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Existence Tests for uncertain functions with parameters

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1 Introduction

2 Existence Tests

3 Testcase

4 Discussion
A mobile robot is moving on an horizontal plane:

Figure: Redermor underwater robot.

Detecting Loops is an important topic in SLAM!

Figure: Tube enclosing the trajectory $p$ of the robot.
Algorithm LOOP [ADJ] explore the $t$-plane to find inner/outer approximation of the set:

$$\mathbb{T}^* = \left\{ (t_1, t_2) \in [0, t_{\text{max}}]^2, p(t) = \int_0^t v(\tau) d\tau \text{ and } t_1 < t_2 \right\} \quad (1)$$

The algorithm return $t$-boxes classified in $\mathbb{T}^{out}, \mathbb{T}^{in}$ and $\mathbb{T}^?$.

$$\mathbb{T}^{in} \subset \mathbb{T} \subset (\mathbb{T}^{in} \cup \mathbb{T}^?) \quad (2)$$

with $\mathbb{T}$ enclosing $\mathbb{T}^*$
**Introduction**

*Figure*: Tube enclosing the trajectory $p$ of the robot.

*Figure*: $t$-plane
1 Introduction

2 Existence Tests
   - Newton test
   - Krawczyc test
   - Miranda test

3 Testcase

4 Discussion
Consider a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $[x] \in \mathbb{R}^n$. Denote by $J_f$ is Jacobian Matrix.

**Definition (Newton test, Moore [MKC09])**

$$\mathcal{N}(f, [J_f], [x]) = \hat{x} - [J_f]^{-1}([x]) \cdot f(\hat{x})$$  \hspace{1cm} (1)

If $\mathcal{N}([x]) \subset [x]$ then $[x]$ contains a unique zero $x^*$ of $f$. It is also in $\mathcal{N}([x])$.

The Newton test applied in LOOP algorithm prove the unicity and existence of 14 loops over 28. And we wants more!
Consider a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $[x] \in \mathbb{R}^n$.

**Theorem (Krawczyk test, Moore [MKC09])**

Let $Y$ be a nonsingular matrix approximating $J_f(\text{center}([x]))^{-1}$. Let $\hat{x} \in [x]$ a real vector.

$$K([x]) = \hat{x} - Y \cdot f(\hat{x}) + \{ I - Y \cdot J_f([x]) \} \cdot ([x] - \hat{x}) \quad (2)$$

If $K([x]) \subseteq [x]$ then $[x]$ contains a zero $x^*$ of $f$. It is also in $K([x])$. 

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**Krawczyk test**

Clément Aubry
Consider a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$

**Theorem (Miranda[Mir40])**

Define: $[x]^{i\pm} = ([x_1], \ldots, [x_{i-1}], \inf / \sup([x_i]), [x_{i+1}], \ldots, [x_n])$.

If

$$
\begin{align*}
  f_i(x) &\geq 0 \forall x \in [x]^{i-}, \\
  f_i(x) &\leq 0 \forall x \in [x]^{i+},
\end{align*}
$$

then $[x]$ contains a zero $x^*$ of $f$. 

(3)
When the functions involved are not well conditionned, Miranda can’t prove anything.
Even if we’re well conditionned, Miranda can fail as in the example at the left where:

\[ f_2([x_1], \inf([x_2])) \not\leq 0 \]

That happens cause of small variations of \( \nabla f_2 \) (in magenta) and \( \text{width}([x]) \). If we can’t change \( \nabla f_2 \), we can work on \([x]\).
The idea is to bissect \([x]\) in Miranda test.
1. Introduction

2. Existence Tests

3. Testcase
   - On A Robot Trajectory

4. Discussion
A mobile robot is moving on a trajectory defined by:

\[
f(t, [p]) = \begin{pmatrix} x(t, [p]) \\ y(t, [p]) \end{pmatrix} = \begin{pmatrix} \sin(t + p_1 * t) \cdot \cos(2t + p_2 * t) \\ \sin(t^2 + p_3 * t) \cdot \cos(t + p_4 * t) \end{pmatrix}
\]

with

\[
p = ([0.0050, 0.015][0.105, 0.115][−0.095, −0.085][0.205, 0.215])^T
\]

Now assume LOOP algorithm on the velocity tubes of this trajectory return t-boxes \(([t_1], [t_2])\) \(\in [0, t_{max}]^2\) that satisfies:

\[
g(t_1, t_2) = f(t_2) - f(t_1) = 0
\]

and

\[0 < t_1 < t_2\]

We want to prove the existence and uniqueness of \(g(t_1, t_2) = 0\) with \(t_1 \in [t_1], t_2 \in [t_2]\).
Figure: Trajectory of the mobile robot.
We apply existence tests presented before to our trajectory and the following table answer to the question "Did the test guarantee the existence of \( f(x) = 0 \) for the \( i_{th} \) intersection?"

<table>
<thead>
<tr>
<th>intersection</th>
<th>( \mathcal{N}([x]) )</th>
<th>( K([t]) )</th>
<th>Miranda</th>
<th>Miranda B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
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<td>no</td>
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</tr>
<tr>
<td>6</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>
More and more test for Miranda-Bissection method.
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The Question is : How Bissection Could Improve Existence Tests?
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