Real-time control allocation using zonotopes

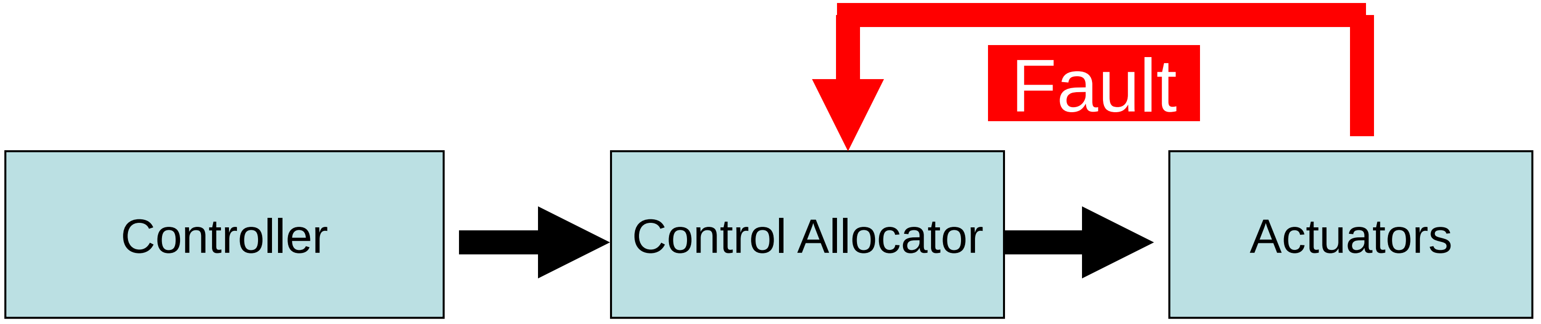
Max Demenkov[1]

[1]Institute of Control Sciences, Russian Academy of Sciences
Moscow, Russia

Part of this research is done in collaboration with
De Montfort University (UK), QinetiQ Ltd (UK) and Central Aerohydrodynamic Inst. (TsAGI, Russia)

What is control allocation ?

- Future mechanical systems are designed to be over-actuated (i.e. they have more actuators than needed for control)
- This is done to add additional control capabilities and guarantee reliability in case of actuator fault
- In order to simplify control and/or avoid control redesign and tuning when faults occur, the main controller produces the desired vector of moments, or angular rates, considered as virtual actuators
- Then a control allocation algorithm commands physical actuators to provide the desired moments or angular rates



The origins of control allocation problem

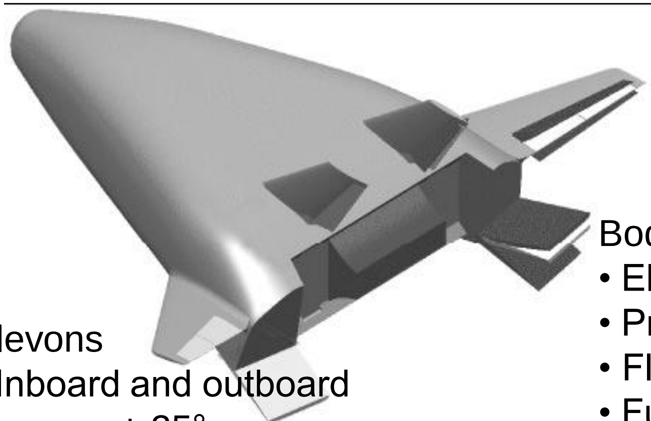
- Started from the paper of Durham (1993) in AIAA Journal of Guidance, Navigation and Control. Followed by extensive research within the aerospace community

- Later, the problem attracted attention in general control journals and meetings, where it is mostly considered together with the system and actuator dynamics

- We consider here only stand-alone problem of control allocator design for 2D/3D case, the same way as it was posed within aerospace community- without consideration of dynamics

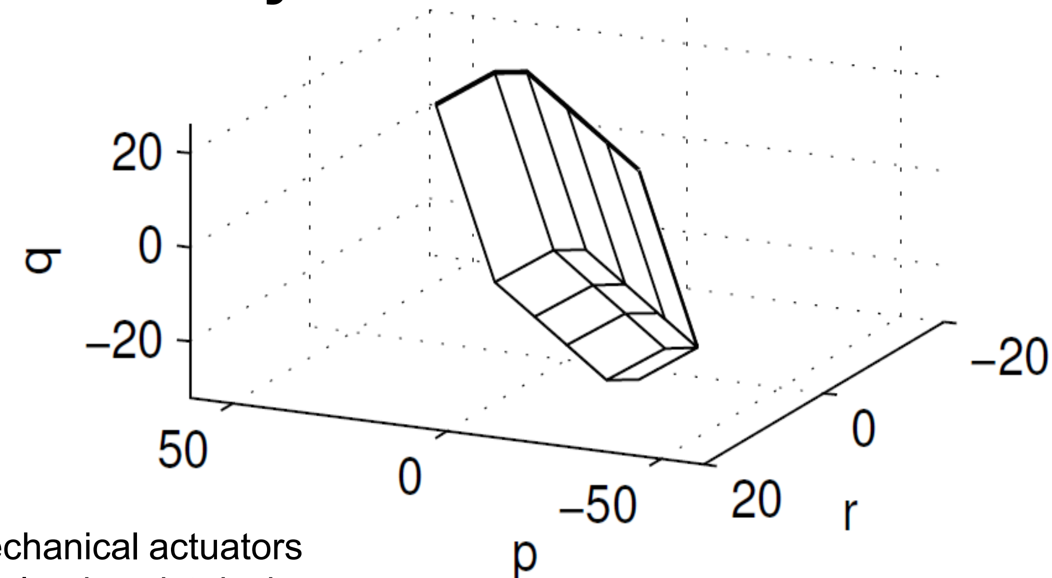
An example: X-33, prototype of unmanned reentry vehicle

Vertical rudders
• Rudders: 60° outboard and
 30° inboard deflection
• Electromechanical actuators
• Function: yaw control and pitch trim bias



Body flaps
• Electromechanical actuators
• Pneumatic load assist device
• Flaps: – 15°, 26°
• Function: pitch control at all speeds, yaw control and entry

Elevons
• Inboard and outboard elevons: ± 25°
• Electromechanical actuators
• Function: pitch control and roll control at all speeds



Attainable angular rates set

The eight control surfaces of X-33 vehicle have control power capable of providing redundant pitch, roll, and yaw restoring moments such that if one surface fails, the potential exists for an alternate control scheme that will maintain control of the vehicle.

An example: Control Moment Gyroscopes (CMG)

Angular momentum vector

h

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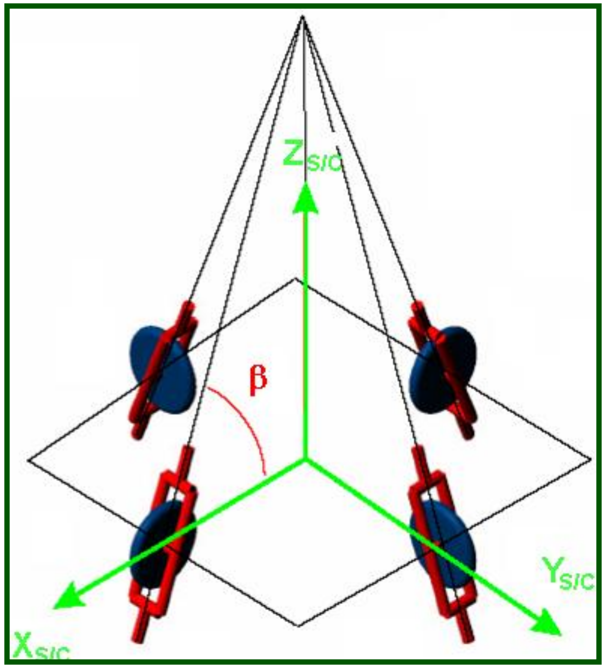
Gimbal angles vector

δ



CMG Cluster

Realized Torque

CMG is useful for attitude control of a space station or satellite: MIR, International Space Station, imaging satellites, etc…

Control allocation solution designed at CNES (Toulouse, France) and implemented on board of high resolution European satellite "Pleyades" (Thieuw and Marcille, 2007):

$$h = F(\delta), \quad \mathrm{J}(\delta) - \text{Jacobian matrix}$$

$$\dot{h} = \mathrm{J}(\delta)\dot{\delta} \;\Rightarrow\; \dot{\boldsymbol{\delta}} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1} h$$

Moore-Penrose Pseudo Inverse: the simplest control allocation solution

$$y = Bu,\; u \in R^m,\; y \in R^n,\; m \gg n \Rightarrow u = B^T (BB^T)^{-1} y$$

- Minimum normed vector; the solution that requires minimum energy
- In case *B* is a Jacobian matrix - excessively large controls near a singular state, where rank of the matrix decreases
- Maximum possible set *Y* of attainable moments/rates cannot be realized due to interval constraints on *u* :

$$u_{\min} \le u \le u_{\max}$$

$$Y \supseteq \{ y : u_{\min} \le B^T (BB^T)^{-1} y \le u_{\max} \}$$

Linear constraint satisfaction problem

$$y=Bu,\ U=\{u,\ u_{min} \le u \le u_{max}\},\ 0 \in U$$

$$y_{des} \rightarrow u_0 \in U, \text{ fixed time + on-line reconfigurable}$$

Can be easily solved by classical LP or multi-parametric LP, but:

1) safety certification problems in some cases (like in civil aviation) - for a large number of runs, you'll find several cases when LP cannot find a solution in reasonable time

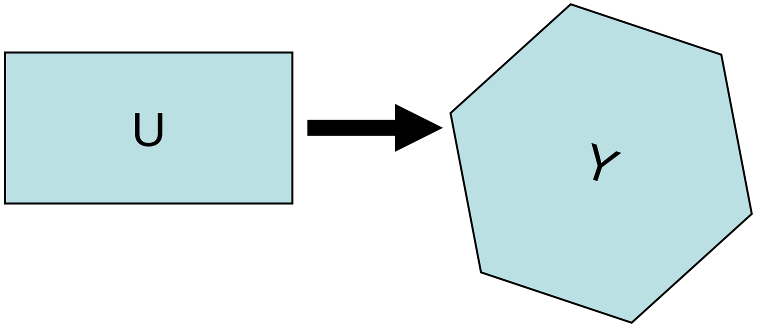
2) Complexity of on-board software implementation – not many aerospace engineers are also experts in optimization

3) Multi-parametric closed-form solution is good from the viewpoint of certification and quite simple in implementation, but lacks in reconfiguration capability (changing **B** requires off-line recomputation of the solution tables)

Geometry of the problem

$$Y=BU,\ U=\{u,\ u_{min} \leq u \leq u_{max}\},\ 0 \in U$$

Y is a zonotope – an image of a cube U under affine projection B



Zonotopes have been previously used in:

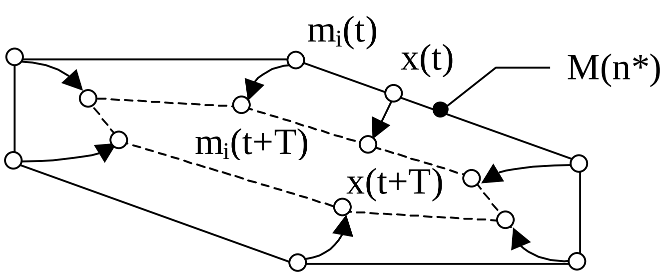
- Affine arithmetics within interval analysis community
- Reachable set estimation (e.g. Antoine Girard, etc…) and state estimation within control community, including people here at SWIM !

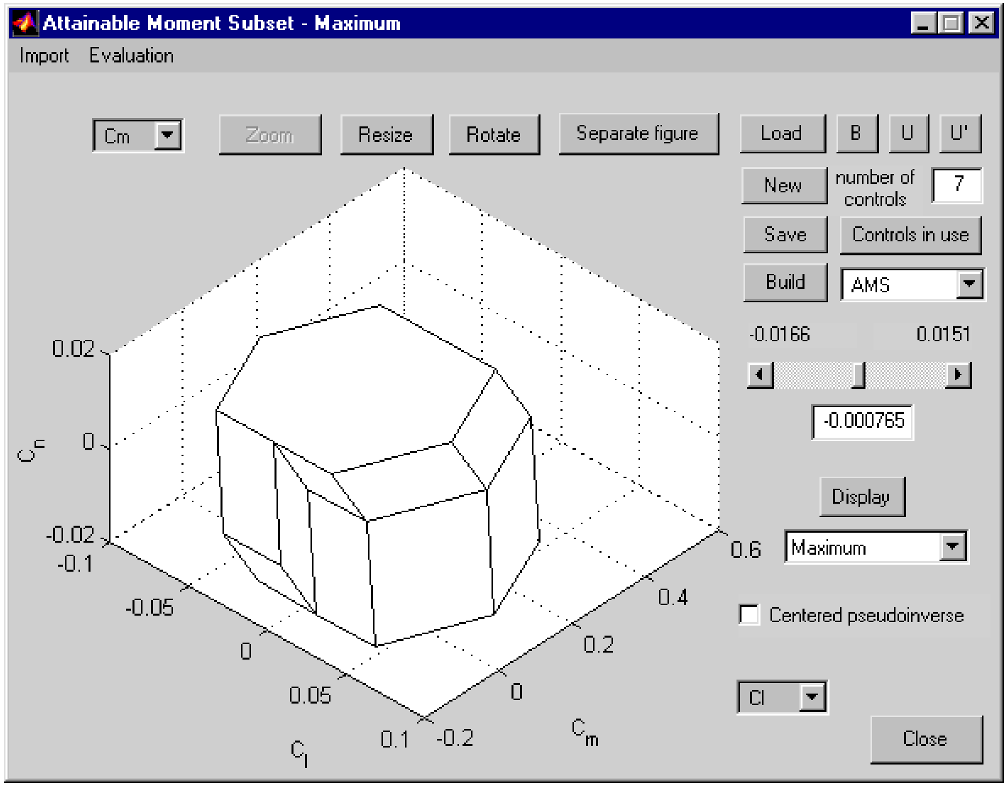
See book of G.M. Ziegler «Lectures on Polytopes» (1995) to study zonotope properties

Attainable Moment/Rate Sets

Can be computed in the same way as controllable/reachable regions for linear discrete systems: convex hull, Fourier-Motzkin elimination or other off-line methods can be used.

$$Y = \sum_{i=1}^{m} b_i [u_{i\ \min}, u_{i\ \max}]$$

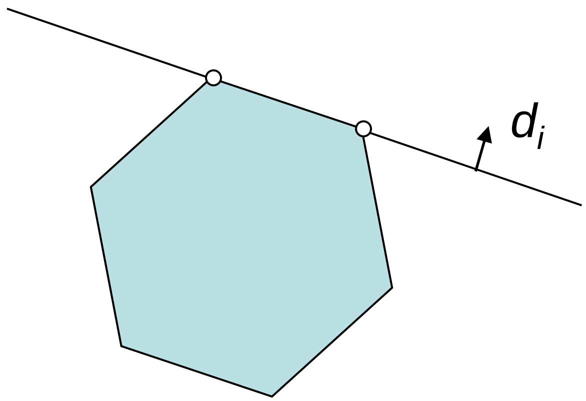




MATLAB GUI for attainable moment sets analysis – developed at De Montfort University for QinetiQ

Real-time construction of zonotope in the half-plane form

Suppose that we want to maximize a normal to a facet over a zonotope:



$$d_i^T Bu \to \max,$$
$$u_{\min} \le u \le u_{\max}$$

$d_i^T B$ is just a linear function → control takes values at the corners of U

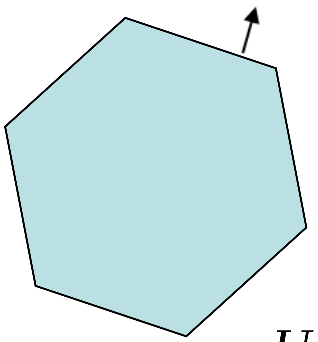
$$d_i^T Bu \to \max \Rightarrow d_i^T Bu = \sum_{j=1}^{m} d_i^T b_j \begin{cases} u_{i\min}, \text{if } d_i^T b_j < 0 \\ u_{i\max}, \text{if } d_i^T b_j > 0 \\ u_{i\min} \text{ or } u_{i\max}, \text{ if } d_i^T b_j = 0 \end{cases}$$

2D(3D) case: to allow more than 1(2) vertices to be a solution of our maximization problem , we need d_i **to be orthogonal to (more than) one column of** *B* **matrix ⇒ Any facet normal is a cross product of two columns of *B in 3D case and orthogonal to a column* in 2D case !**

Real-time construction of zonotope in the half-plane form

$$B=[b_1,\ldots,b_m]$$

$$\text{3D case: } d_k=b_i\times b_j=\begin{bmatrix} b_i(2)b_j(3)-b_i(3)b_j(2) \\ b_i(3)b_j(1)-b_i(1)b_j(3) \\ b_i(1)b_j(2)-b_i(2)b_j(1) \end{bmatrix},\ k=\overline{1,M},\quad M=\frac{m!}{2(m-2)!}$$



$$\text{2D case: } d_k=\begin{bmatrix} b_k(2) \\ -b_k(1) \end{bmatrix},\ k=\overline{1,m}.$$

$$U=\{u:\ |u|\le u_{\max}\}\Rightarrow \max_{u\in U}(d_k{}^T Bu)=\sum_{i=1}^{m} d_k{}^T b_i \operatorname{sign}(d_k{}^T b_i)\,u_{\max}$$

System of linear inequalities:

$$Y=\{y: d_k{}^T y\le \max_{u\in U}(d_k{}^T Bu),\ k=1,\ldots,M\}$$

Normalization:

$$d_k=d_k./\max_{u\in U}(d_k{}^T Bu)\Rightarrow Y=\{y: d_k{}^T y\le 1,\ k=1,\ldots,M\}$$

Generalized zonotope construction in the form of linear inequalities (post-conference slide)

How to build half-plane representation of a zonotope in *n*-dimensional case ?
Basically, in the same way, but d_i is orthogonal now to *n-1* columns of *B*…

The algorithm has been proposed (independently) in:

M. Demenkov. (2007). Interval bisection method for control allocation. In *Proceedings of the 17th IFAC Symposium on Automatic Control in Aerospace, Toulouse, France.* (also in PhD thesis of M. Demenkov, same year)

M. Althoff, O. Stursberg, and M. Buss. (2010). Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes. Nonlinear Analysis: Hybrid Systems, 4:233–249. (also in Dr.-Ing. thesis of M. Althoff, same year)

M. Gouttefarde and S. Krut. (2010). Characterization of parallel manipulator available wrench set facets. Advances in Robot Kinematics: Motion in man and machine, Part 7, 475-482.

Each facet of a zonotope is constructed totally separately from the others, leading to easy parallelization on multi-core and GPU processors. Note that with increasing of dimensionality, the number of zonotope facets is growing very rapidly…

Thanks to SWIM participants ***Vu Tuan Hieu Le*** *and* ***Xin Chen !***

Two ways of determining control vector

Problem: $y \rightarrow u \in U$

- **Convex cone method**

See Demenkov M. (2011). Reconfigurable direct control allocation for overactuated systems. In *Proceedings of 18th IFAC World Congress, Milan, Italy.*

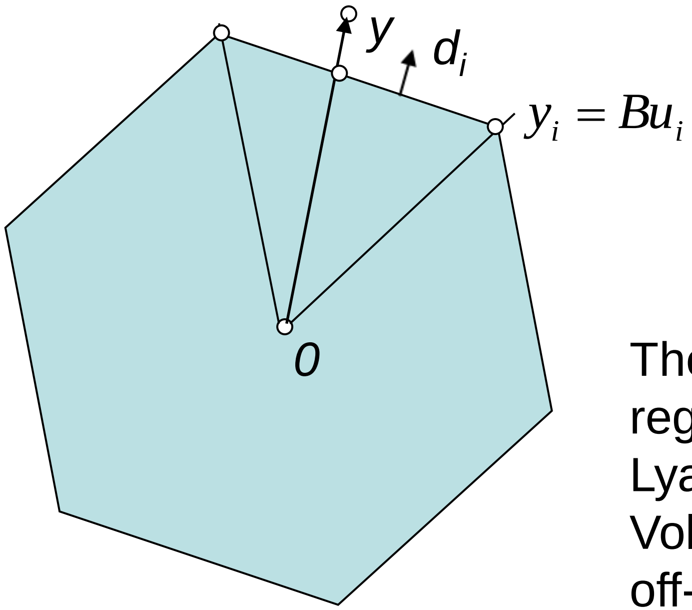
- **Bisection method**

See Demenkov M. (2007). Interval bisection method for control allocation. In *Proceedings of the 17th IFAC Symposium on Automatic Control in Aerospace, Toulouse, France.*

There are, of course, much more possible ways…. but, here I'm talking about my own ones…

Real-time determination of control: convex cone method

1) To determine in which cone we are (i.e. in which **y** lies), multiply **y** to a matrix of facet normals and then take maximum. The cone for which the maximum is achieved, is the one where **y** lies:



$$k = \arg \max_i d_i^T y \ , \ i = \overline{1, M}$$

$$y \notin Y \Rightarrow y = y./d_k^T y$$

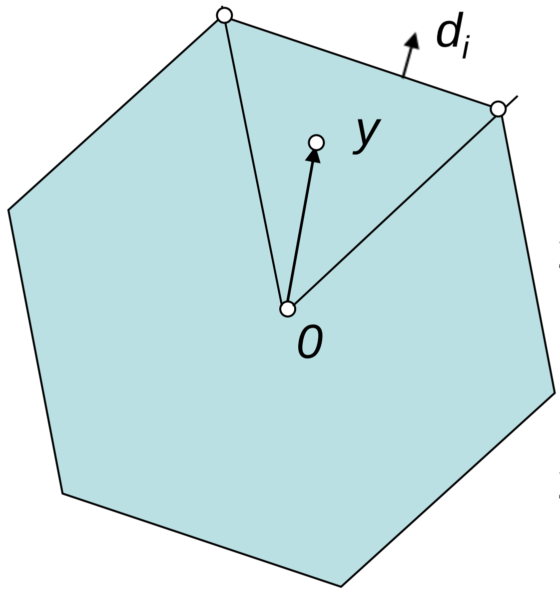
The procedure was proposed in the context of regulator design using piecewise-linear Lyapunov functions in Blanchini (Automatica, Vol. 31, 1995) - the regulator is designed off-line and is linear in every cone

An algorithm for splitting ***Y*** into convex cones is described e.g. in:

Demenkov M.(2007). *Geometric algorithms for input constrained systems with application to flight control*. Ph.D. thesis, De Montfort University, UK, 2007.

Real-time determination of control: convex cone method

1) To determine in which cone we are (i.e. in which **y** lies), multiply **y** to a matrix of facet normals and then take maximum. The cone for which the maximum is achieved, is the one where **y** lies:



$$k = \arg\max_i \, d_i^T y \,, \quad i = \overline{1, N}$$

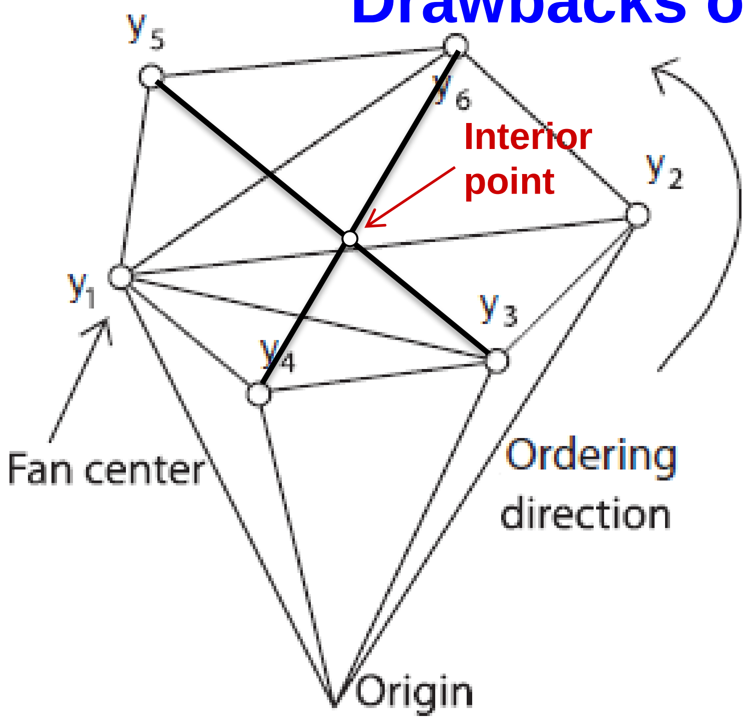
2) Find all vertices $y_i = Bu_i$ attributed to *k*-th cone (can be derived from optimization equations considered previously)

2) Find all possible combinations of such vertices. For each combination, solve a system of linear equalities to determine weights λ

3) Find ***u*** as the linear combination of vertices:

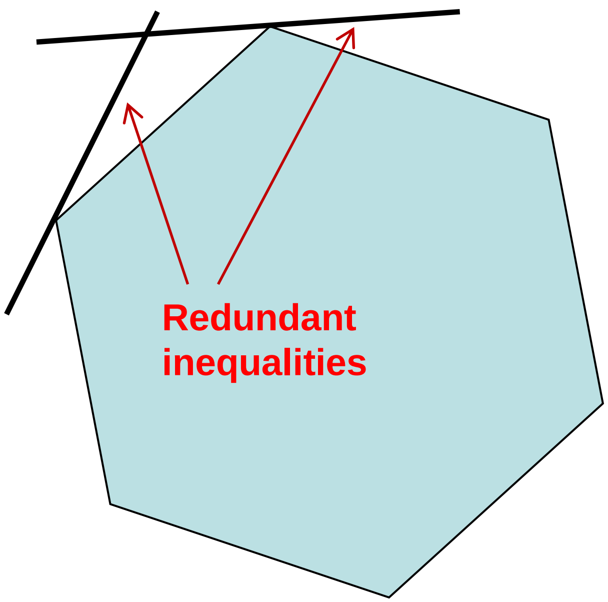
$$\lambda_1 y_1 + \lambda_2 y_2 \, (+\lambda_3 y_3) = y \quad \Rightarrow \quad u = \lambda_1 u_1 + \lambda_2 u_2 \, (+\lambda_3 u_3)$$

Drawbacks of convex cone method



- "Interior points" – it is possible to delete them and "order" corner points of a facet, but in this case algorithm depends on tolerances !

- Redundant inequalities- they can be removed, but in this case, as well as with interior points, additional processing is needed



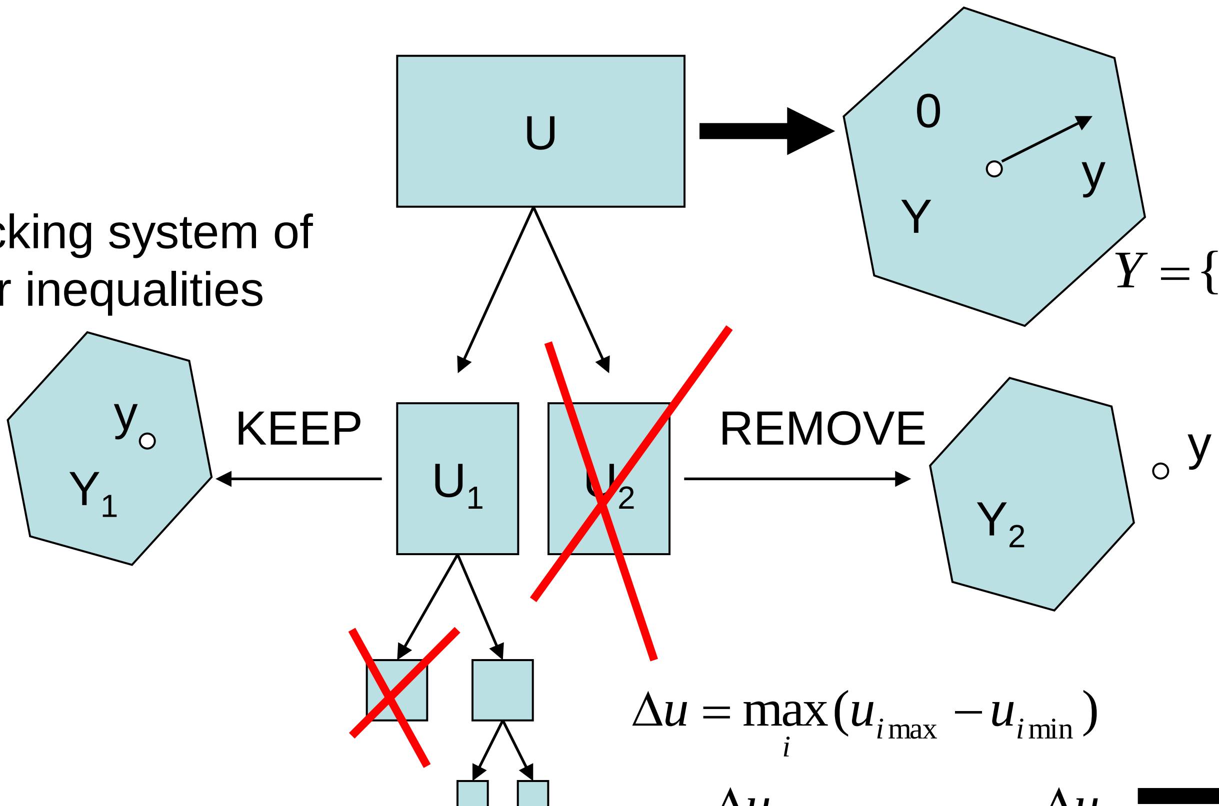
- Each zonotope cone consists of several subcones which can intersect each other

- It is not clear how to pick up solution if it belongs to several subcones – for example one can take point in U which is nearest to the previously computed

No possibility of optimization !

Bisection algorithm

Instead of dividing the zonotope into cones, divide the control hypercube into smaller ones and determine whether the given vector ***y*** belongs to their images or not



Checking system of linear inequalities

$$Y = \{y : d_i^T y \le 1,\ i = 1,..,M\}$$

Exact description of the cube image – no overestimation problem !

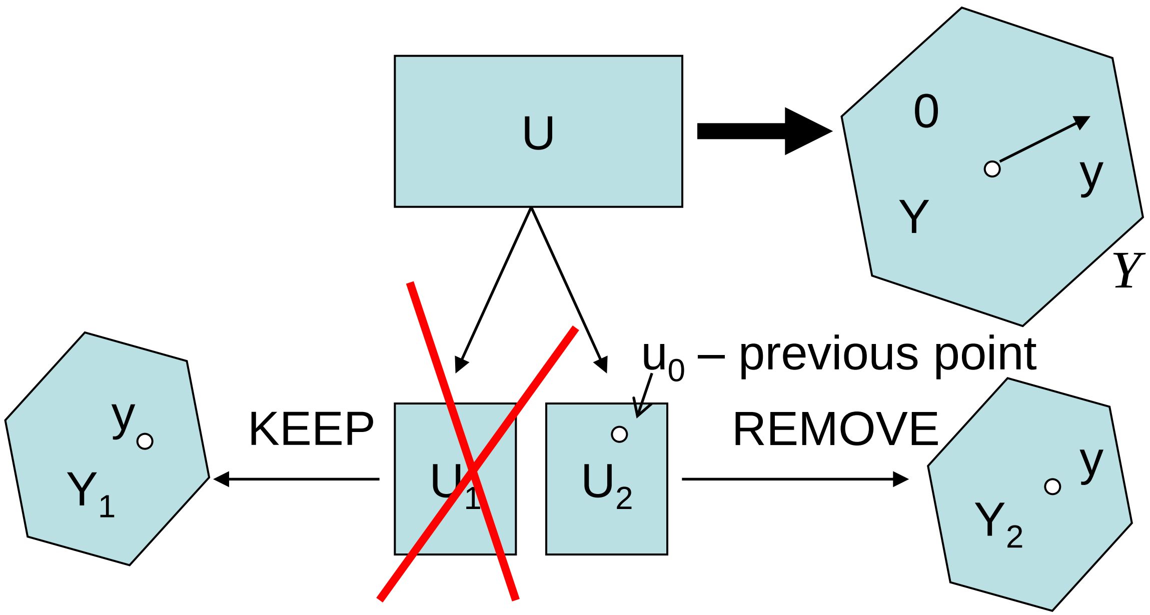
$$\Delta u = \max_i (u_{i\max} - u_{i\min})$$

$$\varepsilon \ge \frac{\Delta u}{2^N} \Rightarrow N \ge \log_2 \frac{\Delta u}{\varepsilon}$$

Tolerance ε defines time of computation

Bisection algorithm: additional criterion

Suppose images of both boxes contain y – which one we have to discard ?



$$Y = \{ y : d_i^T y \leq 1,\ i = 1, .., M \}$$

Checking if previously computed control vector is inside

An example: back to CMG Control Scheme

Torque Commands

$\dot{h}$

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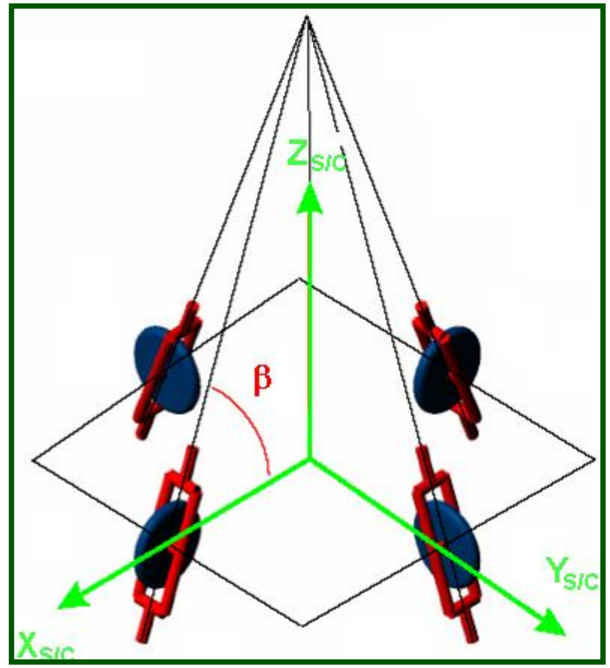
Rate Command to each CMG

$\dot{\delta}$



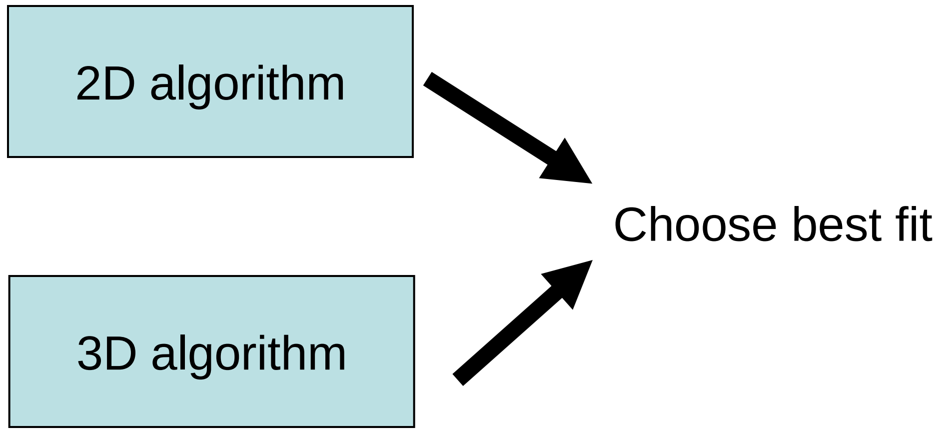
CMG Cluster

Realized Torque

$$h = F(\delta),\quad \mathrm{J}(\delta) - \text{Jacobian matrix}$$

$$\dot{h} = \mathrm{J}(\delta)\dot{\delta}\ ,\quad |\dot{\delta}| \le \dot{\delta}_{\max}$$

What if Jacobian rank < 3 ?



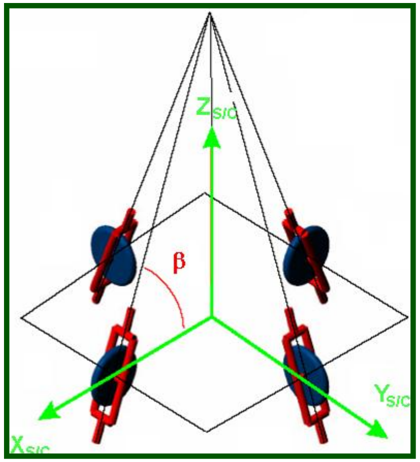
In case of degeneracy, cross product of any two Jacobian columns gives us normal to a plane

The desired vector of momentum rates can be projected onto that plane and the problem can be solved in that plane in 2D

Numerical (2D) example

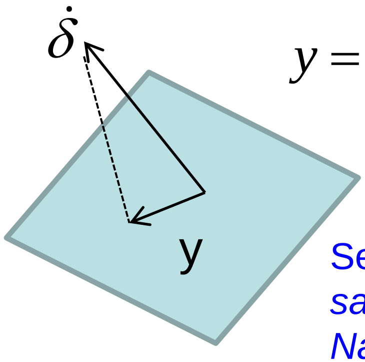
$$J(\delta)=\begin{bmatrix} -\cos(\beta)\cos(\delta_1) & \sin(\delta_2) & \cos(\beta)\cos(\delta_3) & -\sin(\delta_4) \\ -\sin(\delta_1) & -\cos(\beta)\cos(\delta_2) & \sin(\delta_3) & \cos(\beta)\cos(\delta_4) \\ \sin(\beta)\cos(\delta_1) & \sin(\beta)\cos(\delta_2) & \sin(\beta)\cos(\delta_3) & \sin(\beta)\cos(\delta_4) \end{bmatrix}$$

$$\delta=[90^o \quad 0 \quad -90^o \quad 0] \Rightarrow J(\delta)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -0.6 & -1 & 0.6 \\ 0 & 0.8 & 0 & 0.8 \end{bmatrix}$$



Pyramid type CMG cluster with skew angle β=53.13

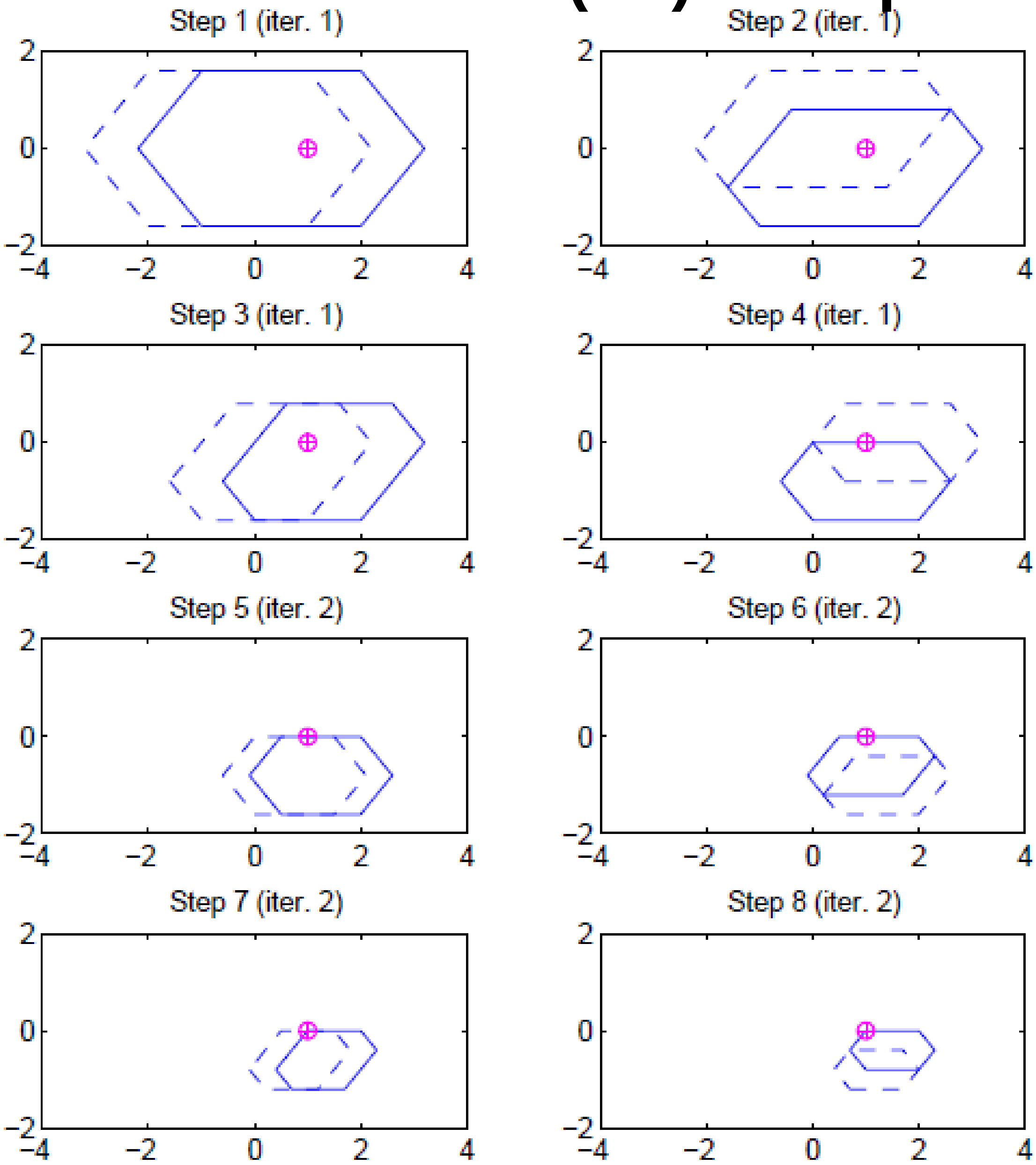
Projection onto 2D plane:



$$y=[1 \quad 0]^T,\ B=\begin{bmatrix} -1 & -0.6 & -1 & 0.6 \\ 0 & 0.8 & 0 & 0.8 \end{bmatrix}$$

See Demenkov M., Kryuchenkov E. (2009). Bisection method for CMG steering logic in satellite attitude control. Proceedings of IFAC Workshop in Aerospace Guidance, Navigation and Flight Control Systems, Samara, Russia.

Numerical (2D) example: bisection method



By solid lines we depict the attainable set ***Y*** for the box that has been chosen by the algorithm at this iteration, while dashed lines represent the one for the box that has been deleted

The cross inside the circle represents the commanded vector ***y***

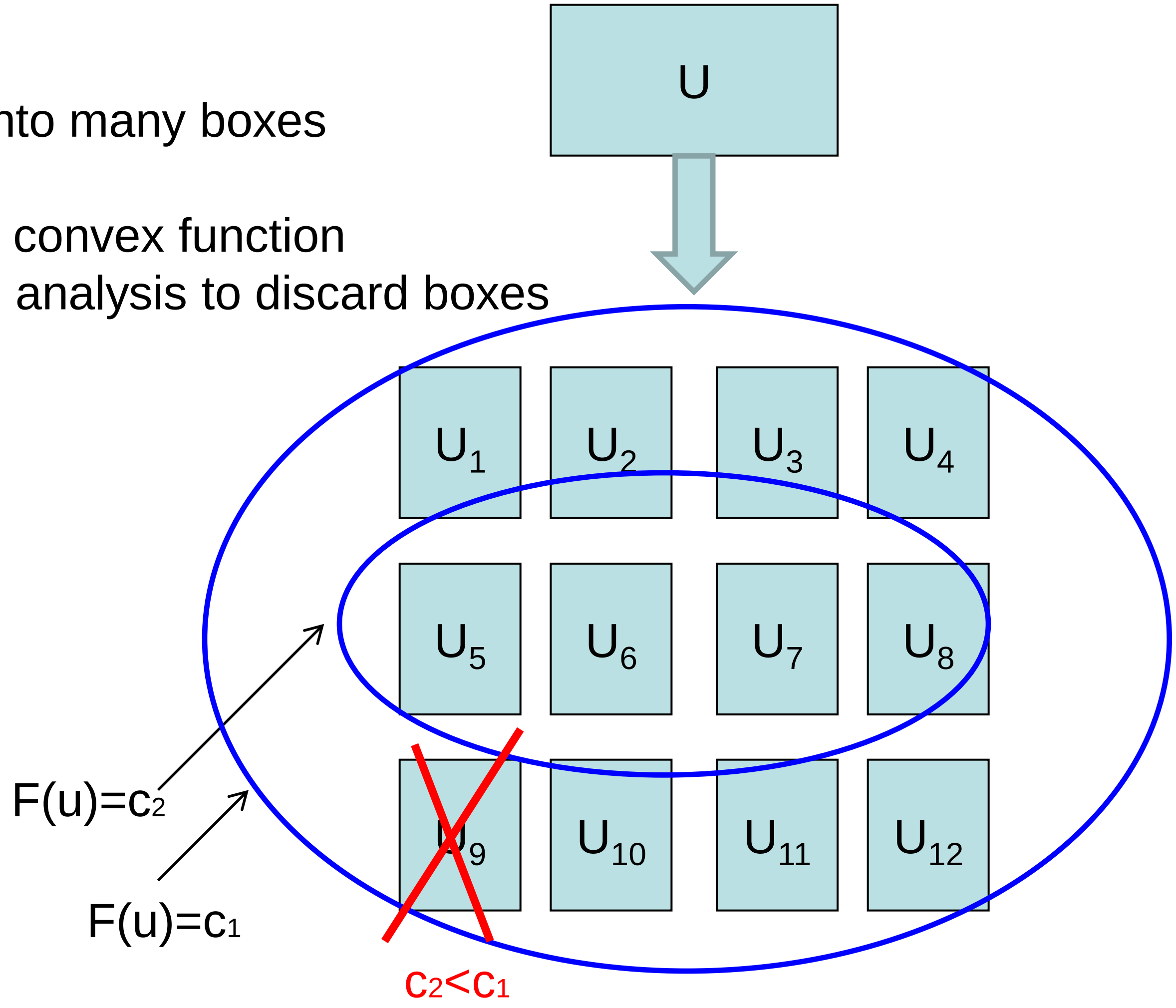
The solution after 32 bisections:

$$\begin{bmatrix} -0.5078 \\ -0.0078 \\ -0.5078 \\ -0.0078 \end{bmatrix} \leq \dot{\delta} \leq \begin{bmatrix} -0.5 \\ 0 \\ -0.5 \\ 0 \end{bmatrix}$$

Current research: optimization

• Divide U into many boxes

• Use some convex function
and interval analysis to discard boxes



Conclusion

• It is possible to construct 2D/3D zonotopic sets in real time, in the form of linear inequalities and compute control in real-time from the given matrix B and control constraints (Fault Detection and Isolation is required)

• Bisection-based control allocation algorithm is discussed

• It has a guarantee of obtaining the solution for every given desired vector, control effectiveness matrix and the set of control constraints in a finite and known in advance number of steps

• The proposed method possess the property of utilizing the whole attainable set and has relatively low algorithmic complexity

Further research is connected with adding optimization capabilities and higher-dimensional extension

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