Guaranteed state estimation by zonotopes for systems with interval uncertainties

Vu Tuan Hieu Le*, Cristina Stoica*, Teodoro Alamo**, Eduardo F. Camacho** and Didier Dumur*

* SUPELEC Systems Sciences (E3S) - Automatic Control Department, Gif sur Yvette, France
{vutuanhieu.le,cristina.stoica,didier.dumur}@supelec.fr
** Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Escuela Superior de Ingenieros, Sevilla, Spain
alamo@cartuja.us.es, eduardo@esi.us.es

SWIM 2012 - Oldenburg, Germany

Outline

Outline

Mathematical notations

Mathematical notations

- Interval: $[a;b] = \{x : a \leq x \leq b\}$
  $mid([a;b]) = \frac{a+b}{2}$, $rad([a;b]) = \frac{b-a}{2}$
  Unitary interval: $\mathbf{B} = [-1;1]$
  Interval matrix : $[A]$ with A_{ij} are intervals.

Mathematical notations

- Interval: $[a; b] = \{x : a \leq x \leq b\}$
  $mid([a; b]) = \frac{a+b}{2}$, $rad([a; b]) = \frac{b-a}{2}$
  Unitary interval: $\mathbf{B} = [\text{-}1;1]$
  Interval matrix : $[A]$ with A_{ij} are intervals.
- Zonotope: a convex symmetric polytope
  m-zonotope: the set $p \oplus H\mathbf{B}^m = \{p + Hz, z \in \mathbf{B}^m\}$,
  with a vector $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n \times m}$

Mathematical notations

- Interval: $[a;b] = \{x : a \leq x \leq b\}$
  $mid([a;b]) = \frac{a+b}{2}$, $rad([a;b]) = \frac{b-a}{2}$
  Unitary interval: $\mathbf{B}$ = [-1;1]
  Interval matrix : $[A]$ with A_{ij} are intervals.
- Zonotope: a convex symmetric polytope
  m-zonotope: the set $p \oplus H\mathbf{B}^m = \{p + Hz, z \in \mathbf{B}^m\}$,
  with a vector $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n \times m}$
- *P-radius* of a zonotope $X = p \oplus H\mathbf{B}^m$:
  $d(x) = \max(\|x - p\|_P^2)$, with $x \in X$

Mathematical notations

Example: $p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$d(x) = \max(\|x\|_P^2)$, with $x \in X = p \oplus H\mathbf{B}^3$

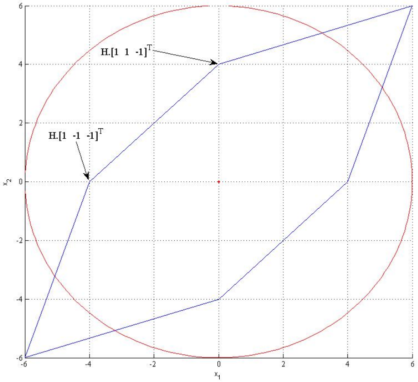


Figure: Zonotope and ellipsoid representing the associated P-radius

Outline

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \tag{1}$$

where

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \quad (1)$$

where

- $x_k \in \mathbb{R}^{n_x}$ state of system

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \quad (1)$$

where

- $x_k \in \mathbb{R}^{n_x}$ state of system
- $y_k \in \mathbb{R}^{n_y}$ measured output

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \tag{1}$$

where

- $x_k \in \mathbb{R}^{n_x}$ state of system
- $y_k \in \mathbb{R}^{n_y}$ measured output
- $\omega_k \in \mathbb{R}^{n_x}$ state disturbances

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \tag{1}$$

where

- $x_k \in \mathbb{R}^{n_x}$ state of system
- $y_k \in \mathbb{R}^{n_y}$ measured output
- $\omega_k \in \mathbb{R}^{n_x}$ state disturbances
- $v_k \in \mathbb{R}^{n_y}$ measurement perturbation

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \quad (1)$$

where

- $x_k \in \mathbb{R}^{n_x}$ state of system
- $y_k \in \mathbb{R}^{n_y}$ measured output
- $\omega_k \in \mathbb{R}^{n_x}$ state disturbances
- $v_k \in \mathbb{R}^{n_y}$ measurement perturbation
- Assumptions
  1. $A_k \in [A]$ Schur stable.
  2. $\omega_k \in W, v_k \in V$, with W a zonotope, V a box (for simplicity, W, V can be centered in the origin).
  3. $x_0 \in X_0$, where X_0 is a zonotope.

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \tag{1}$$

where

- $x_k \in \mathbb{R}^{n_x}$ state of system
- $y_k \in \mathbb{R}^{n_y}$ measured output
- $\omega_k \in \mathbb{R}^{n_x}$ state disturbances
- $v_k \in \mathbb{R}^{n_y}$ measurement perturbation
- Assumptions
  1. $A_k \in [A]$ Schur stable.
  2. $\omega_k \in W, v_k \in V$, with W a zonotope, V a box (for simplicity, W, V can be centered in the origin).
  3. $x_0 \in X_0$, where X_0 is a zonotope.

Goal: Find a guaranteed set which bounds the system state

Outline

Guaranteed state estimation by zonotopes

General algorithm:

Guaranteed state estimation by zonotopes

General algorithm:

- Consider $x_{k-1} \in \hat{X}_{k-1}$.

Guaranteed state estimation by zonotopes

General algorithm:

- Consider $x_{k-1} \in \hat{X}_{k-1}$.
- Step 1: (Prediction step) Compute a zonotope $\bar{X}_k$ that offers a bound for the uncertain trajectory of the system ($\bar{X}_k = A_k \hat{X}_{k-1} \oplus W$).

Guaranteed state estimation by zonotopes

General algorithm:

- Consider $x_{k-1} \in \hat{X}_{k-1}$.
- Step 1: (Prediction step) Compute a zonotope $\bar{X}_k$ that offers a bound for the uncertain trajectory of the system ($\bar{X}_k = A_k \hat{X}_{k-1} \oplus W$).
- Step 2: (Measurement) Compute the consistent state set X_{y_k} by using the measurement.

Guaranteed state estimation by zonotopes

General algorithm:

- Consider $x_{k-1} \in \hat{X}_{k-1}$.
- Step 1: (Prediction step) Compute a zonotope $\bar{X}_k$ that offers a bound for the uncertain trajectory of the system ($\bar{X}_k = A_k \hat{X}_{k-1} \oplus W$).
- Step 2: (Measurement) Compute the consistent state set X_{y_k} by using the measurement.
- Step 3: (Correction step) Compute an outer approximation $\hat{X}_k$ of the intersection between X_{y_k} and $\bar{X}_k$.

Single-output case

- Given a system:

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = c^T x_k + v_k \end{cases} \quad (2)$$

$v_k \in V = \sigma \mathbf{B}^1$, an interval.

- Consistent state set $X_{y_k} = \{x \in \mathbb{R}^n : |c^T x - y_k| \leq \sigma\}$ becomes a strip.
- State estimation $\hat{X}_k$ the intersection between a zonotope and a strip.

Intersection of a zonotope and a strip

Property 1: [1]

Given:

- zonotope $X = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$
- strip $S = \{x \in \mathbb{R}^n : |c^T x - d| \leq \sigma\}$
- vector $\lambda \in \mathbb{R}^n$

define:

- a vector $\hat{p}(\lambda) = p + \lambda(d - c^T p) \in \mathbb{R}^n$
- matrix $\hat{H}(\lambda) = [(I - \lambda c^T)H \ \sigma\lambda] \in \mathbb{R}^{n\times(m+1)}$.

Then the following expression holds:
$X \cap S \subseteq \hat{X}(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)B^{r+1}$.

Goal: Find a vector λ that offers the best approximation

[1] T. Alamo et al. (2005) Guaranteed state estimation by zonotopes

How to choose λ?

How to choose λ?

Two existing approaches to compute λ

How to choose λ?

Two existing approaches to compute λ

- Minimizing the segments of the zonotope: simple but not efficient.

How to choose λ?

Two existing approaches to compute λ

- Minimizing the segments of the zonotope: simple but not efficient.
- Minimizing the volume of the zonotope: more accurate and more complex.

$\rightarrow$ The proposed approach combines the advantages of the two existing approaches.

Guaranteed state estimation by zonotopes

At the time instant k:

- Guaranteed state set at $k-1$: $\hat{X}_{k-1} = p \oplus H\mathbf{B}^r$
- The measured output: $d = y_k$
- Rewrite $W = FB^{n_x}$, $V = \sigma B$

- Prediction step

$$\bar{X}_k = A_k p \oplus \begin{bmatrix} A_k H & F \end{bmatrix} \mathbf{B}^{r+n_x} \tag{3}$$

- Measurement: $\{x \in \mathbb{R}^n : |c^T x - y_k| \leq \sigma\}$
- Correction step using Property 1

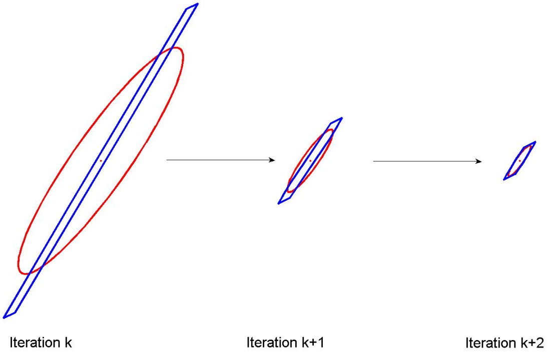
$$\hat{X}_k(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+n_x+1} \tag{4}$$

with $\hat{p}(\lambda) = A_k p + \lambda(d - c^T A_k p)$
and $\hat{H}(\lambda) = \begin{bmatrix} (I - \lambda c^T) \begin{bmatrix} A_k H & F \end{bmatrix} & \sigma\lambda \end{bmatrix}$

Guaranteed state estimation by zonotopes

New criterion to compute λ

Compute a matrix $P = P^T \succ 0$ and a vector λ such that at each sample time, the P-radius of the zonotopic state estimation set is decreased.



Guaranteed state estimation by zonotopes

Equivalent optimization problem:

$$\max_{\hat{z}} \|\hat{H}\hat{z}\|_P^2 \leq \max_{z} \beta \|Hz\|_P^2 + \max_{s} \|Fs\|_2^2 + \sigma^2 \qquad (5)$$

with $\hat{z} = \begin{bmatrix} z \\ s \\ \eta \end{bmatrix} \in \mathbf{B}^{r+n_x+1}$, $z \in \mathbf{B}^r$, $s \in \mathbf{B}^{n_x}$, $\eta \in \mathbf{B}^1$, and $\beta \in [0; 1)$.

Sufficient condition:

$$\hat{z}^T \hat{H}^T P \hat{H} \hat{z} - \beta z^T H^T P H z - s^T F^T F s - \sigma^2 + \\ + \sigma^2(1 - \eta^2) \leq 0 \qquad (6)$$

Guaranteed state estimation by zonotopes

Denoting $v = H\tilde{z}$, then the inequality (6) can be written in the matrix formulation:

$$\begin{bmatrix} v \\ s \\ \eta \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ * & A_{22} & A_{23} \\ * & * & A_{33} \end{bmatrix} \begin{bmatrix} v \\ s \\ \eta \end{bmatrix} \leq 0 \tag{7}$$

with the following notations:

$$\begin{cases} A_{11} = ((I - \lambda c^T)A_k)^T P((I - \lambda c^T)A_k) - \beta P \\ A_{12} = ((I - \lambda c^T)A_k)^T P(I - \lambda c^T)F \\ A_{13} = ((I - \lambda c^T)A_k)^T P\sigma\lambda \\ A_{22} = ((I - \lambda c^T)F)^T P(I - \lambda c^T)F - F^T F \\ A_{23} = ((I - \lambda c^T)F)^T P\sigma\lambda \\ A_{33} = \sigma^2 \lambda^T P \lambda - \sigma^2 \end{cases} \tag{8}$$

with $A_k \in [A]$

Guaranteed state estimation by zonotopes

It is equivalent to:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ * & A_{22} & A_{23} \\ * & * & A_{33} \end{bmatrix} \preceq 0, \quad \forall \begin{bmatrix} v \\ s \\ \eta \end{bmatrix} \neq 0 \qquad (9)$$

Using the explicit notations (8) and doing some manipulations a BMI (Bilinear Matrix Inequality) problem is derived as:

$$\begin{bmatrix} \beta P & 0 & 0 & A_k^T P - A_k^T c Y^T \\ * & F^T F & 0 & F^T P - F^T c Y^T \\ * & * & \sigma^2 & Y^T \sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \qquad (10)$$

with β, P and $Y = P\lambda$ as decision variables.

Guaranteed state estimation by zonotopes

Because $A_k \in [A]$ which is a convex set, a sufficient condition for (10) is:

$$\begin{bmatrix} \beta P & 0 & 0 & S_i^T P - S_i^T c Y^T \\ * & F^T F & 0 & F^T P - F^T c Y^T \\ * & * & \sigma^2 & Y^T \sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \qquad (11)$$

for $i = 1, \ldots, q$, with:

- β, P and $Y = P\lambda$ as decision variables
- q number of interval elements in $[A]$
- S_i vertices of $[A]$

Minimization of the P-radius

- Denoting

$$\begin{cases} d_k = \max_{x \in \hat{X}_k}(\|x - p_k\|_P^2) \\ const = \max_{s \in \mathbf{B}^{n_x}} \|Fs\|_2^2 \end{cases} \quad (12)$$

  the contractiveness condition on the P-radius is: $d_{k+1} \leq \beta d_k + const + \sigma^2$.

- At infinity: $d_\infty = \beta d_\infty + const + \sigma^2 \Leftrightarrow d_\infty = \frac{\sigma^2 + const}{1-\beta}$

Consider an ellipsoid
$E = \{x : x^T P x \leq \frac{\sigma^2 + const}{1-\beta}\} \Leftrightarrow E = \{x : x^T \frac{(1-\beta)P}{\sigma^2 + const} x \leq 1\}$

To minimize the size of the guaranteed set, the ellipsoid of the smallest diameter must be found $\Rightarrow$ Linear Matrix Inequality (LMI) problem:

$\max_{\tau, P} \tau$
s.t. the LMI

$$\frac{(1-\beta)P}{\sigma^2 + const} \succeq \tau I \qquad (13)$$

Guaranteed state estimation by zonotopes

Algorithm
Solve $\max_{\tau,P,Y} \tau$
subject to the BMIs

$$\left\{ \begin{array}{l} \frac{(1-\beta)P}{\sigma^2+const} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & S_i^T P - S_i^T c Y^T \\ * & F^T F & 0 & F^T P - F^T c Y^T \\ * & * & \sigma^2 & Y^T \sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \end{array} \right. \quad (14)$$

$i = 1, \ldots, q.$
End

Remarks

A_k unknown $\rightarrow \bar{X}_k = A_k p \oplus \begin{bmatrix} A_k H & F \end{bmatrix} \mathbf{B}^{r+n_\omega}$ is unknown
$\bar{X}_k$ approximated by $Z = mid([A])p_{k-1} \oplus G\mathbf{B}^{r+3n_x}$
with

$$G = \begin{bmatrix} mid([A])\hat{H}_{k-1} & rs(rad([A])|\hat{H}_{k-1}|) & rs(rad([A])|\hat{p}_{k-1}|) & F \end{bmatrix}$$

- A_k stable $\rightarrow rs(rad([A])|\hat{p}_{k-1}|)$ is bounded
- λ is independent of $\hat{H}_{k-1}$

$\rightarrow$ The contractiveness of the P-radius is ensured.

Multi-output case

- Given a system

$$\begin{cases} x_{k+1} = A_k x_k + \omega_k \\ y_k = C^T x_k + v_k \end{cases} \tag{15}$$

$v_k \in V$ a box $\in \mathbb{R}^{n_y}$
$V = \Sigma \mathbf{B}^{n_y}$ with $\Sigma = diag(\sigma_1, \ldots, \sigma_{n_y})$

- First idea: Using the proposed algorithm for each of n_y outputs

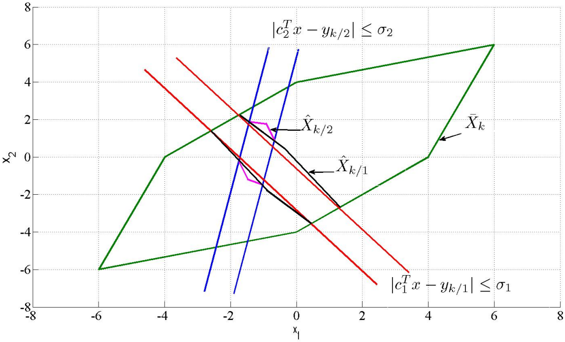


Figure: State estimation of the 2-outputs system

- Advantage: direct application of single-output case.
- Inconvenient: conservative result due to the coupling effect of multi-outputs.

Proposed solution

- Consistent state set created by the measurement: $X_{y_k} = \{x \in \mathbb{R}^{n_x} : |C^T x - y_k| \in V\} \Rightarrow$ polytope
- State estimation is the intersection between a zonotope and a polytope

Find a zonotopic outer approximation of the intersection between a zonotope and a polytope

Intersection of a zonotope and a polytope

Property 2:

Given

- zonotope $X = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$
- polytope $P = \{x \in \mathbb{R}^n, d \in \mathbb{R}^m : |C^T x - d| \leq \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{bmatrix}\}$

  $(\sigma_i \in \mathbb{R}^+)$
- matrix $\Lambda \in \mathbb{R}^{n \times m}$

Define

- vector $\hat{p}(\Lambda) = p + \Lambda(d - C^T p) \in \mathbb{R}^{n_x}$
- matrix $\hat{H}(\Lambda) = \begin{bmatrix} (I - \Lambda C^T)H & \Lambda\Sigma \end{bmatrix}$ with $\Sigma = diag(\sigma_1, \ldots, \sigma_{n_y})$

Then $X \cap P \subseteq \hat{X}(\Lambda) = \hat{p}(\Lambda) \oplus \hat{H}(\Lambda)B^{r+m}$

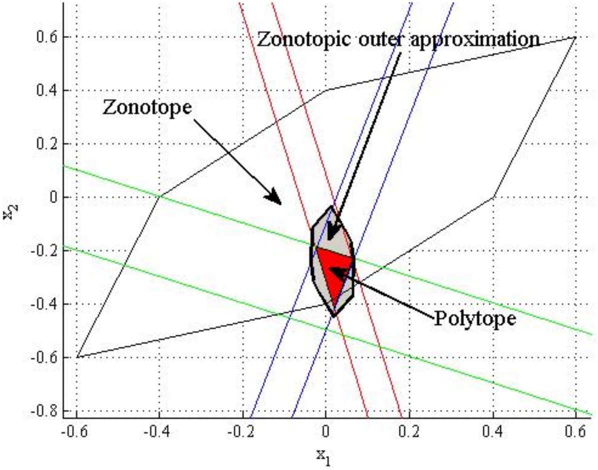


Figure: Zonotopic approximation of the intersection between a zonotope and a polytope

Similar to single-output case:
Based on the estimation at $k-1$: $\hat{X}_{k-1} = p \oplus H\mathbf{B}^r$, the state estimation set at k is:

$$\hat{X}_k(\Lambda) = \hat{p}(\Lambda) \oplus \hat{H}(\Lambda)\mathbf{B}^{r+n_x+n_y} \tag{16}$$

with $\hat{p}(\Lambda) = A_k p + \Lambda(y_k - C^T A_k p)$
and $\hat{H}(\Lambda) = \left[(I - \Lambda C^T)\begin{bmatrix} A_k H & F \end{bmatrix} \quad \Lambda\Sigma\right]$

- Single-output case: λ is a vector.
- Multi-output case: Λ is a matrix.

$\Lambda \in \mathbb{R}^{n_x \times n_y}$ computed to ensure the contractiveness of the P-radius

$$\max_{\tau,\beta,P,Y} \tau$$
subject to

$$\begin{cases} \frac{(1-\beta)P}{\sigma_1^2+\ldots+\sigma_{n_y}^2+const} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & S_i^T P - S_i^T C Y^T \\ * & F^T F & 0 & F^T P - F^T C Y^T \\ * & * & \Sigma^T \Sigma & Y^T \Sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \\ \tau > 0 \end{cases} \tag{17}$$

with

- β, P and $Y = P\Lambda$ as decision variables
- q number of interval elements in $[A]$
- S_i vertices of $[A]$

Outline

- Single-output case

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+0.3\delta_k \end{bmatrix} x_k + 0.02 \begin{bmatrix} -6 \\ 1 \end{bmatrix} \omega_k \\ y_k = \begin{bmatrix} -2 & 1 \end{bmatrix} x_k + 0.2 v_k \end{cases} \quad (18)$$

with $|\delta_k| \leq 1$, $\|v_k\|_\infty \leq 1$, $\|\omega_k\|_\infty \leq 1$.

The initial state belongs to the box $3\mathbf{B}^2$.

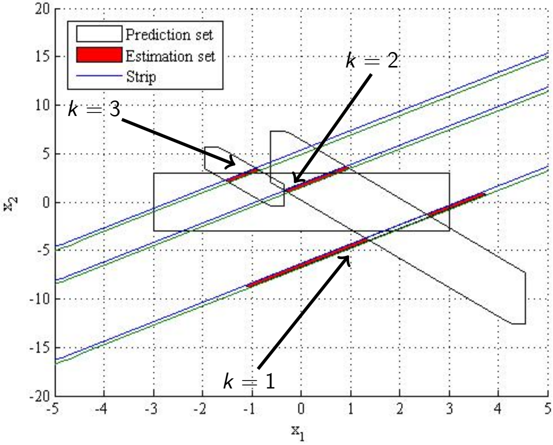


Figure: Intersection $\hat{X}_k$ between the predicted state set $\bar{X}_k$ and the measurement X_{y_k}

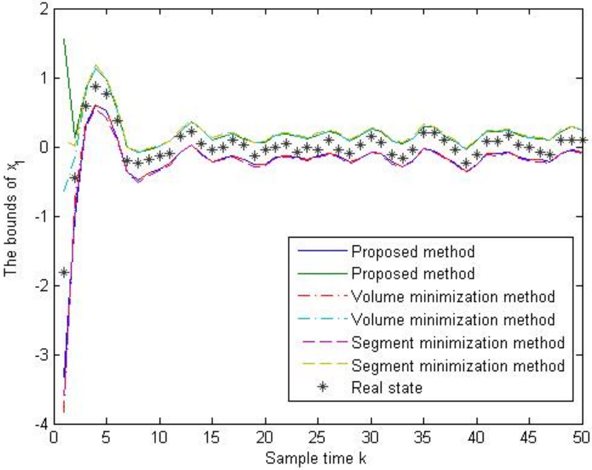


Figure: Guaranteed bound of x_1 obtained by different methods

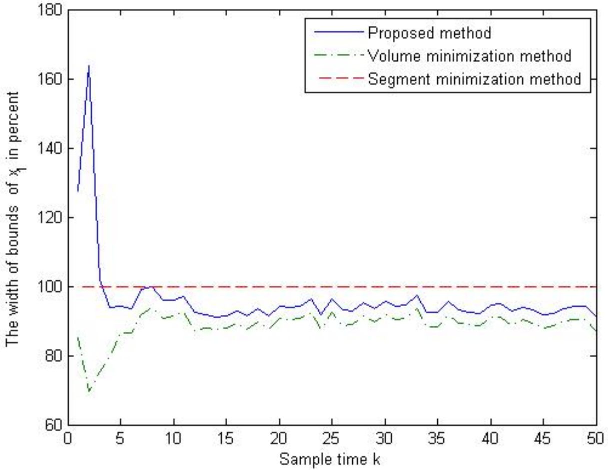


Figure: Comparison of the bound's width of x_1 obtained by different methods

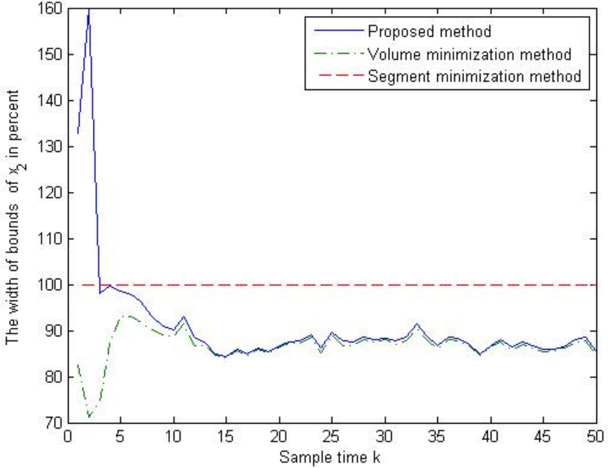


Figure: Comparison of the bound's width of x_2 obtained by different methods

Table: Total computation time after 50 samples

Algorithm	Time(second)
Segment minimization	0.0312
Presented algorithm (without off-line LMI optimization (14) included)	0.0312
Presented algorithm (with off-line LMI optimization (14) included)	0.9828
Volume minimization	10.3273

- Multi-output system

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+0.3\delta_k \end{bmatrix} x_k + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \omega_k \\ \\ y_k = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} v_k \end{cases} \quad (19)$$

with $|\delta_k| \leq 1$, $\|v_k\|_\infty \leq 1, \|\omega_k\|_\infty \leq 1$.

The initial state belongs to the box $3\mathbf{B}^2$.

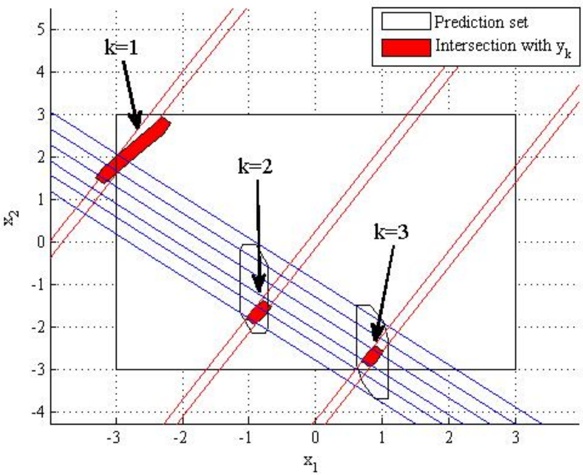


Figure: Evolution of the guaranteed state estimation

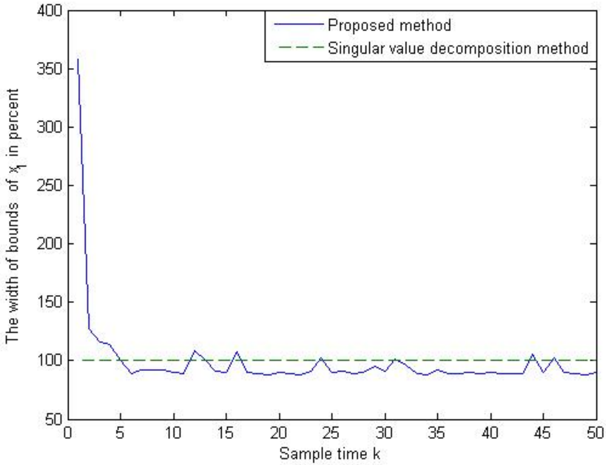


Figure: Bounds of x_1

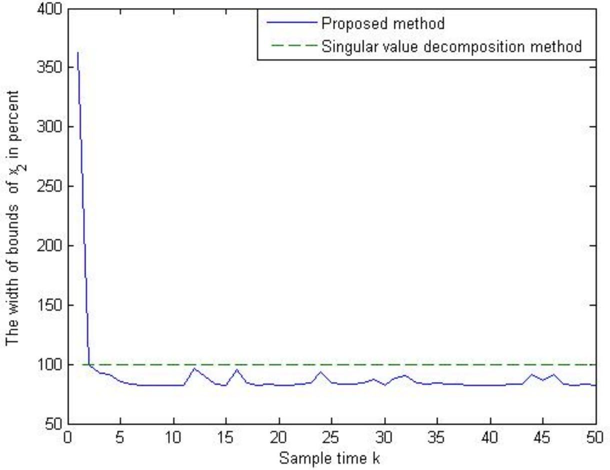


Figure: Bounds of x_2

Table: Total computation time after 50 samples

Algorithm	Time(second)
Presented algorithm (without off-line LMI optimization included)	0.0468
Presented algorithm (with off-line LMI optimization included)	0.2808
Singular value decomposition algorithm [2]	1.5444

[2] C.Combastel (2003) A state bounding observer based on zonotopes

Outline

Conclusion

Conclusion

- The set-membership estimation is recalled.

Conclusion

- The set-membership estimation is recalled.
- A new criterion is used in the correction step in order to obtain good performance and low-complexity.

Conclusion

- The set-membership estimation is recalled.
- A new criterion is used in the correction step in order to obtain good performance and low-complexity.
- A zonotopic approximation of the intersection of a zonotope and a polytope is presented to solve the multi-output case.

Conclusion

- The set-membership estimation is recalled.
- A new criterion is used in the correction step in order to obtain good performance and low-complexity.
- A zonotopic approximation of the intersection of a zonotope and a polytope is presented to solve the multi-output case.
- The simulation results show the advantages of this approach in comparison to existing approaches.

Conclusion

- The set-membership estimation is recalled.
- A new criterion is used in the correction step in order to obtain good performance and low-complexity.
- A zonotopic approximation of the intersection of a zonotope and a polytope is presented to solve the multi-output case.
- The simulation results show the advantages of this approach in comparison to existing approaches.
- Current and further work: use of guaranteed state estimation for control system (Tube MPC) and fault detection.

Thank you!

Questions?