Endpoint and Midpoint Interval Representations Theoretical and Computational Comparison

Tomáš Dzetkulič

Institute of Computer Science
Academy of Sciences of the Czech Republic

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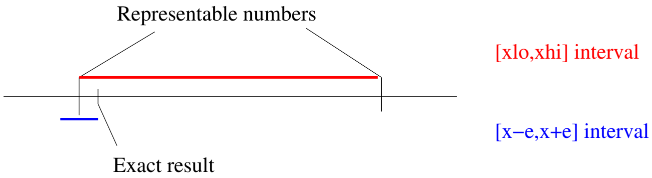

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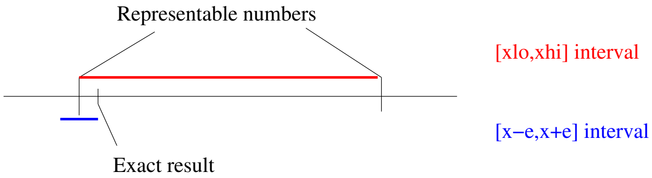

$6.666666666666666657415 \times 10^{-2} + [9.251 \times 10^{-19}, 9.252 \times 10^{-19}]$
Width: $< 10^{-30}$

Interval Types

Let Ω be the set of numbers representable on digital computer

We consider four kinds of intervals:

1. $[x_{lo}, x_{hi}]$ such that $x_{lo}, x_{hi} \in \Omega$
2. $[x - e, x + e]$ such that $x, e \in \Omega$
3. $[x - e_{lo}, x + e_{hi}]$ such that $x, e_{lo}, e_{hi} \in \Omega; e_{lo}, e_{hi} >= 0$
4. $[x - e_{lo}, x + e_{hi}]$ such that $x, e_{lo}, e_{hi} \in \Omega$

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Assumptions: intervals are narrow and the entire mantissa is used

$\oplus, \otimes$ are *round to nearest, ties to even* addition and multiplication

$\overline{+}, \underline{+}$ denote operations rounded up/down

Computing With Midpoint Intervals

In [1], Dekker showed that given $a, b \in \Omega$
$(a \oplus b) - (a + b) \in \Omega$ and $(a \otimes b) - (a \times b) \in \Omega$
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We can then compute $[x_1 - e_1, x_1 + e_1] + [x_2 - e_2, x_2 + e_2]$:
1. $(x, e_3) := \texttt{add}(x_1, x_2)$
2. $e := e_1 \overline{+} e_2 \overline{+} |e_3|$
3. $\textbf{return}[x - e, x + e]$

Addition With Huge Magnitude Difference

Example: $(1.3) + (1.4 \times 10^{-50})$
Exact result mantissa is long (ones in the beginning and in the end)

In classical interval analysis one of the interval bounds changes to next floating point number
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$\rightarrow$ In case small intervals are often added to our interval, the use of second interval kind has a huge advantage over the classical interval

Addition With Medium Magnitude Difference

Example: $(1.3) + (1.4 \times 10^{-10})$

Exact result mantissa is longer than allowed by the standard
$\rightarrow$ rounding occurs

In classical interval analysis the bounds of exact result can lie anywhere in between of two representable numbers

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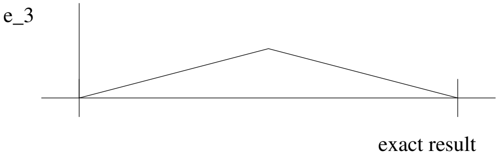

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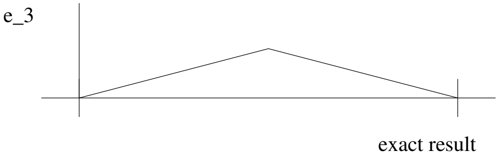

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For multiplication, the same observations are valid

Addition With No Magnitude Difference

Example: $(1.3) + (1.4)$
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In intervals of the second kind, e_3 is $\epsilon/2$ with the probability 0.5

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The Effect of Rounding To Nearest

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Either:

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This effect does not affect intervals of the first kind, since there is never a tie in a directed rounding

Addition Of Opposite Numbers

Example: $(1.3) + (-1.4)$
Result mantissa is shorter than allowed by the standard

There is no error introduced in classical interval analysis

There is a minor error introduced in the directed rounding of $e_1 \overline{+} e_2$ in intervals of the second kind

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Multiplication of wide intervals $[1-1, 1+1] \times [1-1, 1+1]$ yields suboptimal results ($[1-3, 1+3]$)

$\rightarrow$ shift of the interval center is required in intervals of the second kind

Rigorous Polynomial

1. $\sum_i [a_i, b_i] x^i$
2. $(\sum_i a_i x^i) + [-e, e]$
3. $(\sum_i a_i x^i) + [-e_{lo}, e_{hi}]$

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In second and third case we need less memory to store polynomial

Computational Experiments

Test 1: Add 10000 random numbers from interval $[-1.0, 1.0]$
Test 2: Add 10000 random numbers from interval $[0.5, 1.5]$
Test 3: Multiply 10000 random numbers from distribution $e^{[-1.0,1.0]}$

Test versions: Sequential and Divide&Conquer

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	$[a, b]$	$[a - e, a + e]$	$[a - e_{lo}, a + e_{hi}]$
Test	Error		
1 Sequential	8.7×10^{-11}	4.4×10^{-11}	2.2×10^{-11}
1 D&C	1.1×10^{-12}	7.6×10^{-13}	3.8×10^{-13}
2 Sequential	8.3×10^{-9}	4.1×10^{-9}	2.1×10^{-9}
2 D&C	1.1×10^{-11}	8.9×10^{-12}	4.5×10^{-12}
3	1.6×10^{-12}	1.0×10^{-12}	5.0×10^{-13}

Arithmetic Operations Count

	$[a, b]$	$[a-e, a+e]$	$[a-e_{lo}, a+e_{hi}]$
Addition			
Rounding mode change	2	2	2
Add	2	8	9
Time(10^9 operations)	$40s$	$48s$	$51s$
Multiplication			
Rounding mode change	2	2	2
Add	0	14	17
Mul	8	9	18
Min/Max/Abs	6	3	7
Time(10^9 operations)	$57s$	$63s$	$86s$

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Thank you for you attention.

[1] T. Dekker. A floating-point technique for extending the available precision. *Numerische Mathematik*, 18:224—-242, 1971/72.

[2] A. Neumaier. Taylor forms–use and limits. *Reliable Computing*, pages 43–79, 2003.