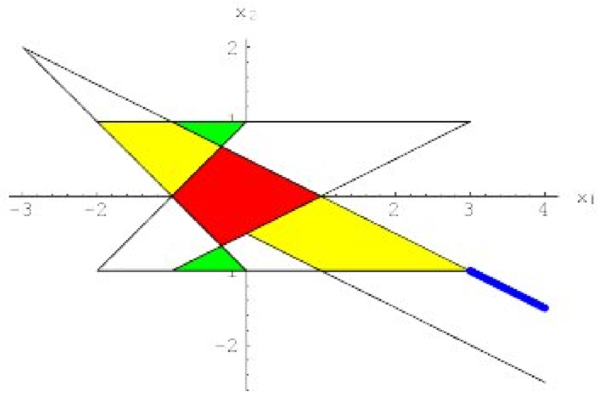
# Properties and Estimations of Parametric $AE$ Solution Sets





Evgenija D. Popova

Institute of Mathematics & Informatics
Bulgarian Academy of Sciences


04–06.06.2012

# Outline

- Parametric $AE$ solution sets ($\Sigma^p_{AE}$) — definitions

- $\Sigma^p_{AE}$ — characterization, properties

- Outer estimations

- Inner estimations

- Examples

- Conclusion

# Parametric Linear Systems

Consider the linear algebraic system

$$A(p) \cdot x \ = \ b(p),$$

where

$$a_{ij}(p) := a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu} p_\mu, \qquad b_i(p) := b_{i,0} + \sum_{\mu=1}^{m} b_{i,\mu} p_\mu$$

$$a_{ij,\mu}, b_{i,\mu} \in \mathbb{R}, \qquad \mu = 0, \ldots, m, \quad i, j = 1, \ldots, n$$

the uncertain parameters $p_\mu$ vary within given intervals

$$p \ \in \ [p] = ([p_1^-, p_1^+], \ldots, [p_m^-, p_m^+])^\top.$$

# Parametric $AE$ Solution Sets

For $\mathcal{A} := \{t \mid \forall p_t \in [p_t]\}$, $\mathcal{E} := \{t \mid \exists p_t \in [p_t]\}$, such that

$$\mathcal{A} \cup \mathcal{E} = \{1, \ldots, m\},\ \mathcal{A} \cap \mathcal{E} = \emptyset,$$

$$\Sigma^{p}_{AE} := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p)x = b(p))\}.$$

$AE$ terminology is after S. Shary.

The quantification of the parameters concerns the solution set, not the system.

For a given $A(p)x = b(p)$, $p \in [p] \in \mathbb{IR}^m$, there are $2^m$ parametric solution sets $\Sigma^{p}_{AE}$.

# Parametric $AE$ Solution Sets — special cases

$$\Sigma^{p}_{uni}\left(A(p), b(p), [p]\right) \quad := \quad \{x \in \mathbb{R}^n \mid \exists p \in [p],\ A(p)\cdot x = b(p)\}$$

$$\begin{aligned} \Sigma^{p}_{tol} \quad &= \quad \Sigma\left(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), [p]\right) \\ &:= \quad \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\} \end{aligned}$$

$$\begin{aligned} \Sigma^{p}_{cont} \quad &= \quad \Sigma\left(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), [p]\right) \\ &:= \quad \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\} \end{aligned}$$

# Parametric $AE$ Solution Sets

GOAL:

explicit representation of $\boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}}$ by means of inequalities

Why?

- exploring the solution set properties,

  which helps designing better (sharp, fast) numerical methods

- finding exact bounds,

  which helps in testing new numerical methods

The problem is related to Quantifier Elimination.

# Classification of the parameters

**Definition 1.** *A parameter is of* **1st class** *if it is involved in only one equation does not matter how many times.*

**Definition 2.** *A parameter is of* **2nd class** *if it is involved in more than one equation of the system.*

$$\begin{pmatrix} p_1 & 1 & 1 \\ p_2 & 2p_1 & p_2 + 1 \\ 1 & 1 & 3p_1 - 1 \end{pmatrix} \cdot x = \begin{pmatrix} p_3 - p_4 \\ p_1 - p_2/3 \\ p_3/2 \end{pmatrix}$$

# Parametric $AE$ Solution Sets

E. D. Popova, W. Krämer, *Characterization of AE Solution Sets to a Class of Parametric Linear Systems*, Compt. rend. Acad. bulg. Sci. 64(3):325-332, 2011.

**Theorem 1.**

$$\Sigma^{p}_{AE} = \bigcap_{p_{\mathcal{A}} \in [p_{\mathcal{A}}]} \bigcup_{p_{\mathcal{E}} \in [p_{\mathcal{E}}]} \{x \in \mathbb{R}^n \mid A(p_{\mathcal{A}}, p_{\mathcal{E}}) \cdot x = b(p_{\mathcal{A}}, p_{\mathcal{E}})\}.$$

# Parametric $AE$ Solution Sets

E. D. Popova, W. Krämer, *Characterization of AE Solution Sets to a Class of Parametric Linear Systems*, Compt. rend. Acad. bulg. Sci. 64(3):325-332, 2011.

**Theorem 2.** *If* $x \in \Sigma^{p}_{AE} \neq \emptyset$,

$$\sum_{\nu \in \mathcal{A}} (A_{\bullet\bullet\nu} x - b_{\bullet\nu})[p_\nu] \subseteq b_{\bullet 0} - A_{\bullet\bullet 0} x + \sum_{\mu \in \mathcal{E}} (b_{\bullet\mu} - A_{\bullet\bullet\mu} x)[p_\mu].$$

*equivallently*

$$|A(\dot{p})x - b(\dot{p})| \quad \leq \quad \sum_{\mu=1}^{m} \delta_\mu |A_{\bullet\bullet\mu} x - b_{\bullet\mu}| \widehat{p}_\mu,$$

*where* $\delta_\mu := \{1 \text{ if } \mu \in \mathcal{E},\ -1 \text{ if } \mu \in \mathcal{A}\}$, $\qquad \dot{p} := mid([p]),\ \widehat{p} := rad([p])$.

# Parametric $AE$ Solution Sets

**Theorem 3.** *Let* $\boldsymbol{A}(\boldsymbol{p})\boldsymbol{x} = \boldsymbol{b}(\boldsymbol{p})$ *involves only 1st class* $\boldsymbol{\mathcal{E}}$*-parameters.* *A point* $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ *belongs to* $\boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}}$, **if and only if**

$$\sum_{\nu\in\mathcal{A}} (\boldsymbol{A}_{\bullet\bullet\nu}\boldsymbol{x} - \boldsymbol{b}_{\bullet\nu})[\boldsymbol{p}_\nu] \subseteq \boldsymbol{b}_{\bullet 0} - \boldsymbol{A}_{\bullet\bullet 0}\boldsymbol{x} + \sum_{\mu\in\mathcal{E}} (\boldsymbol{b}_{\bullet\mu} - \boldsymbol{A}_{\bullet\bullet\mu}\boldsymbol{x})[\boldsymbol{p}_\mu].$$

*equivallently*

$$|\boldsymbol{A}(\dot{\boldsymbol{p}})\boldsymbol{x} - \boldsymbol{b}(\dot{\boldsymbol{p}})| \quad \leq \quad \sum_{\mu=1}^{m} \boldsymbol{\delta}_\mu |\boldsymbol{A}_{\bullet\bullet\mu}\boldsymbol{x} - \boldsymbol{b}_{\bullet\mu}| \widehat{\boldsymbol{p}}_\mu,$$

*where* $\boldsymbol{\delta}_\mu := \{\mathbf{1} \text{ if } \boldsymbol{\mu} \in \boldsymbol{\mathcal{E}},\ -\mathbf{1} \text{ if } \boldsymbol{\mu} \in \boldsymbol{\mathcal{A}}\}$, $\qquad \dot{\boldsymbol{p}} := mid([\boldsymbol{p}])$, $\widehat{\boldsymbol{p}} := rad([\boldsymbol{p}])$.

# Parametric $AE$ Solution Sets

E. D. Popova, *Explicit Description of AE Solution Sets to Parametric Linear Systems*, Preprint No. 7, IMI-BAS, 2011.

If $A(p)x = b(p)$ involves 2nd class $\mathcal{E}$-parameters,
a point $x \in \mathbb{R}^n$ belongs to $\Sigma^p_{AE}$, **if and only if**

$$|A(\dot{p})x - b(\dot{p})| \quad \le \quad \sum_{\mu=1}^{m} \delta_\mu |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\widehat{p}_\mu,$$

and "cross" inequalities

$$\left| w_\lambda(x) + \sum_{\mu\in\mathcal{E}} u_{\lambda,\mu}(x)\dot{p}_\mu + \sum_{\mu\in\mathcal{A}} v_{\lambda,\mu}(x)\dot{p}_\mu \right| \le \sum_{\mu\in\mathcal{E}} |u_{\lambda,\mu}(x)|\widehat{p}_\mu - \sum_{\mu\in\mathcal{A}} |v_{\lambda,\mu}(x)|\widehat{p}_\mu,$$

$$\lambda \in \mathcal{T}$$

obtained by Fourier-Motzkin-type elimination of $\mathcal{E}$-parameters

$$\delta_\mu := \{1 \text{ if } \mu \in \mathcal{E},\ -1 \text{ if } \mu \in \mathcal{A}\}, \quad \dot{p} := \mathrm{mid}([p]),\ \widehat{p} := \mathrm{rad}([p]).$$

# Parametric AE Solution Sets — Properties

- The elimination of $\mathcal{A}$-parameters and 1st class $\mathcal{E}$-parameters

  does not introduce "cross" inequalities.

- The shape of $\Sigma^{p}_{AE}$ is linear w.r.t. these parameters.

  However $\Sigma^{p}_{AE}$ is not convex even in a single orthant.

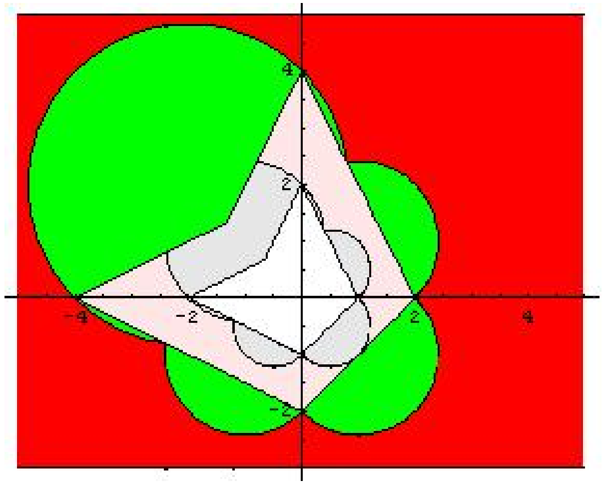
- The boundary of $\Sigma^{p}_{AE}$ involving 2nd class $\mathcal{E}$-parameters

  may consist of polynomials of arbitrary degree.

$$\begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix} x = \begin{pmatrix} 2p_3 \\ 2p_3 \end{pmatrix}$$

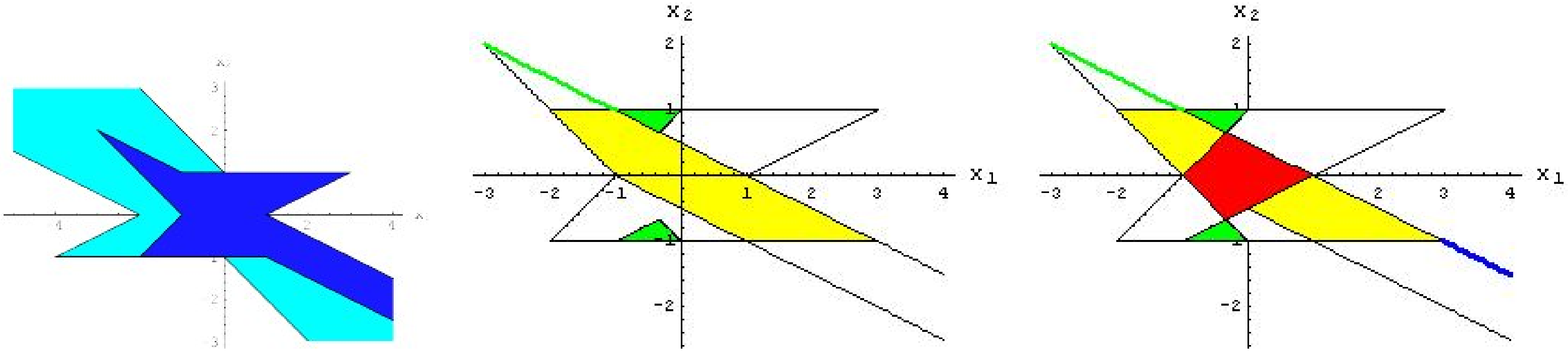
$p_1 \in [-2, 2], p_2 \in [-1, 2], \qquad p_3 \in [1, 2]$

$\Sigma_{\forall p_3 \exists p_1, p_2}$ – red



# Examples

$$\begin{pmatrix} p_1 & p_1+1 \\ p_2+1 & -2p_4 \end{pmatrix} x = \begin{pmatrix} p_3 \\ -3p_2+1 \end{pmatrix}, \quad p_1, p_2 \in [0,1],\ p_3, p_4 \in [-1,1]$$




$\Sigma_{\exists p_{1\ldots4}} \subseteq \Sigma_{\substack{\exists\exists\ \exists \\ \exists\exists\ \exists}}$ $\quad$ $\Sigma_{\forall p_1 \exists p_{2\ldots4}}$ – bounded $\quad$ $\Sigma_{\forall p_3 \exists p_1, p_2, p_4}$ – unbounded

$\Sigma_{\forall p_2 \exists p_1, p_3, p_4}$ – disconnected $\quad$ $\Sigma_{\forall p_4 \exists p_1, p_2, p_3}$

$\Sigma_{\forall p_1, p_2 \exists p_3, p_4}$ $\quad$ $\Sigma_{\forall p_1, p_3 \exists p_2, p_4}$ ? $\quad$ $\Sigma_{\forall p_2, p_4 \exists p_1, p_3}$

$\Sigma_{\forall p_1, p_4 \exists p_2, p_3}$

QE of *Mathematica* gives **311** logical expressions

# Parametric Tolerable Solution Set — Properties

**Theorem 4.** *The parametric tolerable solution set is a* **convex polyhedron**.

**Theorem 5.** *Let* $\boldsymbol{A}_{ri}([\boldsymbol{u}]) = \boldsymbol{A}_{rd}([\boldsymbol{v}]) \subseteq [\boldsymbol{A}]$.
*If* $\boldsymbol{q} \in [\boldsymbol{q}]$ *is 1st class parameter, then*

$$\boldsymbol{\Sigma}_{tol}([\boldsymbol{A}], \boldsymbol{b}([\boldsymbol{q}])) \subseteq \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}([\boldsymbol{u}]), \boldsymbol{b}([\boldsymbol{q}])) = \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}_{ri}(\boldsymbol{u}), [\boldsymbol{u}], \boldsymbol{b}([\boldsymbol{q}])) \subseteq \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}_{rd}(\boldsymbol{v}), [\boldsymbol{v}], \boldsymbol{b}([\boldsymbol{q}])).$$

*If* $\boldsymbol{A}(\boldsymbol{v})$ *involves more dependencies than* $\boldsymbol{A}(\boldsymbol{u})$ *and* $\boldsymbol{A}([\boldsymbol{u}]) = \boldsymbol{A}([\boldsymbol{v}])$, *then*

$$\boldsymbol{\Sigma}_{tol}(\boldsymbol{A}(\boldsymbol{u}), \boldsymbol{b}(\boldsymbol{q}), [\boldsymbol{u}], [\boldsymbol{q}]) \quad \subseteq \quad \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}(\boldsymbol{v}), \boldsymbol{b}(\boldsymbol{q}), [\boldsymbol{v}], [\boldsymbol{q}]).$$

# Parametric Controllable Solution Set — Properties

If $q_\nu$ are 1st class parameters for all $\nu \in \mathcal{A}$, then

$$\Sigma_{cont}(A(p_{\mathcal{E}}), b([q_{\mathcal{A}}]), [p_{\mathcal{E}}]) = \Sigma_{cont}(A(p_{\mathcal{E}}), b(q_{\mathcal{A}}), [p_{\mathcal{E}}], [q_{\mathcal{A}}]).$$

However, in the general case of 2nd class $\mathcal{A}$-parameters:

**Theorem 6.** *If there are two equations $\alpha$, $\beta$ of the parametric system which involve simultaneously an existentially quantified parameter $p_k$ and an universally quantified parameter $q_l$ such that*

$$sign(f_{k\beta} b_{\alpha,l}) = sign(f_{k\alpha} b_{\beta,l}),$$

*where $f_{k\lambda}(x) = A_{\lambda\bullet k}x$, $\lambda \in \{\alpha, \beta\}$, then*

$$\Sigma_{cont}(A(p_{\mathcal{E}}), b([q_{\mathcal{A}}]), [p_{\mathcal{E}}]) \subseteq \Sigma_{cont}(A(p_{\mathcal{E}}), b(q_{\mathcal{A}}), [p_{\mathcal{E}}], [q_{\mathcal{A}}]).$$

# Outer and Inner Estimations

For a given index set $\boldsymbol{I}$, define the set $\boldsymbol{\mathcal{B}_I}$ of end-points (vertices) of $\boldsymbol{p_{\mathcal{I}}}$.

**Theorem 7.** *It holds*

$$\boldsymbol{\Sigma^p_{AE}} = \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}} \boldsymbol{\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), [p_{\mathcal{E}}])}.$$

**Corollary 1.** *For* $\boldsymbol{\Sigma^p_{AE}} \neq \emptyset$,

$$\Box\boldsymbol{\Sigma^p_{AE}} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}} \Box\boldsymbol{\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), [p_{\mathcal{E}}])}.$$

$[\boldsymbol{u}] \subseteq \boldsymbol{\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), [p_{\mathcal{E}}])} \subseteq [\boldsymbol{v}]$ by any parametric solver for $\boldsymbol{\Sigma^p_{uni}}$.

# Outer and Inner Estimations

based on the characterization

$$|A(\dot{p})x - b(\dot{p})| \quad \leq \quad \sum_{\mu=1}^{m} \delta_\mu |A_{\bullet\bullet\mu} x - b_{\bullet\mu}| \widehat{p}_\mu,$$

where $\delta_\mu := \{1 \text{ if } \mu \in \mathcal{E}, \ -1 \text{ if } \mu \in \mathcal{A}\}$, $\qquad \dot{p} :=$ mid$([p])$, $\widehat{p} :=$ rad$([p])$.

# Outer Estimations $\Sigma^{p}_{AE} \subseteq [u]$

E. D. Popova, M. Hladík, *Outer Enclosures to Parametric AE Solution Set*, submitted, 2012.

**Theorem 8.** *(Bauer–Skeel generalization) Let* $A(\dot{p})$ *be regular and define*

$$C := A^{-1}(\dot{p}), \qquad x^* := Cb(\dot{p}), \qquad M := \sum_{k=1}^{m} |CA_k|\hat{p}_k.$$

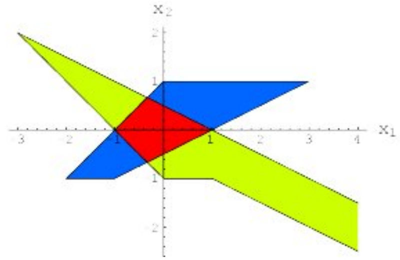
*If* $\rho(M) < 1$*, then every* $x \in \Sigma^{p}_{AE}$ *satisfies*

$$|x - x^*| \leq (I - M)^{-1} \left( \sum_{k \in \mathcal{E}} |C(A_k x^* - b_k)|\hat{p}_k - \sum_{k \in \mathcal{A}} |C(A_k x^* - b_k)|\hat{p}_k \right).$$

# Outer Estimations — Properties

For $\Sigma^{p}_{tol}$:

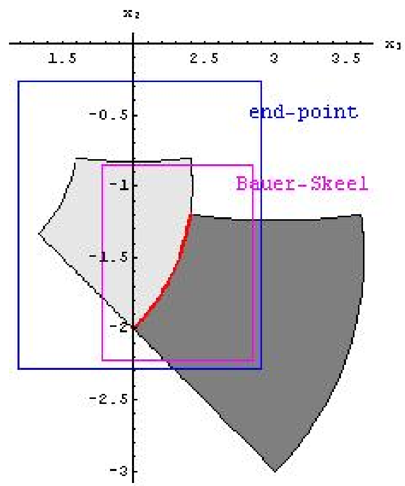
| | | |
|---|---|---|
| For $\Sigma^{p}_{tol}$: | End-Point approach | – best enclosures but not the hull |
| | LP approach | – good for row-dependencies |
| | Bauer–Skeel | – worse enclosures. |




For $\Sigma^{p}_{con}$, Bauer–Skeel method provides always better enclosures.

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix}$$

$p_1 \in [0, \frac{1}{2}]$, $p_2 \in [1, \frac{3}{2}]$, $q \in [1, \frac{3}{2}]$
$\Sigma^{p}_{con}$ – red.



## Inner Estimation: $[v] \subseteq \Sigma_{tol}(A(p_{\mathcal{A}}), [b], [p_{\mathcal{A}}])$

following Neumaier, Fr.Interv.Berichte 86/9.

Let $[e] = ([-1, 1], \ldots, [-1, 1])^\top$.

For a given $\tilde{x} \in \mathrm{int}\Sigma_{tol}(A(p), [b], [p])$, compute $\max \eta > 0$, such that

$$\eta \left( A_{\bullet\bullet 0}[e] + \sum_{\nu=1}^{k} (A_{\bullet\bullet\nu}[e])[p_\nu] \right) \subseteq [b] - A_{\bullet\bullet 0}\tilde{x} \ominus \sum_{\nu=1}^{k} (A_{\bullet\bullet\nu}\tilde{x})[p_\nu], \qquad (1)$$

where $[a_1, a_2] \ominus [b_1, b_2] := [a_1 - b_1, a_2 - b_2]$.

**Theorem 9.** *For* $\tilde{x} \in \mathit{int}\Sigma_{tol}(A(p), [b], [p])$ *and* $\eta > 0$, *such that (1) holds,*

$$\tilde{x} + \eta[e] \subseteq \Sigma_{tol}(A(p), [b], [p]).$$

# Inner Estimation:

S.Shary, 1996: The "end-point" approach provides the best $[v] \subseteq \Sigma_{tol}([A],[b])$

with comp. complexity $O(2^{n^2})$

By a complicated search-like algorithm he reduces the comp. complexity to $O(2^n)$.

Since $\qquad \Sigma_{tol}([A],[b]) = \Sigma_{tol}(A_{ri}(p),[b]),$

consider $A(p) = A^0 + \sum_{\nu=1}^{n} A^{\nu} p_{\nu},$

where $A^0 = \text{mid}[A]$, $A^{\nu} = \text{rad}[A]_{\bullet\nu}$, $p_{\nu} \in [-1,1]$, $\nu = 1,\ldots,n$

and apply the "end-point" approach to

the parametric system with comp. complexity $O(2^n)$.

# Examples

Consider the Lyapunov matrix equation

$$\boldsymbol{AX} + \boldsymbol{XA}^\top = \boldsymbol{F},$$

where $\boldsymbol{A} \in [\boldsymbol{A}]$, $\boldsymbol{F} \in [\boldsymbol{F}]$, or $\boldsymbol{A}$, $\boldsymbol{F}$ have linear uncertainty structure.

A common approach is to transform a matrix equation into linear system

$$\boldsymbol{Px} = \boldsymbol{f},$$

where $\boldsymbol{P} = \boldsymbol{I_n} \bigotimes \boldsymbol{A} + \boldsymbol{A} \bigotimes \boldsymbol{I_n}$, $\boldsymbol{x} = \text{vec}(\boldsymbol{X})$, $\boldsymbol{f} = \text{vec}(\boldsymbol{F})$.

In both cases above, $\boldsymbol{P}$ has a linear uncertainty structure.

Therefore, a $\boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}}$ must be considered

depending on the context of the particular problem.

# Examples — Controllability

Sokolova S., Kuzmina, E., 2008.

Consider

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})$$

where $\boldsymbol{A} \in [\boldsymbol{A}] \in \mathbb{IR}^{n\times n}$, $\boldsymbol{B} \in [\boldsymbol{B}] \in \mathbb{IR}^{n\times m}$.

Let $[\boldsymbol{A}]$ be assimptotically stable.

The interval object is completely controllable if and only if

$$\text{rank}[\boldsymbol{V}] = \boldsymbol{n}, \qquad [\boldsymbol{V}] \subseteq \boldsymbol{\Sigma}_{tol}([\boldsymbol{A}], [\boldsymbol{B}]),$$

where

$$\boldsymbol{\Sigma}_{tol}([\boldsymbol{A}], [\boldsymbol{B}]) := \{\boldsymbol{V} \in \mathbb{R}^{n\times n} \mid (\forall \boldsymbol{A} \in [\boldsymbol{A}])(\exists \boldsymbol{B} \in [\boldsymbol{B}])(\boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A}^{\top} = -\boldsymbol{B}\boldsymbol{B}^{\top})\}.$$

# Examples — Controllability

Controllability analysis reduces to

$$\text{finding} \qquad [\boldsymbol{v}] \subseteq \boldsymbol{\Sigma_{tol}}(\boldsymbol{P}(\boldsymbol{a_{ij}}), \boldsymbol{f}(\boldsymbol{f_{ij}}), [\boldsymbol{A}], [\boldsymbol{F}]),$$

where

$$\boldsymbol{P}(\boldsymbol{a_{ij}}) := \boldsymbol{I_n} \otimes \boldsymbol{A} + \boldsymbol{A} \otimes \boldsymbol{I_n}, \qquad \boldsymbol{a_{ij}} \in [\boldsymbol{a_{ij}}]$$

$$[\boldsymbol{v}] = \text{vec}([\boldsymbol{V}]), \qquad \boldsymbol{f}(\boldsymbol{f_{ij}}) := \text{vec}(\boldsymbol{F} = -\boldsymbol{B}\boldsymbol{B}^\top).$$

# Examples – Controllability

For

$$\text{mid}([A]) = \begin{pmatrix} -1 & -1 & 2 \\ 3 & -2 & -5 \\ -2 & 1 & -5 \end{pmatrix}, \qquad \text{rad}([a_{ij}]) = 3/100,$$

$$[B] = \text{diag}([-\tfrac{25}{8}, -\tfrac{5}{8}], [1, \tfrac{6}{5}], [1, \tfrac{3}{2}]),$$

we obtain

$$[V] = \begin{pmatrix} [1.3042, 1.3078] & [0.8377, 0.8413] & [-0.1986, -0.1950] \\ [0.8377, 0.8413] & [2.0175, 2.0211] & [-0.1838, -0.1802] \\ [-0.1986, -0.1950] & [-0.1838, -0.1802] & [0.2030, 0.2066] \end{pmatrix}$$

# Conclusion

The description of $\Sigma^p_{AE}$ by F-M elimination of $\mathcal{E}$-parameters

is feasible, much faster & compact than by Quantifier Elimination.

The inclusion relations between $\Sigma^p_{AE}$ are determined by the type of dependencies.

A single-step Bauer–Skeel method provides outer enclosure with pros and cons.

Inner inclusion for $\Sigma^p_{AE}$ involving 1st class $\mathcal{E}$-parameters is easy.

We know much about the Parametric Tolerable SSets.

There exists a large room for Further Research on general $\Sigma^p_{AE}$.