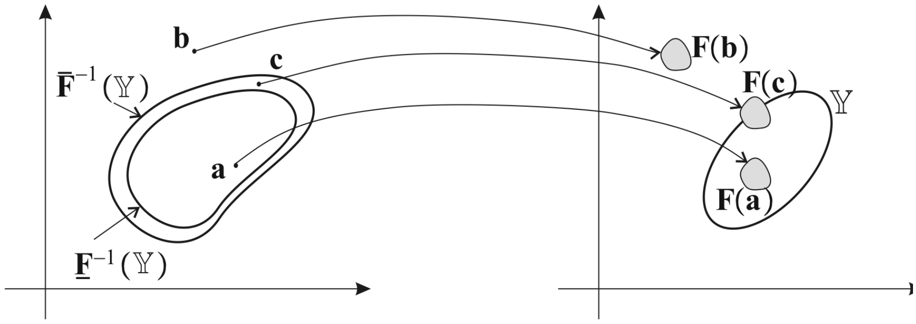


Differential inclusion for the sailboat; $\zeta = \frac{\pi}{3}$ and $\gamma_\infty = \frac{\pi}{2}$.

Set inversion of thick functions. Given the thick function $\mathbf{F}:\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^p)$ and a set $\mathbb{Y} \subset \mathbb{R}^p$, we define

$$\begin{array}{rcl} \underline{\mathbf{F}}^{-1}(\mathbb{Y}) & = & \{\mathbf{x} \mid \mathbf{F}(\mathbf{x}) \subset \mathbb{Y}\} \\ \overline{\mathbf{F}}^{-1}(\mathbb{Y}) & = & \{\mathbf{x} \mid \mathbf{F}(\mathbf{x}) \cap \mathbb{Y} \neq \emptyset\}. \end{array}$$

Interval analysis can be used to solve this problem.

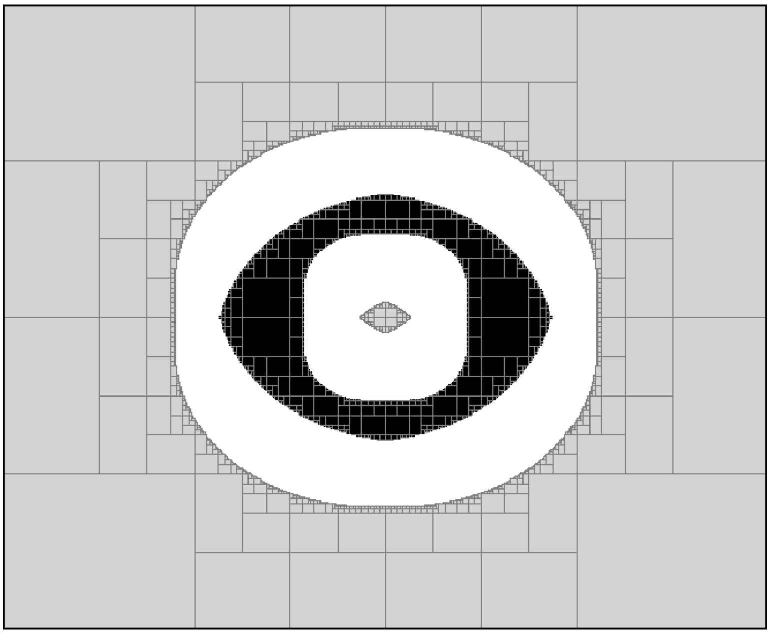


b
c
$\bar{\mathbf{F}}^{-1}(\mathbb{Y})$
a
$\underline{\mathbf{F}}^{-1}(\mathbb{Y})$
F(b)
F(c)
$\mathbb{Y}$
F(a)

Example. Consider the thick function

$$F(x_1, x_2) = (x_1 - [-1, 1])^2 + (x_2 - [-2, 2])^2$$
$$= \left\{ (x_1 - a)^2 + (x_2 - b)^2 , a \in [-1, 1] , b \in [-2, 2] \right\}.$$

For $\mathbb{Y} = [10, 100]$, we get the following picture for $\underline{\mathbf{F}}^{-1}(\mathbb{Y})$ and $\overline{\mathbf{F}}^{-1}(\mathbb{Y})$.



Lower and upper set inversion of a thick function.

Theorem. If $G(\mathbf{x})$ is the thick function defined by

$$G(\mathbf{x}) = \min\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{F}(\mathbf{x}), V(\mathbf{x})\right)$$

We have

(a) $\overline{G}^{-1}([0,\infty[) = \emptyset \;\Rightarrow\; \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ is V-stable
(b) $\underline{G}^{-1}([0,\infty[) \neq \emptyset \;\Rightarrow\; \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ is V-unstable.

4 Parametric case

Consider

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, \mathbf{p}).$$

We want to characterize the set $\mathbb{P}$ of all $\mathbf{p}$ such that the system is V-stable.

Define the thick function

$$
\begin{aligned}
G_{\mathbf{p}}(\mathbf{x}) &= \mathsf{min}\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{F}(\mathbf{x},\mathbf{p}), V(\mathbf{x})\right) \\
&= \left\{\mathsf{min}\left(\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{f}, V(\mathbf{x})\right), \mathbf{f} \in \mathbf{F}(\mathbf{x},\mathbf{p})\right\}.
\end{aligned}
$$

We have

$$\begin{array}{ll}\text{(a)} & \overline{G}_{\mathbf{p}}^{-1}([0,\infty[) = \emptyset \;\Rightarrow\; \mathbf{p} \in \mathbb{P} \\ \text{(b)} & \underline{G}_{\mathbf{p}}^{-1}([0,\infty[) \neq \emptyset \;\Rightarrow\; \mathbf{p} \notin \mathbb{P}.\end{array}$$

As a consequence, if

$$\begin{aligned}\mathbb{P}^- &= \left\{\mathbf{p}, \overline{G}_{\mathbf{p}}^{-1}([0,\infty[) = \emptyset\right\} \\ \mathbb{P}^+ &= \left\{\mathbf{p}, \underline{G}_{\mathbf{p}}^{-1}([0,\infty[) = \emptyset\right\},\end{aligned}$$

then

$$\mathbb{P}^- \subset \mathbb{P} \subset \mathbb{P}^+.$$

5 Test-case

Assumption. The closed loop system satisfies

$$\dot{\mathbf{x}} = \begin{pmatrix} \cos\left(\bar{\theta} + w\right) \\ \sin\left(\bar{\theta} + w\right) \end{pmatrix}, \text{ with } \bar{\theta} = \bar{\theta}\left(\mathbf{x}, \psi, \gamma_\infty, r, \zeta\right), \; w \in \mathbb{W}.$$

Property 1. If $|e(\mathbf{x})| < r_{\mathsf{max}}$ then, it will be the case for ever.

Property 2. If $|e(\mathbf{x})| > r_{\mathsf{max}}$ then $|e(\mathbf{x})|$ will decrease until $|e(\mathbf{x})| < r_{\mathsf{max}}$.

Property 3. The course should be feasible, i.e.,

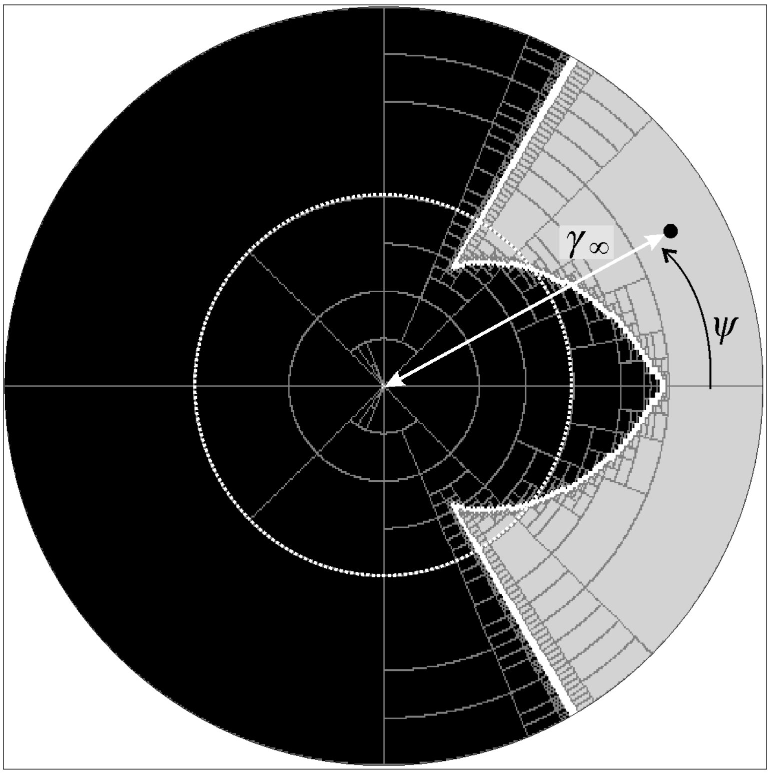
$$\cos\left(\psi - \bar{\theta}\right) + \cos\zeta \geq 0.$$

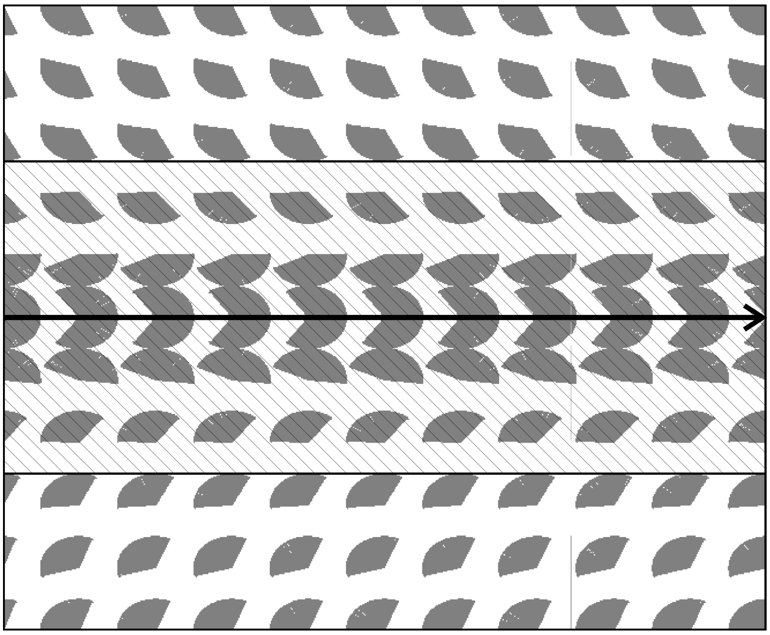
Property 4. The robot always moves toward the right direction, i.e., $\dot{x}_1 > 0$.

Case 1. To take into account Properties 1,2 and 3. Take $V(\mathbf{x}) = x_2^2 - r_{\mathsf{max}}^2$.

The parameter vector is $\mathbf{p} = (\gamma_\infty, \psi)$.

With $\zeta = \frac{\pi}{3}$, $\mathbb{W} = \{0\}$, we get.

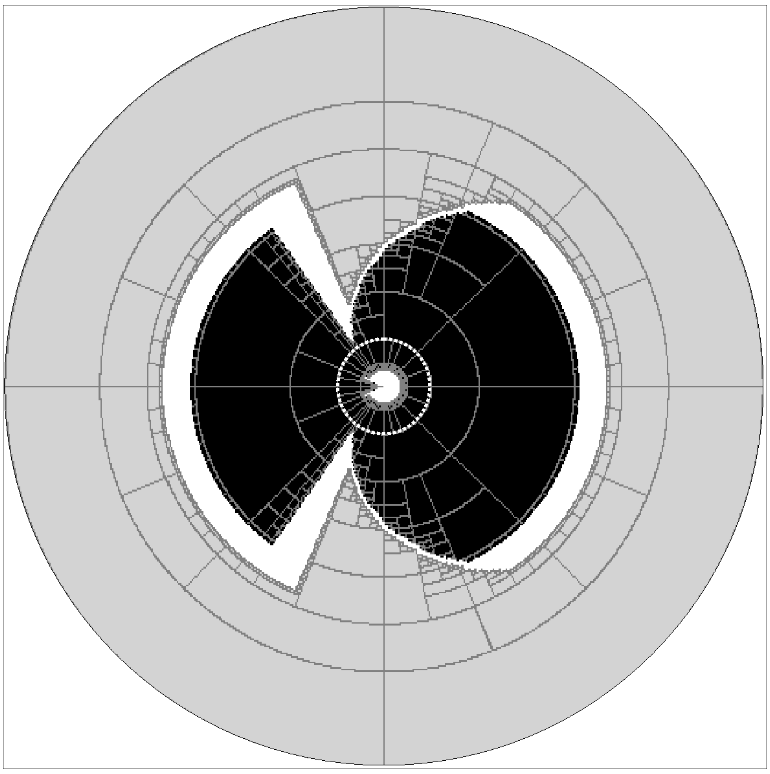


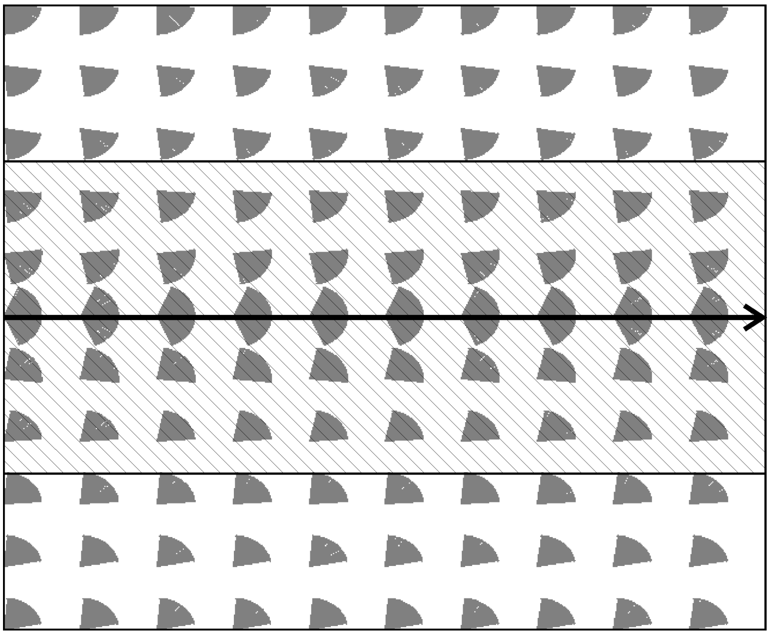


Differential inclusion for the controlled sailboat;

$\zeta = \frac{\pi}{3}$, $\gamma_\infty = \frac{\pi}{2}$.

Case 2. Assume that we want also that $\dot{x}_1 > 0$. Moreover, $\zeta = \frac{\pi}{6}$, $\mathbb{W} = \pm 5^{\circ}$.





Differential inclusion for the sailboat;
$\zeta = \frac{\pi}{6}, \gamma_\infty = \frac{\pi}{8}, [w] = \pm 5^\circ$