Theorem 17.23 The Petri net persistence problem is decidable
Problem 17.22 is decidable.

Proof: Let $N = (S, T, F, M_0)$ be an arbitrary marked Petri net. The idea is to approximate the reachability set of N set "from below" and to test, at each step, whether a violation of persistence has already been found or whether the semilinear set $\mathcal{R}$ from (17.2) has been reached. To this end, let us define for each $k \in \mathbb{N}$ the set of extended markings

$$\mathcal{ER}_k = \{(\mathcal{P}(\sigma), M) \mid \exists \sigma \in T^* : M_0 \xrightarrow{\sigma} M \wedge |\sigma| \leq k\}$$

All the markings in the second components of $\mathcal{ER}_k$ are reachable from M_0 in at most k steps (they correspond, in some sense, to a breadth first exploration of the reachability graph of N), and therefore the reachability set of N is underapproximated by them. This set is finite and may effectively be constructed.
Consider some stage at which

$$\mathcal{ER}_n \;=\; \{\; (0, M_0)\;,\; (x_1, M_1)\;,\; \ldots\;,\; (x_m, M_m)\;\}$$

has been computed thus far. Let S_n be the semilinear set

$$\begin{aligned} S_n &= \textstyle\bigcup_{0 \leq i \leq m} \left\{\, (x_i, M_i) \; + \mathbb{N}{\cdot}F_i^{(n)} \,\right\} \quad \text{where} \\ F_i^{(n)} &= \min\left(E_i^{(n)}\right) \quad \text{and} \\ E_i^{(n)} &= \left\{\, (\mathcal{P}(\xi), C{\cdot}\mathcal{P}(\xi)) \;\middle|\; \xi \in T^* \wedge |\xi| \leq n \wedge M_i \xrightarrow{\xi} \wedge C{\cdot}\mathcal{P}(\xi) \geq 0 \right\} \end{aligned}$$

Since T is finite, $E_i^{(n)}$ and $F_i^{(n)}$ are also finite and can be constructed effectively. Moreover, since the sequence of sets $\mathcal{ER}_n$ is weakly increasing, $F_i^{(n)} \subseteq F_i^{(n+1)}$. Indeed, if (x_i, M_i) belongs to $\mathcal{ER}_n$, it also belongs to $\mathcal{ER}_{n+1}$ and $E_i^{(n)} \subseteq E_i^{(n+1)}$. In general, min is not monotonic, since it may happen that $A \subseteq B \subseteq \mathbb{N}^{|T|+|S|}$ while $\neg\big(\min(A) \subseteq \min(B)\big)$, if $B \setminus A$ contains a member smaller than a minimal one in A. But this does not occur here since the extra members in $E_i^{(n+1)}$ have larger Parikh vectors than the ones in $E_i^{(n)}$.

Referring back to Definition 17.16, $\lim_{n\to\infty} F_i^{(n)} = F_{M_i}$ since any non-decreasing firing sequence from M_i will eventually occur in some $E_i^{(n)}$. Since F_{M_i} is finite and composed of integer vectors, we even have that, for some n_i, $F_i^{(n_i)} = F_{M_i}$ ($=F_i^{(l)}$ for $l \geq n_i$). Note that, while F_{M_i} is finite, it may not in general be constructed effectively (we know that n_i exists, but we cannot compute it easily: it may in particular happen that $F_i^{(n)} = F_i^{(n+1)} \neq F_i^{(n+2)}$).

As a consequence, we also have that $\bigcup_{0\leq i\leq m} F_i^{(n)} \subseteq \bigcup_{M\in\mathbb{N}^{|S|}} F_M = F$ and, since F is finite (see Lemma 17.18) and composed of integer vectors, for some $\widehat{n}$ and the corresponding $\widehat{m}$, $\bigcup_{0\leq i\leq \widehat{m}} F_i^{(\widehat{n})} = \lim_{n\to\infty} \bigcup_{0\leq i\leq m} F_i^{(n)} = \bigcup_{M\in[M_0\rangle} F_M$. But in general, it is not guaranteed that $\bigcup_{0\le i\le m} F_i^{(n)}$ grows to F since the latter uses all the possible markings while the first one only uses the markings reachable from M_0. For each n, S_n may be infinite, of course, but it is semilinear and can effectively be constructed from $\mathcal{ER}_n$ and the various $F_i^{(n)}$'s. Moreover, $\mathcal{ER}_n \subseteq S_n$, for all n, since $0 \in \mathbb{N}$.

The second components of $F_i^{(n)}$ are the effects of minimal, nondecreasing firing sequences firable from M_i with a length bounded by n. Thus, S_n is the finite union of the linear sets having (x_i, M_i) as a base and all linear combinations of pairs from $F_i^{(n)}$ as corresponding periods. Since all sequences $\xi \in F_{M_i}$ are firable from M_i, and M_i is reachable from M_0, all markings occurring as the second components of S_n are reachable from M_0, so that again they approximate the reachable set of N from below. In effect, they are a subset of the set $\mathcal{R}$ in (17.2).

Now the extended markings in S_n will be tested by means of two Presburger formulae, (17.3) and (17.4). The sentence

$$\begin{array}{l}\exists(x,M)\in S_n\ \exists M'\in\mathbb{N}^S\ \exists t\in T:\\ (M\xrightarrow{t}M')\ \wedge\ \neg((x+\mathcal{P}(t),M')\in S_n)\end{array}\tag{17.3}$$

is a Presburger formula. It checks whether some marking $M' \notin S_n$ can be reached by firing a single transiton t from a marking $M \in S_n$. By Theorem 17.6, (17.3) is decidable. The sentence

$$\begin{array}{l}\exists(x,M)\in S_n\ \exists(x',M')\in S_n\ \exists t\in T\ \exists t'\in T:\\ t\neq t'\ \wedge\ (M\xrightarrow{t})\ \wedge\ (M\xrightarrow{t'}M')\ \wedge\ \neg(M'\xrightarrow{t})\end{array}\tag{17.4}$$

is also a Presburger formula. It checks whether, within S_n, there is a transition t which is enabled at some marking M but becomes disabled at M' by the firing of another transition t'.

If (17.4) is true, then we can conclude that "N is not persistent" and stop the algorithm. This is justified by the definition of persistence, since we just identified a situation of non-persistence amongst reachable markings.

If both (17.3) and (17.4) are false, this means that the second components of S_n already comprise all reachable markings, amongst which no situation of non-persistence has been identified. Thus we may conclude that "N is persistent" and stop the algorithm.

If (17.3) is true and (17.4) is false, the algorithm proceeds by producing $\mathcal{ER}_{n+1}$.

Finally, we show that for any Petri net N, one of the first two cases must arise.

If N is not persistent, its non-persistence will eventually be detected by the sentence (17.4); indeed, the witness M of non-persistence belongs to some $\mathcal{ER}_n$ so that (17.4) will detect the non-persistence at last at step n.

If N is persistent, then (17.4) never becomes true. However, from the remarks above and from (the proof of) Theorem 17.21, at some point S_n contains all reachable markings, sentence (17.3) evaluates to false, and the algorithm terminates with the output "N is persistent". ■ 17.23