Using Taylor Models in the Reachability Analysis of Non-linear Hybrid Systems

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SWIM 2012

Outline

1. Verification of hybrid systems
2. Taylor model method
3. Intersections of Taylor models and guards
4. Experimental results
5. Future work

Outline

Motivations

Given a hybrid system on which we would like to verifiy

1. **Reachability** – A given state set is reachable.
2. **Safety** – No unsafe state is reachable.
3. **Liveness** – Some good properties *keep happening*.
4. **...**

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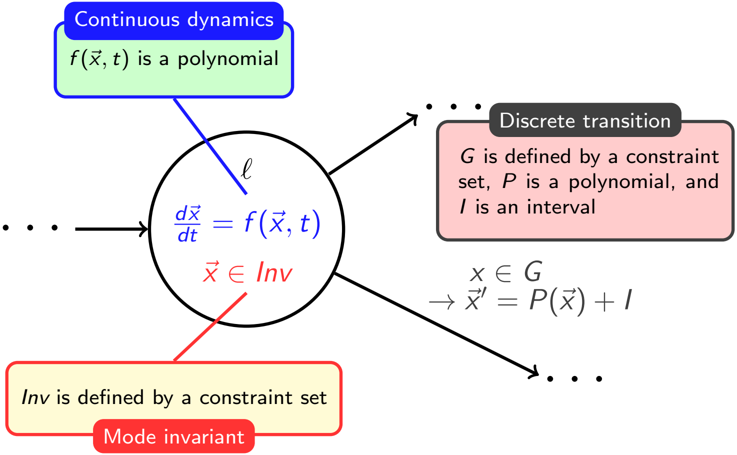
Many important properties can be reduced to reachability.

Hybrid automata



Reachability computation by flowpipe construction

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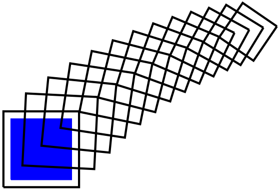

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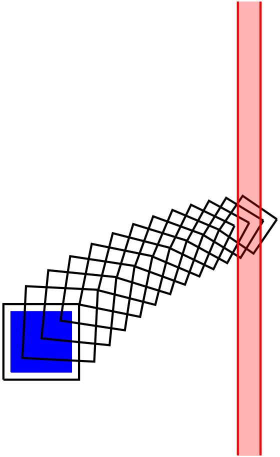

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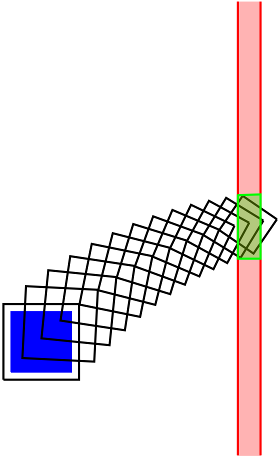

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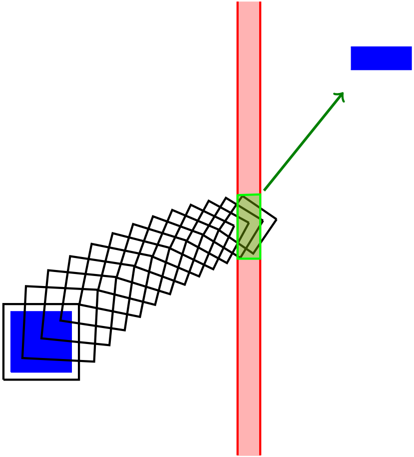

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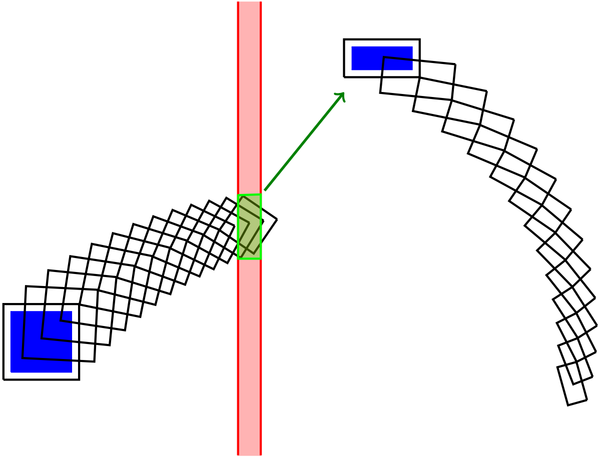

Difficulties

Continuous part:

1. Overestimation is *heavily* accumulated.
2. Global accuracy improvement is inefficient.

Discrete part:

1. Flowpipe/guard and flowpipe/invariant intersections are difficult to compute.
2. Reducing the complexity of intersections is also hard.

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We try to find new trade-off between accuracy and efficiency.

Outline

High order approximations

Polynomial p is a *k-order approximation* of a smooth function $f : D \to \mathbb{R}$ iff

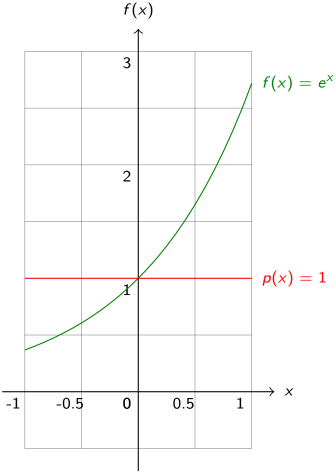
(a) $f \in C^k$

(b) $f(\vec{c}) = p(\vec{c})$ for the center point $\vec{c}$ of D and for each $0 < m \leq k$:

$$\left.\frac{\partial^m f}{\partial \vec{x}^m}\right|_{\vec{x}=\vec{c}} = \left.\frac{\partial^m p}{\partial \vec{x}^m}\right|_{\vec{x}=\vec{c}} .$$

Examples

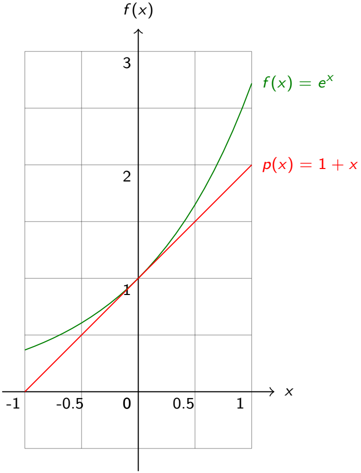
Several high order approximations of $f(x) = e^x$ with $x \in [-1, 1]$.



0-order approximation

Examples

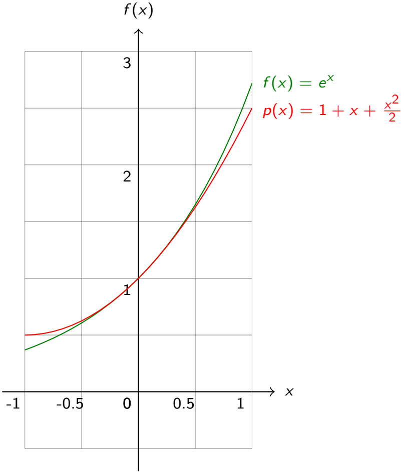
Several high order approximations of $f(x) = e^x$ with $x \in [-1, 1]$.



1-order approximation

Examples

Several high order approximations of $f(x) = e^x$ with $x \in [-1, 1]$.



2-order approximation

Taylor models

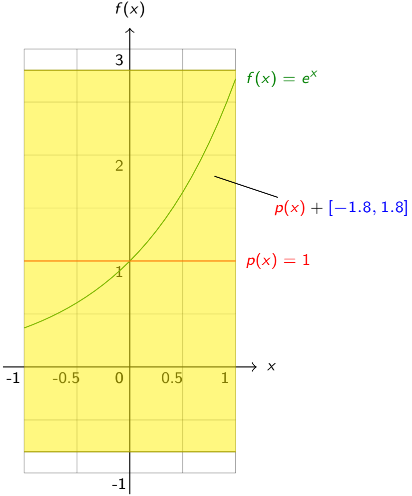
- Introduced by Berz and Makino.
- Taylor model: a pair (p, I) over an *interval* domain D. It defines the set

$$p + I \;=\; \{\vec{x} = p(\vec{x}_0) + \vec{y} \mid \vec{x}_0 \in D \wedge \vec{y} \in I\}$$

- Closed under many basic operations.

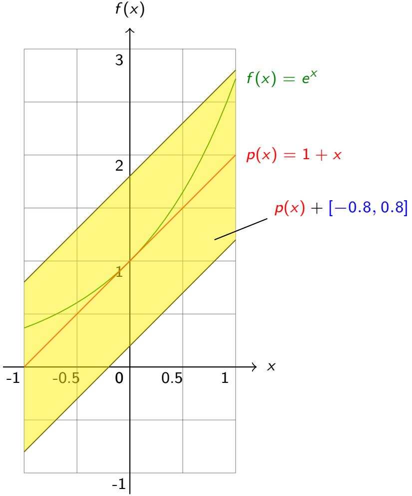
Berz. **Modern Map Methods in Particle Beam Physics.** ser. Advances in Imaging and Electron Physics. Academic Press, 1999, vol. 108.

High order over-approximations by Taylor models



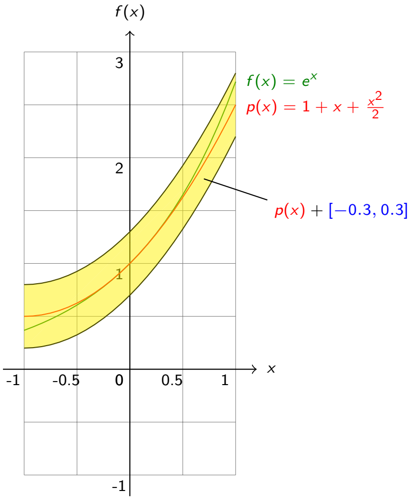
0-order over-approximation

High order over-approximations by Taylor models



1-order over-approximation

High order over-approximations by Taylor models



2-order over-approximation

Taylor model arithmetic

Addition:

$$(p_1, I_1) + (p_2, I_2) = (p_1 + p_2, I_1 \oplus I_2)$$

Multiplication:

$$(p_1, I_1) * (p_2, I_2) = (p_1 * p_2, (\mathtt{B}(p_1) \otimes I_2) \oplus (I_1 \otimes \mathtt{B}(p_2)) \oplus (I_1 \otimes I_2))$$

Truncation:

$$\mathtt{Trunc}_k((p_n, I)) = (p_k, I \oplus \mathtt{B}(p_n - p_k))$$

More operations: anti-derivation, Lie derivation, ...

Makino and Berz. **Taylor models and other validated functional inclusion methods.** *J. Pure and Applied Mathematics*, vol. 4, no. 4, 2003.

Verified integration by Taylor models

Assume the ODE is $\frac{d\vec{x}}{dt} = f(\vec{x}, t)$ and the initial set is given by a Taylor model X_0 over domain D_0. In the i^{th} integration step,

Berz and Makino. **Verified integration of ODEs and flows using differential algebraic methods on high-order Taylor models.** *Reliable Computing*, vol. 4, 1998.
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1. Compute a k-order approximation $p_k(\vec{x}_0, t)$ over $\vec{x}_0 \in X_{i-1}, t \in [0, t_i - t_{i-1}]$ for the flow starting from the Taylor model X_{i-1}.

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2. Evaluate a proper remainder I_k such that $(p_k(\vec{x}_0, t), I_k)$ is an over-approximation of the flow in $[0, t_i - t_{i-1}]$.

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3. Compute the i^{th} flowpipe $X_{[t_{i-1}, t_i]} = (p_k(X_{i-1}, t), I_k)$, and $X_i = (p_k(X_{i-1}, t_i - t_{i-1}), I_k)$.

Berz and Makino. **Verified integration of ODEs and flows using differential algebraic methods on high-order Taylor models.** *Reliable Computing*, vol. 4, 1998.
Berz and Makino. **Rigorous integration of flows and ODEs using Taylor models.** *In Proc. of SNC'09.*

Performance

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Complexity: If p is a polynomial with n variables and k degree, then it could have $\binom{n+k}{k}$ monomials in the worst case.

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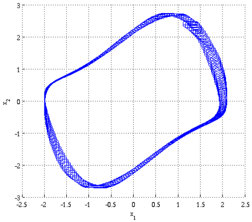
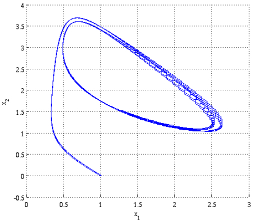
Accuracy:

- The initial set in every step is represented by a *high order model*.
- By Taylor model arithmetic, *overestimation is effectively restricted*.
- Auxiliary methods can be used to further improve the accuracy, such as shrink wrapping, preconditioning, ...

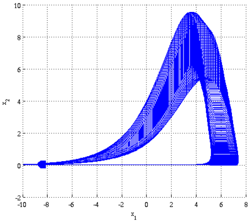
Continuous systems:



Van-der-Pol oscillator

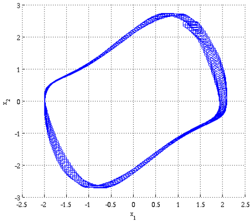


Brusselator



Rössler attractor

Continuous systems:



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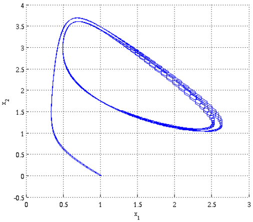

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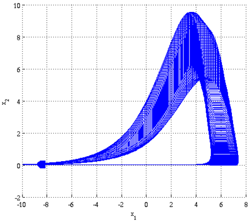

Rössler attractor

How can we apply Taylor models to hybrid systems?

Outline

Taylor model flowpipe construction for hybrid systems

- In a mode:
  - Verified integration by using Taylor models.
  - Compute flowpipe/invariant intersections.
- For a discrete transition:
  - Compute flowpipe/guard intersections.
  - Compute the image of the reset mapping.

Taylor model flowpipe construction for hybrid systems

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- For a discrete transition:
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  - Compute the image of the reset mapping.

Over-approximate an intersection

Intersection of a Taylor model (p, I) and a guard:

$$\underbrace{\vec{x} = p(\vec{x}_0, t) + \vec{y} \ \wedge \ \underbrace{\vec{x}_0 \in D_0 \wedge t \in [0, \Delta]}_{\text{Taylor model domain}} \ \wedge \ \vec{y} \in I}_{\text{Taylor model}} \ \wedge \ \underbrace{\gamma(\vec{x})}_{\text{guard predicate}}$$

Over-approximate an intersection

We propose the following techniques to over-approximate the intersection $(p, I) \cap G$.

- **Domain contraction** -
  Contract the domain of p such that (p, I) with the new domain is the over-approximation.
- **Range over-approximation** -
  Over-approximate the Taylor model (p, I) by a convex set S, then compute $S \cap G$.
- **Template method** -
  Compute a Taylor model over-approximation $(p^*, 0)$ based on a given *template*.

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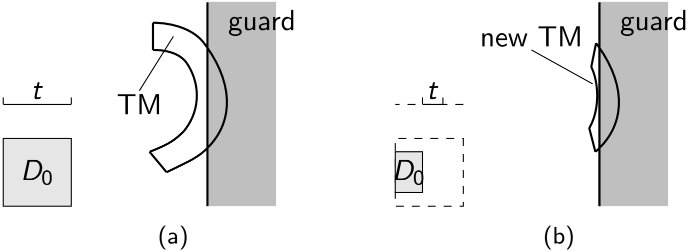
They can be used in a combination.

Domain contraction - An example



Domain contraction - Main procedure

Compute an *interval* contraction by *interval constraint propagation*:

1. Contract the interval in one dimension.
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Accuracy can be further improved by

- Perform the contraction through all dimensions for several times.
- Determine an order on the dimensions and perform the contraction for one time.

Domain contraction - Contraction in one dimension

Compute the lower bound for the contracted domain in the i^{th} dimension.

while the size of $[lo, up]$ is larger than ϵ **do**
  Split $[lo, up]$ into $[lo, a]$ and $[a, up]$ wherein $a = \frac{lo+up}{2}$;
  if $(p(\vec{y}), I)$ with $\vec{y} \in D$ and $(\vec{y})_i \in [lo, a]$ intersects the guard **then**
    $up \leftarrow a$;
  else
    $lo \leftarrow a$;
  end if
end while
return lo;

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return lo;

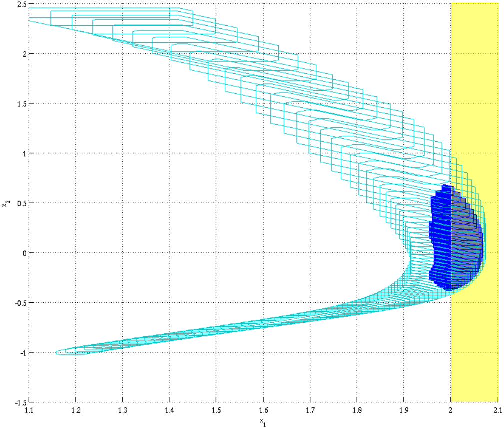
Domain contraction - Emptiness checking

Check the emptiness of

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{x} = p(\vec{y}) + \vec{z} \ \wedge \ \vec{y} \in D \ \wedge \ \vec{z} \in I \ \wedge \ \gamma(\vec{x})\}$$

Approximation methods: *Conservative simplification* on p and γ.

Domain contraction - An example



Domain contraction

Advantages:

- Polynomial computation time.
- The result is still a Taylor model.

Disadvantages:

- The contracted domain is always an interval.
- Large overestimation might be introduced when either p or γ is of high degree.

Range over-approximation

1. Over-approximate (p, I) by a set S which can be a
   - **support function** -
     Conservative over-approximation of a polynomial.
   - **zonotope** -
     Zonotopes are order 1 *Taylor models and vise versa.*
2. Over-approximate $S \cap G$ by a Taylor model.

Sankaranarayanan, Dang and Ivancic. **Symbolic Model Checking of Hybrid Systems Using Template Polyhedra.** In *Proc. of TACAS'08.*
Le Guernic and Girard. **Reachability Analysis of Hybrid Systems Using Support Functions.** In *Proc. of CAV'09.*

Range over-approximation

Advantages:

- Available algorithms can be used.
- The orientation of the over-approximation can be chosen.

Disadvantages:

- Multi-time over-approximation.
- Best orientation is not easy to find.

Template method

We call $p^*(\vec{x}_0)$ a *template polynomial* if its domain is given by

$$\vec{x}_0 \in D_u = [\ell_1, u_1] \times [\ell_2, u_2] \times \cdots \times [\ell_n, u_n]$$

wherein $\ell_1, \ldots, \ell_n, u_1, \ldots, u_n$ are unknown parameters.

Sankaranarayanan, Sipma and Manna. **Constructing invariants for hybrid systems.** *Formal Methods in System Design.* 2008.

Gulwani and Tiwari. **Constraint-Based Approach for Analysis of Hybrid Systems.** In *Proc. of CAV'08.*

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We find the parameters such that the Taylor model $(p^*, 0)$ contains the intersection $(p, I) \cap G$:

$$\forall \vec{x}.((\vec{x} = p(\vec{y}) + \vec{z} \wedge \vec{y} \in D \wedge \vec{z} \in I \wedge \vec{x} \in G)$$
$$\rightarrow \exists \vec{x}_0.(\vec{x} = p^*(\vec{x}_0) \wedge \vec{x}_0 \in D_u))$$

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We may also add more constraints to limit the overestimation.

Sankaranarayanan, Sipma and Manna. **Constructing invariants for hybrid systems.** *Formal Methods in System Design.* 2008.
Gulwani and Tiwari. **Constraint-Based Approach for Analysis of Hybrid Systems.** In *Proc. of CAV'08.*

Template method

Advantages:

- Flexibility.
- The unknown parameters can be found by SMT solving.

Disadvantages:

- Bad scalability because of the quantifier elimination.
- Best template is not easy to find.

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5. Future work

Glycemic Control in Diabetic Patients

$$\begin{cases} \frac{dG}{dt} & = & -p_1 G - X(G + G_B) + g(t) \\ \frac{dX}{dt} & = & -p_2 X + p_3 I \\ \frac{dI}{dt} & = & -n(I + I_b) + \frac{1}{V_I} i(t) \end{cases}$$

Typical parameters:

$$\begin{array}{llll} p_1 = 0.01, & p_2 = 0.025, & p_3 = 1.3 \times 10^{-5}, & V_I = 12, \\ n = 0.093, & G_B = 4.5, & I_b = 15. & \end{array}$$

G	plasma glucose concentration above the basal value G_B
I	plasma insulin concentration above the basal value I_B
X	insulin concentration in an interstitial chamber

Glycemic Control in Diabetic Patients

$g(t)$ infusion of glucose into the bloodstream
$i(t)$ infusion of insulin into the bloodstream
The infusion of glucose after a meal:

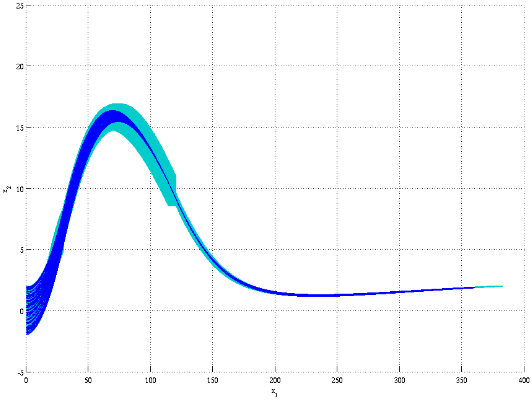
$$g(t) = \begin{cases} \frac{t}{60} & t \le 30 \\ \frac{120-t}{180} & t \in [30, 120] \\ 0 & t \ge 120 \end{cases}$$

Two control schemes:

$$i_1(t) = \begin{cases} \frac{25}{3} & G(t) \le 4 \\ \frac{25}{3}(G(t) - 3) & G(t) \in [4, 8] \\ \frac{125}{3} & G(t) \ge 8 \end{cases}$$

$$i_2(t) = \begin{cases} 1 + \frac{2G(t)}{9} & G(t) < 6 \\ \frac{50}{3} & G(t) \ge 6 \end{cases}$$

Glycemic Control in Diabetic Patients

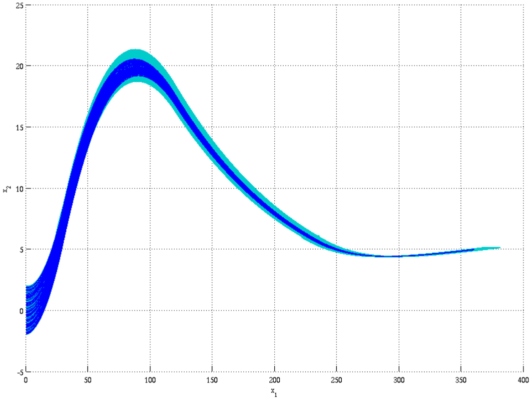


Our platform: CPU i7-860 2.8 GHz Memory: 4 GB

Order of the Taylor models: 9 Time step: 0.02 Time horizon: [0,360]

Total time: 2138 s Time of intersection: 663 s Memory: 430 MB

Glycemic Control in Diabetic Patients



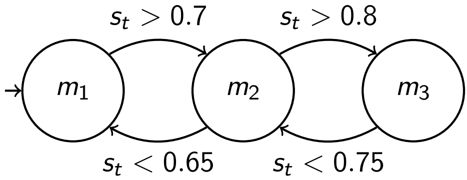
Order of the Taylor models: 9 Time step: 0.02 Time horizon: [0,360]

Total time: 1804 s Time of intersection: 443 s Memory: 410 MB

Vehicle model

The dynamics of a non-holonomic vehicle is given as follows,

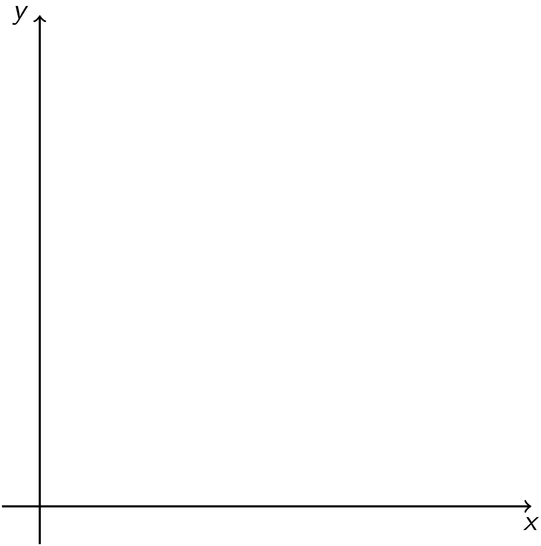
$$
\begin{array}{lll}
\frac{dx}{dt} = vc_t & \frac{dy}{dt} = vs_t & \frac{dv}{dt} = \color{red}{u_1} \\
\frac{dc_t}{dt} = \sigma v^2 s_t & \frac{ds_t}{dt} = -\sigma v^2 c_t & \frac{d\sigma}{dt} = \color{red}{u_2}
\end{array}
$$



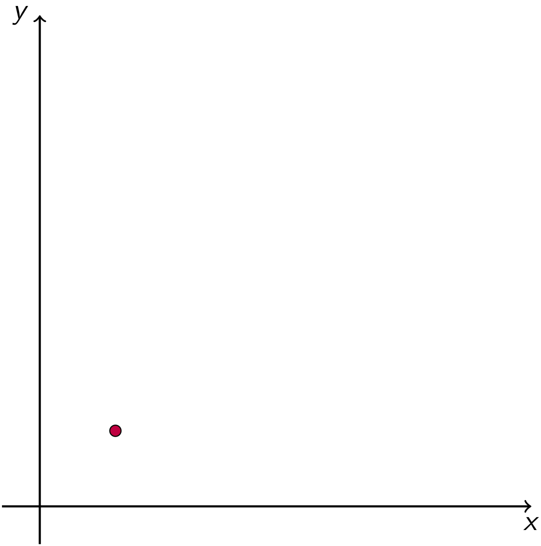
Initial set:

$$
\begin{array}{lll}
x \in [1, 1.2] & y \in [1, 1.2] & v \in [0.8, 0.81] \\
s_t \in [0.7, 0.71] & c_t \in [0.7, 0.71] & \sigma \in [0, 0.05] \,.
\end{array}
$$

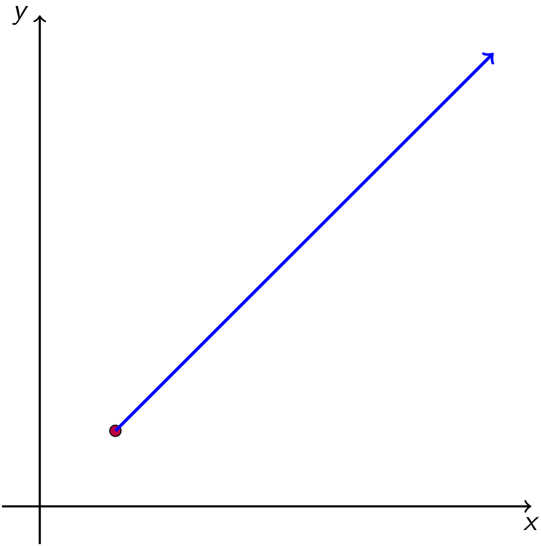
Vehicle model



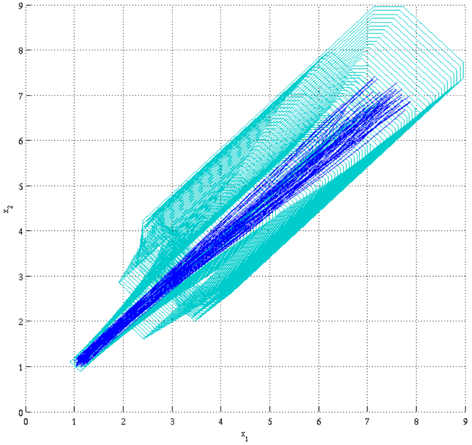
Vehicle model



Vehicle model



Vehicle model



Order of the Taylor models: 9 Time step: 0.1 Time horizon: [0,10]

Total time: 85 s Time of intersection: 40 s Memory: 4 MB

Scalability tests

D = 6							
deg = 2				deg = 4			
size	order	T	mem	size	order	T	mem
3	4	8	4	3	4	18	4
	9	211	12		9	449	12
5	4	13	4	5	4	27	4
	9	298	12		9	587	12
7	4	18	4	7	4	36	4
	9	373	12		9	687	12
D = 8							
deg = 2				deg = 4			
size	order	T	mem	size	order	T	mem
3	4	14	4	3	4	23	4
	9	1389	28		9	2430	28
5	4	18	4	5	4	27	4
	9	1433	28		9	2446	28
7	4	22	4	7	4	28	4
	9	2074	28		9	2474	28
D = 10							
deg = 2				deg = 4			
size	order	T	mem	size	order	T	mem
3	4	16	4	3	4	27	8
	9	t.o.	-		9	t.o.	-
5	4	22	8	5	4	46	8
	9	t.o.	-		9	t.o.	-
7	4	35	8	7	4	62	8
	9	t.o.	-		9	t.o.	-

Dang and Testylier. **Hybridization domain construction using curvature estimation.** In *Proc. of HSCC'11.*

More examples

				Our tool								Ariadne	
Benchmark	DEG	LOC	VAR	δ	T	ORD	T.T.	T.I.	MEM	D.C.	R.M.	T.T	MEM
Brusselator	3	1	2	0.05	[0,10]	4	77	0	4	-	-	34	12
Brusselator	3	1	2	0.03	[0,15]	4	152	0	8	-	-	EXC	-
Watertank	1	4	2	0.1	[0,80]	3	9	5	8	$\surd$	Z	7	24
Van-der-PolE	3	2	3	0.01	[0,6]	4	71	37	4	$\surd$	Z	EXC	-
Lotka-Volterra	2	1	3	0.01	[0,3]	4	14	0	8	-	-	52	8
Hallstah	3	1	2	0.01	[0,7]	3	19	0	≤ 1	-	-	0.6	≤ 1
B.B. no drag	1	1	4	0.02	[0,3]	3	0.8	0.3	≤ 1	$\surd$	-	0.2	≤ 1
B.B. const drag	1	2	4	0.02	[0,3]	3	2	0.8	≤ 1	$\surd$	-	0.6	≤ 1
B.B. Stokes-Einstein	2	2	4	0.02	[0,3]	3	8	2	≤ 1	$\surd$	-	EXC	-
Diabetic [16]	2	9	4	0.02	[0,360]	9	2138	663	430	$\surd$	Z	EXC	-
Diabetic [17]	2	6	4	0.02	[0,360]	9	1804	443	410	$\surd$	Z	EXC	-
Watt governor [39]	4	1	5	1e-4	[0,15]	5	NR	-	-	-	-	EXC	-
Vehicle	4	3	6	0.1	[0,10]	9	85	40	4	$\surd$	S.F.	EXC	-
Angiogenesis [38]	2	1	12	1e-8	[0,2e-6]	4	34	0	12	-	-	EXC	-
Coll-avoid-2 [40]	2	3	12	0.01	[0,10]	3	27	8	3	$\surd$	-	EXC	-

http://systems.cs.colorado.edu/research/cyberphysical/taylormodels/

Outline

Future work

- Taylor model flowpipe construction with varying orders and time steps.
- Split Taylor models according to equilibriums.
- Taylor model contraction for domains.
- Heuristics for choosing templates.

Thank you!

Questions?