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Generalised Composition Operations for High-level Petri Nets *

Eike Best

Alexander Lavrov

Department of Parallel Systems

Faculty of Computer Science

Carl von Ossietzky Universität Oldenburg,

D-26111 Oldenburg, Germany

{eike.best,alexander.lavrov}@informatik.uni-oldenburg.de

Abstract. We propose generic schemes for basic composition operations (sequential composition, choice, iteration, and refinement) for high-level Petri nets. They tolerate liberal combinations of place types (equal, disjoint, intersecting) and, owing to a parameterised scheme of type construction, allow for weak and strong versions of compositions. Properties such as associativity, commutativity, and coherence with respect to unfolding, are preserved.

Keywords: High-level Petri nets, Composition operators, Process algebras.

1. Introduction

The lack of compositionality, often attributed to Petri net models, has inspired intensive research aimed at overcoming this disadvantage. Part of the efforts is concerned with introducing algebraic features into net formalisms [2, 12]. Operators defined on place-transition nets usually share intuitively well-founded, uniform schemes of node connection (as a rule, based on cross-products of participating frontier node sets) [3], while the diversity of high-level models

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induces an even more noticeable multitude of corresponding operator schemes. Usually these schemes either involve complex type formation constructs [7], or are accompanied by substantial restrictions on structural and/or notational characteristics of the interface nodes [6, 8], or are not concerned with algebraic treatment [11]. Such schemes usually represent polar views of the dichotomy 'strong vs. weak (layered [9]) composition' (the more common being the 'strong' case).

The work presented in this paper aims at systematically developing a set of place-based composition operations for high level Petri nets, which:

(1) impose as few as possible restrictions on the types of places (allowing, in particular, for equal, disjoint, and intersecting types);
(2) allow a wider interpretation of the interface place assignment (permitting, in particular, side places next to refined transitions, as well as combined entry/exit-places of a net);
(3) possess a well justified and uniform intuition regarding the causality aspects;
(4) are able to represent weak and strong composition aspects simultaneously;
(5) are still tractable in algebraic contexts, preserve reasonable properties such as commutativity, associativity, and are coherent with respect to net unfolding.

The above goals are reached in the following way. A liberal place typing (goal (1)) and an integrated treatment of weak and strong compositionality (goal (4)) become possible thanks to the proposed universal parameterised type construction scheme (Section 3). This scheme is incorporated in auxiliary net-based operations (Section 4) that serve as a repertoire of the elementary building blocks of which the main operators are composed (Section 5). These operations are defined in such a manner that a number of desirable algebraic (goal (5)), causality (goal (3)), and interface-related (goal (2)) properties are obtained. As an auxiliary uniform and consistent means for expressing and proving properties of the operations, an 'M-nets with a basis' kit is developed (Section 4).

This paper is an expanded version of [4] and an abridged version of [5]. It contains several typical proofs, some more (counter-)examples, and a technical simplification (net union and place addition are merged into a single operation).

2. Basic notions

$\mathbb{N}$ is the set of natural numbers. A multiset μ over a set U is a function $\mu : U \to \mathbb{N}$; $u{\in}\mu$ iff $\mu(u){>}0$, and μ is a set iff $\forall u{\in}U{:}\,\mu(u){\leq}1$. The support of a multiset μ is defined as the set $sp(\mu) = \{u \mid u{\in}\mu\}$, and μ is finite if its support is finite. The set of all finite multisets over U is denoted by $\mathcal{M}_f(U)$. Moreover:

$\mu_1{\leq}\mu_2$ iff $\forall u{\in}U{:}\,\mu_1(u){\leq}\mu_2(u)$ (this is $\subseteq$ if both μ_1 and μ_2 are sets),

$\mu_1{+}\mu_2$ is a multiset such that $\forall u{\in}U{:}\,(\mu_1{+}\mu_2)(u) = \mu_1(u){+}\mu_2(u)$,

$\mu_1{-}\mu_2$ is a multiset such that $\forall u{\in}U{:}\,(\mu_1{-}\mu_2)(u) = \max\{0, \mu_1(u){-}\mu_2(u)\}$,

$\mu_1 \backslash \mu_2$ is a multiset such that $\forall u{\in}U$: $(\mu_1 \backslash \mu_2)(u) = \textbf{if}\ \mu_2(u){=}0\ \textbf{then}\ \mu_1(u)\ \textbf{else}\ 0\ \textbf{fi}$,
$\mu_1 \cup \mu_2$ is a multiset such that $\forall u{\in}U$: $(\mu_1 \cup \mu_2)(u) = \max\{\mu_1(u), \mu_2(u)\}$,
$\mu_1 \cap \mu_2$ is a multiset such that $\forall u{\in}U$: $(\mu_1 \cap \mu_2)(u) = \min\{\mu_1(u), \mu_2(u)\}$.

When using the generalised union $\bigcup$ (and similarly $\bigcap$), we consider this as a unary operator on a set of arguments. Thus, $\bigcup\{\mu_i \mid i{\in}\mathbb{N}\}$ is tantamount to $\mu_0 \cup \mu_1 \cup \mu_2 \cup \ldots$, and $\bigcap\{\mu_1, \mu_2\}$ equals $\mu_1 \cap \mu_2$. For the sum, we also use the traditional notation; thus, for instance, $\sum_{i \in I} \mu_i$ equals (in the same convention as for union and intersection) $\sum\{\mu_i \mid i{\in}I\}$. Restriction of a function f to the domain U is denoted by $f|_U$. We write $f(U) = \{f(u) \mid u{\in}U\}$.

The high-level net model we use is close to that from the standard [1] and represents a relation-oriented version of modular (M-)nets [3]. Its definition (given next) is parameterised by pairwise disjoint sets of values (VAL), modes (MOD), and actions ($\mathcal{A}$). However, MOD and $\mathcal{A}$ play very minor roles in the subsequent parts of the paper (because we confine our attention to place-based operations), and their involvement in the definition is mainly for the sake of completeness. These sets are often omitted in examples.

Definition 2.1. (M-net) An M-net is a triple (S, T, ι), where S is a set of places, T is a set of transitions, with $S \cap T = \emptyset$, and ι is an inscription function with domain $S \cup (S \times T) \cup (T \times S) \cup T$ such that:

- $\forall s{\in}S$: $\iota(s) = (\lambda(s) \mid \alpha(s))$, where $\lambda(s) \in \{\{e\}, \{x\}, \{e,x\}, \{i\}\}$ is called the *status* of s, and $\alpha(s) \subseteq VAL$ is the *type* of s;
- $\forall t{\in}T$: $\iota(t) = (\lambda(t) \mid \alpha(t))$, where $\lambda(t) \subseteq \mathcal{A}$ is called the *label* of t, and $\alpha(t) \subseteq MOD$ is the *set of modes* (or *type*) of t;
- $\forall (s,t){\in}S{\times}T$: $\iota((s,t)) \in \mathcal{M}_f(\{(v,m) \mid v \in \alpha(s) \wedge m \in \alpha(t)\})$;
- $\forall (t,s){\in}T{\times}S$: $\iota((t,s)) \in \mathcal{M}_f(\{(m,v) \mid v \in \alpha(s) \wedge m \in \alpha(t)\})$. ■ 2.1

In the following, we will avoid double parentheses; e.g., we will write $\iota(t,s)$ instead of $\iota((t,s))$.

Definition 2.2. (Subnet) An M-net $N' = (S', T', \iota')$ is a subnet of M-net $N = (S, T, \iota)$ if $S' {\subseteq} S$, $T' {\subseteq} T$, and $\iota' = \iota|_{S' \cup (S' \times T') \cup (T' \times S') \cup T'}$. ■ 2.2

Nets $N_1 = (S_1, T_1, \iota_1)$ and $N_2 = (S_2, T_2, \iota_2)$ are *disjoint* if S_1, S_2, T_1, T_2 (and hence, ι_1 and ι_2) are mutually disjoint. The notion of a type $\alpha(.)$ is extended to the sets of nodes: if $Y {\subseteq} S$ or $Y {\subseteq} T$ then $\alpha(Y) = \bigcup\{\alpha(y) \mid y{\in}Y\}$. Given a net N, we define entry places ${}^\bullet N = \{s \mid s{\in}S \wedge e{\in}\lambda(s)\}$, exit places $N^\bullet = \{s \mid s{\in}S \wedge x{\in}\lambda(s)\}$, combined entry-exit places $S_N^{\{e,x\}} = \{s \mid s{\in}S \wedge \lambda(s){=}\{e,x\}\}$ $(= {}^\bullet N \cap N^\bullet)$, pure entry places ${}^\circ N = {}^\bullet N \setminus N^\bullet$, pure exit places $N^\circ = N^\bullet \setminus {}^\bullet N$, interface places ${}^\bullet N \cup N^\bullet$. The other places with status $\{i\}$ are called internal.

Figure 1 illustrates possible labellings: N_1 represents a structure-oriented labelling. N_2 illustrates a labelling which seems to contradict the (structural) intuition, however this will be permitted in our framework because we separate a net's structure from its use in operators. N_3 exemplifies a mixed case. Table 1 contains interface place sets of the nets from Figure 1.

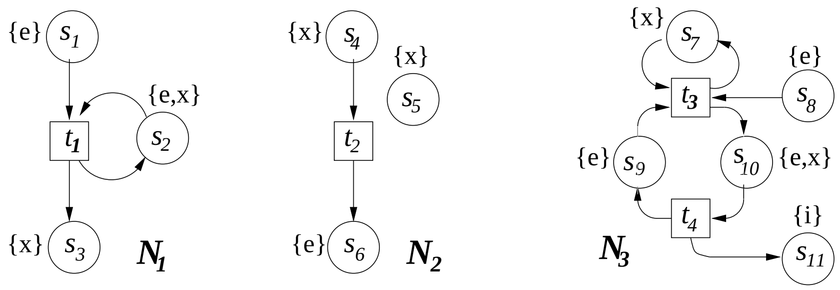


Figure 1. Illustrating place labelling in an M-net.

N_i	$^\bullet N_i$	$N_i^\bullet$	$S_{N_i}^{\{e,x\}}$	$^\circ N_i$	N_i°
N_1	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_2\}$	$\{s_1\}$	$\{s_3\}$
N_2	$\{s_6\}$	$\{s_4, s_5\}$	$\emptyset$	$\{s_6\}$	$\{s_4, s_5\}$
N_3	$\{s_8, s_9, s_{10}\}$	$\{s_7, s_{10}\}$	$\{s_{10}\}$	$\{s_8, s_9\}$	$\{s_7\}$

Table 1. Interface place sets of nets from Figure 1.

Given a net N and a node $y \in S \cup T$, we denote by $^\bullet y$ and $y^\bullet$ the sets of entry and exit nodes of y, respectively: $^\bullet y = \{y' \mid y' \in S \cup T \wedge \iota(y', y) \neq \emptyset\}$, $y^\bullet = \{y' \mid y' \in S \cup T \wedge \iota(y, y') \neq \emptyset\}$, also, $^\circ y = {}^\bullet y \setminus y^\bullet$ and $y^\circ = y^\bullet \setminus {}^\bullet y$. In addition, for $t \in T$ we denote by $S_t^{(side)} = {}^\bullet t \cap t^\bullet$ the set of side places of t. In Table 2, the interface node sets of some nodes from N_1 are given.

Node y	$^\bullet y$	$y^\bullet$	$S_y^{(side)}$	$^\circ y$	y°
s_1	$\emptyset$	$\{t_1\}$	–	$\emptyset$	$\{t_1\}$
t_1	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_2\}$	$\{s_1\}$	$\{s_3\}$

Table 2. Interface node sets for nodes of N_1 from Figure 1.

Input arcs into entry places, output arcs from exit places, and places that are both entry and exit (i.e., have status $\{e, x\}$) are disallowed in [3]. In [2], $\{e, x\}$ places are also disallowed. On the other hand, $\{e, x\}$ places make perfect sense for unguarded loop operators, as we will see later. Hence, at the present moment, we will allow such places. In order to appreciate some difficulties caused by such generality, the reader may wish to attempt forming the sequential composition of nets N_1 and N_2, or the choice of N_2 and N_3, or the refinement transition t_1 by N_3, in Figure 1.

We will assume that VAL is the powerset of a plain set Ω_{BAS} of symbols, i.e., every value is a set of basic symbols. The rationale for this will by explained later. For simplicity, we will identify Ω_{BAS} with $\mathbb{N}$. We will use v, w as generic names for values from VAL. Examples of values: $v = \{1\}$, $w = \{0, 7, 8\}$, ... As generic names for value sets (i.e., for types) we will use ξ, β, φ. Examples of value sets: $\xi = \{\{1\}\}$, $\beta = \{\{1\}, \{0, 7, 8\}\}$, $\varphi = \{\{n\} \mid n \in \mathbb{N}\}$.

The unfolding of an M-net N is obtained by disassembling N in such a way that every place in the resulting net gets a singleton type, and every transition exactly one mode. The behaviour of N ought to be the same as the behaviour of its unfolding; hence the importance of the preservation of unfolding-related equivalences in composition operations. The places and the transitions of the unfolding are given names s^v and t^m, to remind of the values and modes they arise from.

Definition 2.3. (Unfolding) Let $N = (S, T, \iota)$ be an M-net. $\mathcal{U}(N) = (\mathcal{U}(S), \mathcal{U}(T), \iota_{\mathcal{U}})$, where

- $\mathcal{U}(S) = \{s^v \mid s \in S,\ v \in \alpha(s)\}$, and $\mathcal{U}(T) = \{t^m \mid t \in T,\ m \in \alpha(t)\}$,
- $\forall s^v \in \mathcal{U}(S)$: $\iota_{\mathcal{U}}(s^v) = (\lambda_{\mathcal{U}}(s^v) \mid \alpha_{\mathcal{U}}(s^v))$, and $\lambda_{\mathcal{U}}(s^v) = \lambda(s)$, $\alpha_{\mathcal{U}}(s^v) = \{v\}$,
- $\forall t^m \in \mathcal{U}(T)$: $\iota_{\mathcal{U}}(t^m) = (\lambda_{\mathcal{U}}(t^m) \mid \alpha_{\mathcal{U}}(t^m))$, and $\lambda_{\mathcal{U}}(t^m) = \lambda(t)$, $\alpha_{\mathcal{U}}(t^m) = \{m\}$,
- $\forall (s^v, t^m) \in \mathcal{U}(S) \times \mathcal{U}(T)$: $\iota_{\mathcal{U}}(s^v, t^m) = \iota(s, t)(v, m) \cdot \{(v, m)\}$,
- $\forall (t^m, s^v) \in \mathcal{U}(T) \times \mathcal{U}(S)$: $\iota_{\mathcal{U}}(t^m, s^v) = \iota(t, s)(m, v) \cdot \{(m, v)\}$. ■ 2.3

We will also speak of the unfolding of a (sub)set of nodes of a net, considering this as the set of the corresponding nodes in the net unfolding (for instance, $\mathcal{U}({}^\bullet N) = {}^\bullet\mathcal{U}(N)$).

We will finally define (weak) isomorphism of nets, because some of the results hold only with respect to isomorphism (which is essentially renaming of net elements).

Definition 2.4. (Net isomorphism) M-nets $N=(S, T, \iota)$ and $N'=(S', T', \iota')$ are called *isomorphic* up to node renaming, denoted by $N \cong N'$, if there is a bijection $\rho: S \cup T \to S' \cup T'$ such that $\rho(S) \subseteq S'$, $\rho(T) \subseteq T'$ and $\forall y \in S \cup (S \times T) \cup (T \times S) \cup T: \iota(y) = \iota'(\rho(y))$. ■ 2.4

Definition 2.5. (Weak net isomorphism) M-nets $N = (S, T, \iota)$ and $N'=(S', T', \iota')$ are called *weakly isomorphic*, if there are bijections $\rho: S \cup T \to S' \cup T'$ and $\chi: VAL \cup MOD \to VAL \cup MOD$ such that the following holds:

- $\rho(S) \subseteq S', \rho(T) \subseteq T'$, $\chi(VAL) \subseteq VAL$, and $\chi(MOD) \subseteq MOD$;
- $\forall y \in S \cup T: \lambda(y) = \lambda'(\rho(y)) \wedge \chi(\alpha(y)) = \alpha'(\rho(y))$;
- $\forall y \in (S \times T) \forall m \in MOD \forall v \in VAL: \iota(y)(m, v) = \iota'(\rho(y))(\chi(m), \chi(v))$;
- $\forall y \in (T \times S) \forall v \in VAL \forall m \in MOD: \iota(y)(v, m) = \iota'(\rho(y))(\chi(v), \chi(m))$. ■ 2.5

3. Parameterised type construction

The composition operations presented in this paper are based on merging interface places of the participating nets. The difference between the weak and strong compositions comes into effect by the discipline of token propagation through the 'border' resulting from merging these interface places. A *weak discipline* implies that different values can proceed independently: tokens which correspond to a certain value and which enter a net component are allowed to go further without waiting for those tokens which are consumed by the same component but correspond to other values. A *strong discipline* assumes joint treatment of the values entering or leaving a participating net fragment (and considered as belonging to an initial or final marking).

This distinction is illustrated schematically in Figure 2. Assume that places s_1 and s_2 are initially marked by values v and w, respectively. Under strong sequence, we will require that v, v', w and w' are mutually disjoint, and 'strong' means that the interleaving *abcd* (but not the interleaving *acbd*) is possible in $N_1; N_2$. Under weak sequence, we may have $v{=}v'{\neq}w{=}w'$, and then, interleavings *abcd* and *acbd* (but not *adbc*) are possible. We will formalise this by using a parameter value set ψ, whose members specify the weak composition.

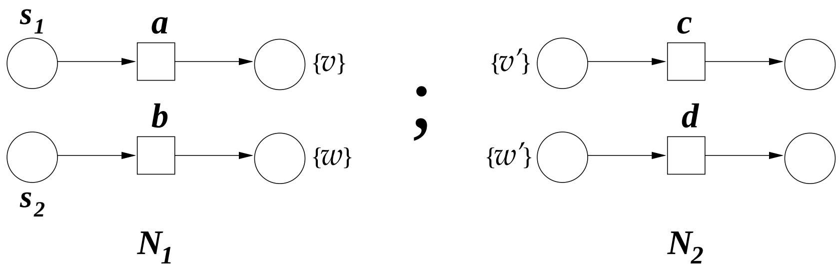


Figure 2. Illustrating the difference between weak and strong sequence.

In the present context of high-level nets, the creation of new places (from old ones by merging) must be accompanied by a *type construction* for these places. For the weak discipline, the values remain untouched; this mechanism is implemented by the intersection part of our type creation operator below. Under the strong composition, however, values must be bound to others, which implies that new values must reflect the corresponding dependency information.

The choice of sets as the form of value representation (i.e., that each value is a subset of Ω_{BAS}) is based primarily on the intention to obtain commutativity and associativity of composition operators (and, in particular, of the refinement, as the most complicated case), as identities rather than isomorphisms. If two transitions, which share some adjacent place(s), are to be refined by different nets, commutativity means that the two different sequences of refinements must result in the same (including names and types) sets of new places. Therefore, type creation must be commutative: each new value must be able to absorb information gradually from the (types of the) contributing places, while the final outcome should not depend on the order of the contributions. Sets are used, since they are simple structured objects, yet – when used in the particular way we propose – are able to meet these wishes.

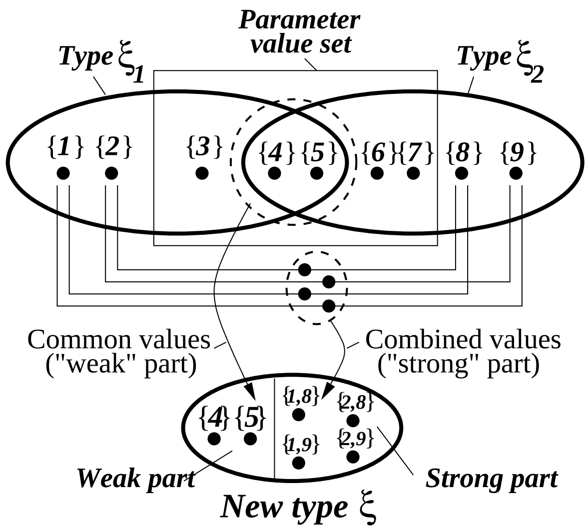


Figure 3. Illustrating the origin of values in a new type.

In general, let $\psi \subseteq VAL$ be a value set with respect to which the following notions will be parameterised. For example, in Figure 3, $\psi = \{\{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$. Values in ψ are called *weak*, others are called *strong*. The extreme cases are that $\psi = \emptyset$ (completely strong environment, Figure 4(a)), and that ψ encompasses all participating type sets (completely weak environment, Figure 4(b)).

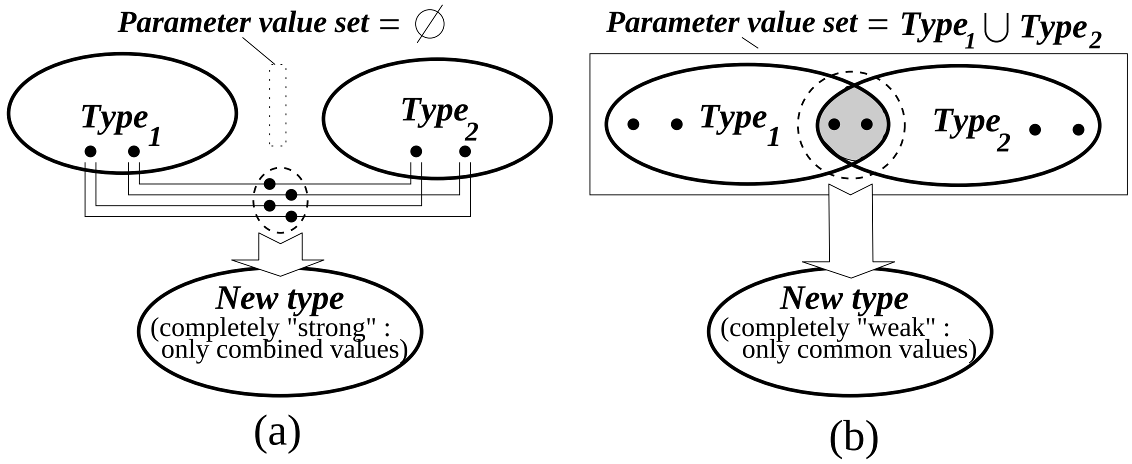


Figure 4. Pure weak (a) and strong (b) cases of type construction.

In Definition 3.3 below, we will describe a type construction facility which, given a parameter set

ψ and a multiset of types $\mu \in \mathcal{M}_f(2^{VAL})$, creates a new type $\underset{\psi}{\bowtie}\, \mu$. In Figure 3, for example, we wish to create $\xi = \underset{\psi}{\bowtie} \{\xi_1, \xi_2\}$ with $\psi = \{\{3\},\{4\},\{5\},\{6\},\{7\}\}$, $\xi_1 = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$ and $\xi_2 = \{\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}\}$. The idea is to let the weak part of ξ arise as the intersection of the original weak types (after which, in Figure 3, values $\{4\}$ and $\{5\}$ survive), and the strong part of ξ as all pairwise combinations of the original strong types (which yields, in Figure 3, the new values $\{1,8\}$, $\{1,9\}$, $\{2,8\}$, and $\{2,9\}$). We have to ensure that the new values can again be characterised as weak or strong with respect to the same ψ. Definition 3.1 and Lemma 3.2 serve this purpose.

Definition 3.1. (Conformable multiset of types) Let $\psi \subseteq VAL$ and $\emptyset \neq \mu \in \mathcal{M}_f(2^{VAL})$.

μ is ψ-conformable iff $\forall \xi \in \mu\ \forall v \in \xi \colon \underbrace{(v \in \psi)}_{\text{(a)}} \vee \underbrace{(\forall w \in \psi \cup (\bigcup_{\xi' \in \mu - \{\xi\}} \xi') \colon v \cap w = \emptyset)}_{\text{(b)}}$. ■ 3.1

Definition 3.1 (illustrated in Table 3) means that each value v contained in one of the original types ξ is contained in ψ by (a) (then it is and remains weak) or, if it is strong, by (b) does not intersect with any value w from ψ or any other set (including ξ itself, if ξ is contained twice or more in μ) in μ and hence, is safely combinable with other strong values. The last condition implies, by the use of multiset minus ($-$) rather than \, that the types with membership in μ exceeding 1 must be completely weak. This is a technical subtlety which guarantees coherence in cases when a type participates more than once in a type construction.

Value sets ξ_1 and ξ_2	$\psi = \{\{1\},\{2\}\}$	**Comments**
$\xi_1 = \{\{1\},\{2\}\}$, $\xi_2 = \{\{1\},\{4\}\}$	$\{\xi_1\}, \{\xi_1,\xi_1\}, \{\xi_2\}, \{\xi_1,\xi_2\}$, $\{\xi_1,\xi_1,\xi_2\}$ are all conformable	
	$\{\xi_1,\xi_2,\xi_2\}$ is not conformable	$\{4\} \notin \psi$ while $\mu(\xi_2) > 1$
$\xi_1 = \{\{1\},\{3\}\}$, $\xi_2 = \{\{1\},\{4\}\}$	$\{\xi_1,\xi_2\}$ is conformable	
$\xi_1 = \{\{1\},\{3\}\}$, $\xi_2 = \{\{1\},\{3,4\}\}$	$\{\xi_1,\xi_2\}$ is not conformable	$\{3\},\{3,4\} \notin \psi$ while $\{3\} \cap \{3,4\} \neq \emptyset$
$\xi_1 = \{\{1,3\},\{2\}\}$, $\xi_2 = \{\{2\}\}\}$	$\{\xi_1,\xi_2\}$ is not conformable	$\{1,3\} \notin \psi$ while $\{1,3\} \cap \{1\} \neq \emptyset$

Table 3. Examples of conformable and not conformable multisets.

From Definition 3.1 it follows that if μ is ψ-conformable then so is each multiset $\emptyset \neq \mu' \leq \mu$. In particular, the set $sp(\mu) = \{\xi \mid \xi \in \mu\}$ is itself ψ-conformable.

Lemma 3.1. (Disjointness of strong values from different types)
Let $\psi \subseteq VAL$, $\emptyset \neq \mu \in \mathcal{M}_f(2^{VAL})$ be ψ-conformable, $\xi_1, \xi_2 \in \mu$, and $\xi_1 \neq \xi_2$.
If $v_1 \in \xi_1$ and $v_2 \in \xi_2$ such that $v_1 \notin \psi$ and $v_2 \notin \psi$ then $v_1 \cap v_2 = \emptyset$.

Proof: Straightforward from part (b) of Definition 3.1. ■ 3.1

The following auxiliary operator yields a value combination from values of a set of value sets, taking each of the participating value sets once.

Definition 3.2. (Value combination)
Let $\nu \subseteq 2^{VAL}$. Then $\biguplus \nu \stackrel{\text{def}}{=} \{v \mid v = \bigcup \{v_\xi \mid \xi \in \nu\}$ with $\forall \xi \in \nu\colon v_\xi \in \xi\}$. ■ 3.2

Note that ν is a set of types but $\biguplus \nu$ is a single type. For instance, if $\nu = \{\{\{1\},\{2\}\},\{\{8\},\{9\}\}\}$, then $\biguplus \nu = \{\{1,8\},\{1,9\},\{2,8\},\{2,9\}\}$.

Definition 3.3. (Type construction) Let $\psi \subseteq VAL$. Let $\emptyset \neq \mu \in \mathcal{M}_f(2^{VAL})$ be ψ-conformable.
Then $\underset{\psi}{\bowtie} \mu \stackrel{\text{def}}{=} \underbrace{\bigcap \{\xi \cap \psi \mid \xi \in \mu\}}_{\text{(A)}} \cup \underbrace{\biguplus \{\xi \setminus \psi \mid \xi \in \mu\}}_{\text{(B)}}$. ■ 3.3

Again, μ is a set of types but $\underset{\psi}{\bowtie} \mu$ is a single type. The idea is that disjunct (A) is the new weak part and disjunct (B) is the new strong part. Figure 3 and Table 4 give examples.

Participating value sets	**Parameter set** ψ	$\underset{\psi}{\bowtie} \mu$
$\xi = \{\{1\}\}$ $\quad (\mu = \{\xi\})$	$\emptyset$	$\{\{1\}\}$
$\xi = \{\{1\}\}$ $\quad (\mu = \{\xi\})$	$\{\{1\},\{2\}\}$	$\{\{1\}\}$
$\xi_1 = \{\{1\},\{2\}\}, \xi_2 = \{\{1\},\{3\}\}$ $\quad (\mu = \{\xi_1,\xi_2\})$	$\{\{1\},\{2\}\}$	$\{\{1\}\}$
$\xi_1 = \{\{1\}\}, \xi_2 = \{\{2\}\}$ $\quad (\mu = \{\xi_1,\xi_2\})$	$\{\{1\},\{2\}\}$	$\emptyset$
$\xi_1 = \{\{1,2\}\}, \xi_2 = \{\{2,3\}\}$ $\quad (\mu = \{\xi_1,\xi_2\})$	$\{\{1,2\},\{2,3\}\}$	$\emptyset$

Table 4. Examples of type construction.

Lemma 3.2. (Consistency of $\bowtie$) *If μ is ψ-conformable, then so is $\{\underset{\psi}{\bowtie} \mu\}$.*

Proof: We need to show the validity of Definition 3.1 for $\{\underset{\psi}{\bowtie} \mu\}$. Since this is a singleton set, the multiset minus $(-)$ is tantamount with $\setminus$. Thus, we need to prove

$$\forall v \in \underset{\psi}{\bowtie} \mu\colon \underbrace{(v \in \psi)}_{\text{(a')}} \vee \underbrace{(\forall w \in \psi \cup (\bigcup_{\xi' \in \{\underset{\psi}{\bowtie} \mu\} \setminus \{\underset{\psi}{\bowtie} \mu\}} \xi')\colon v \cap w = \emptyset)}_{\text{(b')}},$$

that is:

$$\forall v \in \underset{\psi}{\bowtie}\,\mu \colon \underbrace{(v \in \psi)}_{\text{(a')}} \vee \underbrace{(\forall w \in \psi \colon v \cap w = \emptyset)}_{\text{(b')}}.$$

Let $v \in \underset{\psi}{\bowtie}\,\mu$ be arbitrary. If $v \in \psi$, (a') is true and we are done ('v is weak'). Assume $v \notin \psi$. By Definition 3.3, $v \in$(A) or $v \in$(B). However, by $v \notin \psi$, $v \in$(A) is impossible. Hence $v \in$(B), and then, $v = \bigcup\{v_\xi \mid \xi \in \mu\}$ with $v_\xi \in \xi \setminus \psi$ for all $\xi \in \mu$. Hence for every v_ξ, by the ψ-conformability of μ, and since $v_\xi \notin \psi$, we have by (b): $\forall w \in \psi \colon v_\xi \cap w = \emptyset$. Hence also $\forall w \in \psi \colon v \cap w = \emptyset$ by the definition of v (i.e., (b') holds and 'v is strong'). ■ 3.2

Lemma 3.2 ensures that after type construction, the new values can again be categorised into weak or strong ones with respect to the same ψ, and its proof shows that conformability is a necessary precondition for this property to be true.

Lemma 3.3. (Equivalence of the type construction over a multiset and its support)
For any $\emptyset \neq \mu', \mu \in \mathcal{M}_f(2^{VAL})$, if $sp(\mu) \leq \mu' \leq \mu$ and μ is ψ-conformable, then $\underset{\psi}{\bowtie}\,\mu' = \underset{\psi}{\bowtie}\,\mu$.

Proof: From the fact that the multiplicity of completely weak types, if nonzero, does not affect the outcome of the type construction, while types with non-weak values cannot have multiple occurrences in μ by the definition of conformability. ■ 3.3

From Lemma 3.3 it follows that we can use $\underset{\psi}{\bowtie}\{\xi \mid \xi \in \mu\}$ instead of $\underset{\psi}{\bowtie}\,\mu$.

Proposition 3.1. (Properties of the type construction operator)
Let $\psi \subseteq VAL$ and $\emptyset \neq \mu \in \mathcal{M}_f(2^{VAL})$ be ψ-conformable. Then:

(1) *(Associativity) If $\emptyset \neq \kappa \leq \mu$, then* $\underbrace{\underset{\psi}{\bowtie}\,\mu}_{\text{(C)}} = \underbrace{\underset{\psi}{\bowtie}\,((\mu \setminus \kappa) \cup \{\underset{\psi}{\bowtie}\,\kappa\})}_{\text{(D)}}$.

(2) *(Distributivity with respect to set union) If $\forall \xi \in \mu \colon \xi = \bigcup\{\alpha \mid \alpha \in \mu_\xi\}$ (i.e., μ_ξ is a collection of types whose union is ξ), then*

$$\underset{\psi}{\bowtie}\,\mu = \underbrace{\underset{\psi}{\bowtie}\{\bigcup\{\alpha \mid \alpha \in \mu_\xi\} \mid \xi \in \mu\}}_{\text{(C)}} = \underbrace{\bigcup\{\beta \mid \beta = \underset{\psi}{\bowtie}\{\alpha_\xi \mid \xi \in \mu\} \text{ with } \forall \xi \in \mu \colon \alpha_\xi \in \mu_\xi\}}_{\text{(E)}}.$$

Proof: Due to Lemma 3.3, we may assume, w.l.o.g., that μ is a set.

(1) (C) $\subseteq$ (D): Let $v \in$(C). By Definition 3.3, $v \in$(A) or $v \in$(B).
Case $v \in$(A): Then $v \in \xi \cap \psi$ for every $\xi \in \mu \neq \emptyset$; in particular, by $\mu \neq \emptyset$, $v \in \psi$. In particular, also, $v \in \xi \cap \psi$ for every $\xi \in \kappa$, by $\kappa \leq \mu$. Hence $v \in \bigcap\{\xi \cap \psi \mid \xi \in \kappa\}$, and therefore, by Definition 3.3(A), $v \in \underset{\psi}{\bowtie}\,\kappa$. Moreover, $v \in \xi \cap \psi$ for every $\xi \in \mu \setminus \kappa$ as well, whence $v \in \xi \cap \psi$ for every

$\xi\in(\mu\backslash\kappa)\cup\{\underset{\psi}{\bowtie}\ \kappa\}$, and hence, applying Definition 3.3(A) one more time, $v\in$(D).

Case $v\in$(B): Then $v=\bigcup\{v_\xi \mid \xi\in\mu\}$ with $v_\xi\in\xi\backslash\psi$ for all $\xi\in\mu$. (Moreover, by Lemma 3.1, if $\xi_1\neq\xi_2$ then $v_{\xi_1}\cap v_{\xi_2}=\emptyset$.) By $\mu=(\mu\backslash\kappa)\cup\kappa$, $v=\bigcup\{v_\xi \mid \xi\in\mu\backslash\kappa\}\cup\bigcup\{v_\xi \mid \xi\in\kappa\}$, and by Definition 3.3(B), $\bigcup\{v_\xi \mid \xi\in\kappa\} \in \underset{\psi}{\bowtie}\ \kappa$. Hence $v\in$(D).

(D) $\subseteq$ (C): The above reasoning can be reversed in both cases.

(2) (C)$\subseteq$(E): Let $v\in$(C). By Definition 3.3, $v\in$(A) or $v\in$(B).

Case $v\in$(A): Then $v\in\xi\cap\psi$ for every $\xi\in\mu$; in particular, as above, $v\in\psi$. Moreover, by the assumptions about ξ, $v\in\{\alpha \mid \alpha\in\mu_\xi\}$ for every $\xi\in\mu$, and hence, for every $\xi\in\mu$ there is some α (call it α_ξ) in μ_ξ with $v\in\alpha_\xi$. Consider $\beta=\underset{\psi}{\bowtie}\{\alpha_\xi \mid \xi\in\mu\}$. By Definition 3.3(A), $v\in\beta$, and hence also $v\in$(E), since (E) is the union of such β's.

Case $v\in$(B): Then $v=\bigcup\{v_\xi \mid \xi\in\mu\}$ with $v_\xi\in\xi\backslash\psi$ for all $\xi\in\mu$; by the assumption about ξ, for every $\xi\in\mu$ there is some $\alpha_\xi\in\mu_\xi$ with $v_\xi \in \alpha_\xi\backslash\psi$. Hence, by Definition 3.3(B), $v\in \underset{\psi}{\bowtie}\{\alpha_\xi \mid \xi\in\mu\}$, and therefore, $v\in$(E).

(E) $\subseteq$ (C): Let $v\in$(E). Then $v\in\beta$, for some $\beta=\underset{\psi}{\bowtie}\{\alpha_\xi \mid \xi\in\mu\}$ with $\alpha_\xi\in\mu_\xi$ for every $\xi\in\mu$.

Case $v\in\psi$: Then, $v\in\beta$ and Definition 3.3(A) yield $v\in\alpha_\xi$ for every α_ξ, and since $\alpha_\xi\subseteq\xi$, also $v\in\xi$ for every $\xi\in\mu$. By Definition 3.3(A), again, $v\in$(C).

Case $v\notin\psi$: Then $v\in\beta$ and Definition 3.3(B) yield $v=\bigcup\{v_\varphi \mid \varphi\in\{\alpha_\xi \mid \xi\in\mu\}\}$ with $v_\varphi\in\varphi\backslash\psi$ for every φ. (Moreover, by Lemma 3.1, $v_{\varphi_1}\cap v_{\varphi_2}=\emptyset$ whenever $\varphi_1\neq\varphi_2$.) Since $\alpha_\xi\subseteq\xi$, and every φ is such as α_ξ, this implies that $v=\bigcup\{v_\xi \mid \xi\in\mu\}$ with $v_\xi\in\xi\backslash\psi$ for every $\xi\in\mu$. By Definition 3.3(B), again, $v\in$(C). ■ 3.1

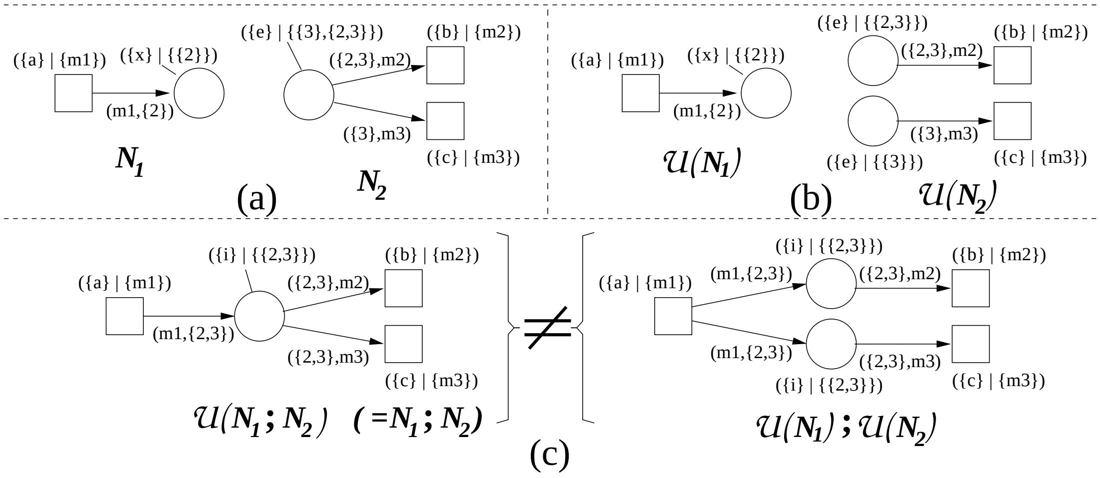


Figure 5. Counterexample to simplification of Definition 3.1 ($\psi = \{\{1\}\}$).

Definitions 3.1 and 3.3 form the technical groundwork on which our results are based. They have been chosen rather judiciously. One might be tempted to simplify Definition 3.1 by calling a type ξ 'good' with respect to ψ if all $v\in\xi$ satisfy $v\in\psi$ or $\forall w\in\psi$: $v\cap w=\emptyset$ (the idea being that $v\in\psi$ means v is weak and $\forall w\in\psi$: $v\cap w=\emptyset$ means v is strong), and then to allow the combination of 'good' types only. Definition 3.3 could be adjusted accordingly. However, this attempt fails in terms of unfolding preservation, as is shown in a counterexample in Figure 5. Figure 5 corresponds to violation of the condition from Definition 3.1 by values from different types (it uses the sequential composition operator, whose formal definition is given in Section 5.1); (a) shows nets N_1 and N_2, (b) gives their unfoldings, (c) illustrates non-equality of the unfolding of the sequence, on the one hand, and the sequence of unfoldings, on the other.

4. Auxiliary operations on nets

Before starting to define net operations and proving their properties, we need to address:

(1) The fact that all operations are based on a similar small set of auxiliary operations.
(2) And a technique for making the proofs of properties as small and uniform as possible.

Point (2) will be addressed in section 4.1, where we will define a framework of *M-nets with basis* (MNB). Such a net is an M-net which is associated with another M-net, called basis. Elements (nodes, types, values) from this basis play the role of an 'alphabet', in terms of which the corresponding elements of the MNB are expressed. The benefit of such a mechanism is that the new (i.e., resulting from some composition) incidence relation can be constructed directly from (new) node names. Had we defined the main operations in terms of only M-nets, we would have had to constantly rename the nodes and to keep showing net isomorphisms while performing transformations. The MNB-framework saves us these efforts, since it does it automatically; this allows to avoid many case distinctions in proofs.

In section 4.2, we address point (1) by defining three auxiliary net operations within the MNB framework: net union, place multiplication, and restriction. These will be enough to express all the main net operations in the rest of the paper.

4.1. M-net with a basis

The MNB mechanism is embodied in Definition 4.2. First, however, we transfer the notion of type set conformability to the types of given place sets. The net in question will be a basis, whence the index $_B$.

Definition 4.1. (Compatible set of place sets) Let (S_B, T_B, ι_B) be an M-net. Let $I\neq\emptyset$. Then $\{S_i\subseteq\mathcal{M}_f(S_B) \mid i\in I\}$ is called ψ-compatible if for any $s=\sum_{i\in I} s_i$ with $\forall i\in I$: $s_i\in S_i$, the multiset $\mu_s=\sum_{s_B\in s} s(s_B)\cdot\{\alpha(s_B)\}$ is ψ-conformable. ■ 4.1

Let, for example, S_B be the place set of net N_B in Figure 6, i.e. $S_B{=}\{s_1, s_2\}$, and let $S'{=}\{\{s_1\}\}$ and $S''{=}\{\{s_2\}\}$. Then $\{S', S''\}$ is $\{\{1\},\{2\}\}$-compatible because $\mu{=}\{\alpha(s_1), \alpha(s_2)\}$ $(=\{\ \{\{1\},\{3\}\}\ ,\ \{\{1\},\{2\},\{4\}\}\ \}$ is $\{\{1\},\{2\}\}$-conformable.

Definition 4.2. (M-net with a basis) Let $N{=}(S, T, \iota)$, $N_B{=}(S_B, T_B, \iota_B)$ be M-nets and let ψ be a parameter set. Then N is called an MNB (M-net with basis) over N_B and ψ if the following conditions are satisfied:

- $\forall s_B{\in}S_B \forall v \in \alpha(S_B){:}|v| = 1$.
- $T{\subseteq}T_B$ and $\forall t{\in}T{:}\,\iota(t){=}\iota_B(t)$.
- $S \subseteq \mathcal{M}_f(S_B)$, $\{S\}$ is ψ-compatible, and $\forall s{\in}S{:}\,\alpha(s){=} \underset{\psi}{\bowtie} \{\alpha(s_B) | s_B{\in}s\}$.
- $\forall(s,t){\in}S{\times}T\ \forall v{\in}\,VAL\ \forall m{\in}MOD{:}\,\iota(s,t)(v,m){=}\sum_{s_B\in s}\sum_{\varrho\in v} s(s_B)\cdot\iota_B(s_B,t)(\{\varrho\},m)$;
  $\forall(t,s){\in}T{\times}S\ \forall v{\in}\,VAL\ \forall m{\in}MOD{:}\,\iota(t,s)(m,v){=}\sum_{s_B\in s}\sum_{\varrho\in v} s(s_B)\cdot\iota_B(t,s_B)(m,\{\varrho\})$. ■ 4.2

Note that the requirement of ψ-compatibility of $\{S\}$ in the above definition is equivalent with the ψ-conformability of $\{\alpha(S)\}$.

Figure 6 presents an example of an MNB N with basis N_B. Note how the type $\{\{1\},\{3,4\}\}$ of the 'combined' place $\{s_1, s_2\}$ arises from the types of s_1,s_2 via ψ and the type construction defined in section 3, and how its connections to t arise through the connections of s_1 and s_2 to t in N_B: the weak value $\{1\}$, contained in the types of both s_1 and s_2, is preserved in the type of $\{s_1, s_2\}$, while the weak value $\{2\}$, contained only in $\alpha(s_2)$, is eliminated via intersection and is not inherited by $\alpha(\{s_1, s_2\})$; the strong values $\{3\}$ and $\{4\}$ are combined into a new strong value $\{3, 4\}$; the coefficients 3 and 1 next to $(\{1\}, m1)$ and $(m1, \{1\})$, respectively, are equal to the corresponding coefficients in N_B; and similarly with coefficients 2 next to $(\{3, 4\}, m2)$ and 4 next to $(m2, \{3, 4\})$.

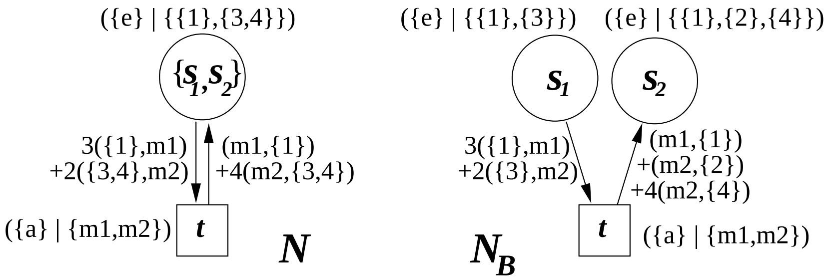


Figure 6. Example of an MNB N over N_B and $\psi = \{\{1\},\{2\}\}$.

Note that, corresponding to the definition of MNB, we do not relate the place labelling $(\{e\}, \{x\}$, $\{e, x\}$ or $\{i\})$ in an MNB with that in its basis; this labelling is assumed to be given explicitly (all our composition operations from Sections 5.1–5.3 will do so). We will sometimes not give N_B and ψ explicitly, if they are clear from the context.

Lemma 4.1. (Relation between the unfoldings of an MNB and its basis) *If N is an MNB over N_B and ψ, then, up to place renaming, $\mathcal{U}(N)$ is an MNB over $\mathcal{U}(N_B)$ and ψ.*

Proof: Using the renaming $\rho{:}\mathcal{U}(S)\cup\mathcal{U}(T)\to\mathcal{M}_f(\mathcal{U}(S_B))\cup\mathcal{U}(T)$ such that $\forall t{\in}\mathcal{U}(T){:}\,\rho(t){=}t$ and $\forall s^v{\in}\mathcal{U}(S){:}\,\rho(s^v){=}\sum_{s_B\in s} s(s_B)\cdot\{s_B^w\}$, where s^v is as in Definition 2.3, and $w{=}v{\cap}(\bigcup\alpha(s_B))$ (note that by Definition 2.3, $s_B^w\in\mathcal{U}(S_B)$). ■ 4.1

The idea of renaming ρ from the above lemma is illustrated by an example. Let $s{=}\{s_{B,1},s_{B,2}\}$, $\alpha(s_{B,1}){=}\{\{1\},\{2\}\}$, $\alpha(s_{B,2}){=}\{\{1\},\{3\}\}$, $\psi{=}\{\{1\}\}$. Thus, $\alpha(s)=\{\{1\},\{2,3\}\}$. Then, $\mathcal{U}(\{s\})=\{\{s_{B,1},s_{B,2}\}^{\{1\}},\ \{s_{B,1},s_{B,2}\}^{\{2,3\}}\}$ and $\mathcal{U}(\{s_{B,1}\})=\{s_{B,1}^{\{1\}},s_{B,1}^{\{2\}}\}$ and $\mathcal{U}(\{s_{B,2}\})=\{s_{B,2}^{\{1\}},s_{B,2}^{\{3\}}\}$. Under ρ we have: $\{s_{B,1},s_{B,2}\}^{\{1\}}\mapsto\{s_{B,1}^{\{1\}},s_{B,2}^{\{1\}}\}$ and $\{s_{B,1},s_{B,2}\}^{\{2,3\}}\mapsto\{s_{B,1}^{\{2\}},s_{B,2}^{\{3\}}\}$.

4.2. Net union, place multiplication, and restriction

Definition 4.3. (MNB union) Let $N_1=(S_1,T_1,\iota_1)$ and $N_2=(S_2,T_2,\iota_2)$ be disjoint MNB's over the same basis N_B and ψ. $N=N_1{\cup}N_2$ is defined to be the MNB $(S_1{\cup}S_2,T_1{\cup}T_2,\iota')$ over the same basis and ψ, where ι arises from $S_1{\cup}S_2$, $T_1{\cup}T_2$ and N_B as in Definition 4.2. ■ 4.3

In this definition, the hidden properties of MNB's come to the fore. It may happen that N_1 and N_2 are disjoint, but that N is still a connected net. Figure 7 illustrates this. N and N' are disjoint, and yet there is an arrow from the place of N' to the transition of N in their union N_1, which is induced by the basis N_B. A similar remark is true for N, N'' and their union, N_2.

Proposition 4.1. (Properties of MNB union) *Let N, N' and N'' be mutually disjoint MNB's over the same basis and ψ. Then,*

(1) *(Idempotence)* $N\cup N=N$.
(2) *(Commutativity)* $N\cup N'=N'\cup N$.
(3) *(Associativity)* $(N\cup N')\cup N''=N\cup(N'\cup N'')$.
(4) *(Coherence w.r.t. unfolding)* $\mathcal{U}(N\cup N')=\mathcal{U}(N)\cup\mathcal{U}(N')$.

Proof: (1), (2) and (3) follow straight from the corresponding set theoretic identities: $S{\cup}S=S$ for (1), $S{\cup}S'=S'{\cup}S$ for (2) and $(S{\cup}S'){\cup}S''=S{\cup}(S'{\cup}S'')$ for (3). To see (4)($\subseteq$), assume s^v is a place in $\mathcal{U}(N\cup N')$. Then place s is in $N\cup N'$; w.l.o.g., assume s is a place of N. Then s^v is also a place in $\mathcal{U}(N)$, and hence in $\mathcal{U}(N)\cup\mathcal{U}(N')$. (4)($\supseteq$) is proved similarly. ■ 4.1

Note that all identities (including coherence with respect to unfolding) hold in the strictest possible sense (with identity rather than isomorphism).

The place multiplication operator $\otimes$ defined next takes an MNB and a collection of subsets of its places, and creates a new set of places through a cartesian product based scheme, by taking all possible combinations of places from the participating sets (one from each of the sets), but

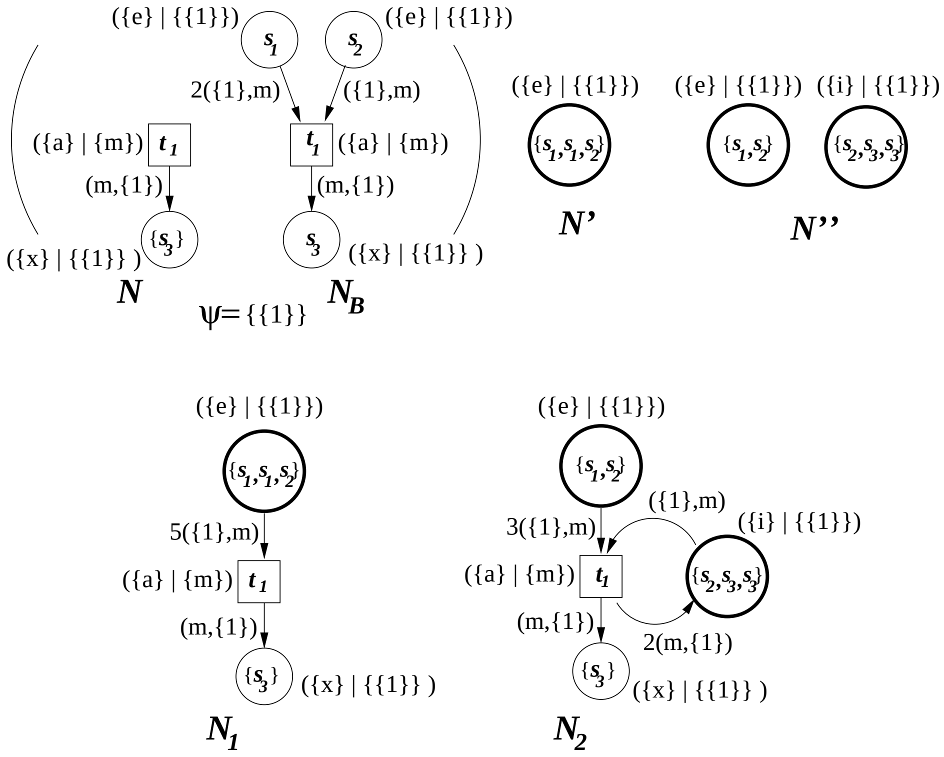


Figure 7. Illustrating MNB union: $N_1 = N \cup N'$ and $N_2 = N \cup N''$.

leaving only those which obtain non-empty types from the type construction operator $\bowtie$. The non-emptiness condition is necessary in order to preserve the coherence of this operator with respect to unfolding, and it just excludes useless places (i.e., ones with empty types).

Definition 4.4. (Place multiplication) Let $N = (S, T, \iota)$ be an MNB, and $I \neq \emptyset$ some index set. Let $\{S_i \mid i \in I\}$ be a ψ-compatible set of subsets of S, and $\tau \in \{\{e\}, \{x\}, \{e, x\}, \{i\}\}$. Then,

$$\overset{\tau}{\underset{\psi}{\otimes}} \{S_i \mid i \in I\} \stackrel{\text{def}}{=} \{\, s \mid s = \sum_{i \in I} s_i, \text{ with } \forall i \in I \colon s_i \in S_i \text{ and } \underset{\psi}{\bowtie} \{\alpha(s_B) \mid s_B \in s\} \neq \emptyset \,\}.$$

Moreover, for $s \in \overset{\tau}{\underset{\psi}{\otimes}} \{S_i \mid i \in I\}$, $\lambda(s) = \tau$ and $\alpha(s) = \underset{\psi}{\bowtie} \{\alpha(s_B) \mid s_B \in s\}$. ■ 4.4

A place set $S' = \overset{\tau}{\underset{\psi}{\otimes}} \{S_i \mid i \in I\}$, produced through a place multiplication in an MNB N over N_B and ψ, can be viewed as an MNB with empty transition and arc sets, $(S', \emptyset, \emptyset)$, over the same N_B and ψ. Depending on the context, we will identify S' either with MNB $(S', \emptyset, \emptyset)$ or with a pure set of nodes, i.e., S' itself. We will also omit ψ and/or τ if they are clear from context. See Table 5 for examples.

$S_i, i \in I$	Parameter set ψ	$\underset{\psi}{\overset{\{e\}}{\otimes}} \{S_i \mid i \in I\}$
$S_1 = \{\{s_1\}\overset{(\{e\}\mid\{\{1\}\})}{\bigcirc}\}$ $(I = \{1\})$	$\{\{1\}\}$	$\{\{s_1\}\overset{(\{e\}\mid\{\{1\}\})}{\bigcirc}\}$ $(= S_1)$
$S_1 = \{\{s_1\}\overset{(\{e\}\mid\{\{1\},\{2\}\})}{\bigcirc}, \{s_2\}\overset{(\{x\}\mid\{\{2\}\})}{\bigcirc}\}$, $S_2 = \{\{s_3\}\overset{(\{e\}\mid\{\{1\},\{3\}\})}{\bigcirc}\}$ $(I = \{1,2\})$	$\{\{1\}\}$	$\{\{s_1, s_3\}\overset{(\{e\}\mid\{\{1\},\{2,3\}\})}{\bigcirc} \quad \{s_2, s_3\}\overset{(\{e\}\mid\{\{2,3\}\})}{\bigcirc}\}$
$S_1 = \{\{s_1\}\overset{(\{e\}\mid\{\{1\},\{2\}\})}{\bigcirc}, \{s_2\}\overset{(\{x\}\mid\{\{2\}\})}{\bigcirc}\}$, $S_2 = \{\{s_3\}\overset{(\{e\}\mid\{\{3\},\{4\}\})}{\bigcirc}\}$ $(I = \{1,2\})$	$\{\{2\}\}$	$\{\{s_1, s_3\}\overset{(\{e\}\mid\{\{1,3\},\{1,4\}\})}{\bigcirc}\}$

Table 5. Examples of place multiplication.

Proposition 4.2. (Properties of place multiplication) *Let (S, T, ι) be an MNB, $I{\neq}\emptyset$, and $\forall i \in I\colon S_i \subseteq S$. Let $\tau, \tau_1 \in \{\{e\}, \{x\}, \{e,x\}, \{i\}\}$. Then,*

(1) *(Associativity) If $\emptyset{\neq}K{\subseteq}I$, then*

$$\underbrace{\overset{\tau}{\otimes} \{S_i \mid i{\in}I\}}_{\text{(C)}} = \underbrace{\overset{\tau}{\otimes} (\{S_j \mid j{\in}I \backslash K\} \cup \{ \overset{\tau_1}{\otimes} \{S_k \mid k{\in}K\}\})}_{\text{(D)}}.$$

(2) *(Distributivity w.r.t. set union) If $\forall i{\in}I\colon S_i = \bigcup\{S_i^{(j_i)} \mid j_i{\in}J_i\}$, where $J_i{\neq}\emptyset$ is an index set, then $\overset{\tau}{\otimes} \{S_i \mid i{\in}I\} = \bigcup\{S' \mid S' {=} \overset{\tau}{\otimes} \{S_i^{(j_i)} \mid i{\in}I\}$ with $\forall i{\in}I\colon j_i{\in}J_i\}$.*

(3) *(Coherence w.r.t. unfolding)* $\underbrace{\mathcal{U}(\overset{\tau}{\otimes} \{S_i \mid i{\in}I\})}_{S} \cong \underbrace{\overset{\tau}{\otimes} \{\mathcal{U}(S_i) \mid i{\in}I\}}_{S'}$.

Proof:

(1) (C) $\subseteq$ (D): Let $s{\in}$(C). By Definition 4.4, $s{=} \sum\limits_{i \in I} s_i$ with $s_i{\in}S_i$ (for all i), $\lambda(s){=}\tau$ and $\alpha(s){=} \underset{\psi}{\bowtie} \{\alpha(s_B) | s_B{\in}s\}$ (which, by a previous remark, equals $\underset{\psi}{\bowtie} (\sum\limits_{s_B \in s} \{\alpha(s_B)\}))$. By $I{=}(I\backslash K){\cup}K$, $s{=} \sum\limits_{j \in I\backslash K} s_j + \underbrace{\sum_{k \in K} s_k}_{s_K}$. Define $\alpha_K {=} \underset{\psi}{\bowtie} \sum\limits_{s_B \in s_K} \{\alpha(s_B)\}$.

Then by Definition 4.4, $s_K \in \overset{\tau_1}{\otimes} \{s_k \mid k \in K\}$ and $\lambda(s_K)=\tau_1$, $\alpha(s_K)=\alpha_K$. By Proposition 3.1(1)($\subseteq$), $\alpha(s) = \underset{\psi}{\bowtie} \big(\bigcup_{j \in I \setminus K} \{\alpha(s_j)\} \cup \{\alpha_K\}\big)$. Hence $s \in$(D) by Definition 4.4, since τ overrides τ_1.

(D) $\subseteq$ (C):] Similar, using Proposition 3.1(1)($\supseteq$).

(2) Proved analogously to the above, with the use of the distributivity of the type construction operator (Proposition 3.1(2)). Notice that the same places may originate from different combinations; however, duplicates are eliminated by the union, due to its idempotence.

(3) We need to construct a renaming $\rho: S \rightarrow S'$ which is a bijection (all other properties of Definition 4.2 are automatically satisfied). Let $s^v \in S$; we determine $\rho(s^v)$. By Definition 4.4, s^v stems from a place s such that $s = \sum_{i \in I} s_i$ with $s_i \in S_i$, and $\lambda(s)=\tau$, and $v \in \underset{\psi}{\bowtie} \big(\sum_{s_B \in s} \{\alpha(s_B)\}\big)$; by Def. 3.3, $v \in$(A) or $v \in$(B) (in the latter case, $v = \bigcup\{v^i \mid i \in I\}$). Define $r^v = \sum_{i \in I} r_i^{v_i}$ with $r_i^{v_i} \in \mathcal{U}(S_i)$, and, for all i, $r_i = v$ if $v \in$(A) or $r_i = v^i$ if $v \in$(B). Then $\forall i$: $v_i \in \sum_{s_B \in s_i} \{\alpha(s_B)\}$; and we may put $\rho(s^v) = r^v$. ■ 4.2

Note that in the case of place multiplication, algebraic properties hold as identities, but coherence with respect to unfolding holds only as isomorphism ($\cong$, as defined in Definition 2.4). The reason is that, when unfolding is performed *after* multiplication, each place name (which is a set) is indexed as a whole by a single value, but if unfolding is done *before* multiplication, each contributing place name is already indexed on its own, hence the parts of the resulting combined name have separate indices.

Definition 4.5. (Restriction) Let $N=(S,T,\iota)$ be an MNB over (S_B,T_B,ι_B), and let Y be a subset of $\mathcal{M}_f(S_B) \cup T_B$. Then N **rs** $Y \overset{\text{def}}{=} (S \setminus Y, T \setminus Y, \iota|_{S' \cup (S' \times T') \cup (T' \times S') \cup T'})$. ■ 4.5

N **rs** Y is again an MNB over the same N_B and ψ as was N. If $Y = \{y\}$, we sometimes write N **rs** y instead of N **rs** $\{y\}$. Figure 8 gives some examples of restriction applied to the MNB N_2 over N_B, ψ (from Figure 7): $N_3 = N_2$ **rs** $\{s_1, s_2\}$, $N_4 = N_2$ **rs** t_1 **rs** $\{s_3\}$.

Proposition 4.3. (Properties of restriction) *Let $N = (S,T,\iota)$ be an MNB over ψ and $N_B = (S_B,T_B,\iota_B)$. If $S' \subseteq \mathcal{M}_f(S_B)$, $Y, Y_1, Y_2 \subseteq S \cup T$ then:*

(1) *(Empty set is a unit element) N **rs** $\emptyset = N$.*
(2) *(Idempotence) $(N$ **rs** $Y)$ **rs** $Y = N$ **rs** Y.*
(3) *(Commutativity) $(N$ **rs** $Y_1)$ **rs** $Y_2 = (N$ **rs** $Y_2)$ **rs** Y_1.*
(4) *(Context insensitivity) If $Y \cap (S \cup T) = \emptyset$ then N **rs** $Y = N$.*
(5) *(Distributivity w.r.t. net union) If N_1 is an MNB and N, N_1 are disjoint, then $(N \cup N_1)$ **rs** $Y = (N$ **rs** $Y) \cup (N_1$ **rs** $Y)$.*
(6) *(Coherence w.r.t. unfolding) $\mathcal{U}(N$ **rs** $Y) \cong \mathcal{U}(N)$ **rs** $\mathcal{U}(Y)$.*

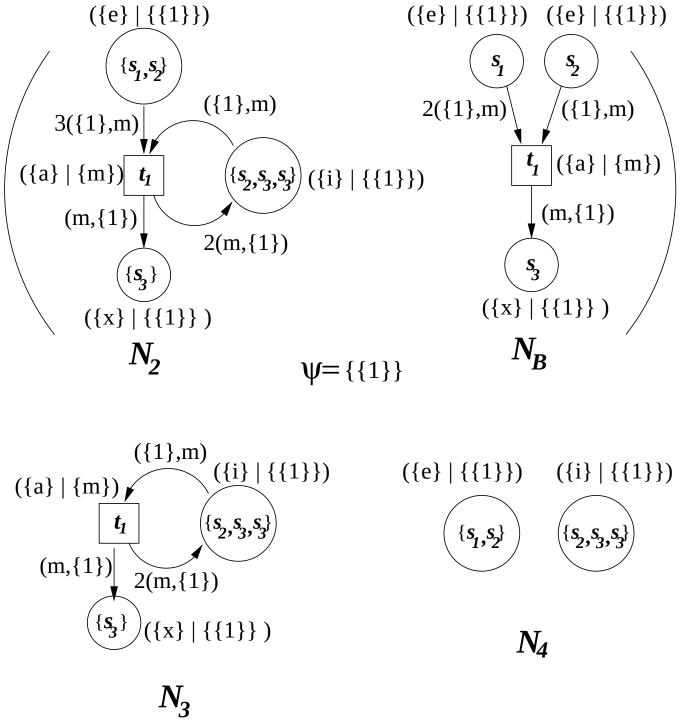


Figure 8. Illustrating node restriction $N_3 = N_2$ **rs** $\{s_1, s_2\}$ and $N_4 = N_2$ **rs** t_1 **rs** $\{s_3\}$.

Proof: (1)–(5) follow from their set-theoretic counterparts: $S\backslash\emptyset = S$ and $T\backslash\emptyset = T$ for (1); $(S\backslash Y)\backslash Y = S\backslash Y$ (and the same for T) for (2); $(S\backslash Y_1)\backslash Y_2 = (S\backslash Y_2)\backslash Y_1$ (and the same for T) for (3); $Y{\cap}S{=}\emptyset \Rightarrow S\backslash Y{=}S$ (and the same for T) for (4); and $(S{\cup}S_1)\backslash Y = (S\backslash Y){\cup}(S_1\backslash Y)$ (and the same for T) for (5). (6) can be proved using the restriction of bijection ρ from the proof of Lemma 4.1 to $\mathcal{U}(S\backslash Y){\times}\mathcal{U}(T\backslash Y)$. ■ 4.3

5. Net composition operations

We consider the most important and widespread – in net theoretic and process algebraic contexts – composition operators: parallel composition ($\|$), sequential composition (;), choice ($\Box$), iteration, and transition refinement. (Transition-based operators such as synchronisation can be considered as orthogonal [2, 3].) The application of each of the above operators, except $\|$ (which is more trivial), can be considered as split into phases, which correspond to the application of the auxiliary operators introduced earlier: $\cup$, $\otimes$ and **rs** . Thus, the general scheme is as follows:

(1) The union of the participating nets is formed by $\cup$.
(2) From the interface places of the participating nets, new places are constructed by place multiplication $\otimes$.

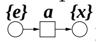
(3) These new places are added to the existing union of the participating nets by another use of $\cup$.

(4) The original interface places of the participating nets, having accomplished their 'mission', are thrown away by restriction **rs** .

Which interface sets are involved in step (2), and how exactly they are impacted under different operations, is determined by the underlying semantics of the intended composition.

To motivate sequential composition, suppose that two disjoint nets N_1 and N_2 have $\{e\}$-places, $\{x\}$-places, but no $\{e,x\}$-places. Then $N_1; N_2$ needs to be constructed by first taking the union of the two nets, and then $\otimes$-multiplying the $\{x\}$-places of N_1 with the $\{e\}$-places of N_2, parameterised by ψ. If N_1 or N_2 or both have $\{e,x\}$-places, then the $\{e,x\}$-places of the first net, while being combined with the pure entry places of the second one, 'exhaust' their exit capability, but retain the entry capability; the resulting places should thus be treated as *entry* for the resulting net. Analogously, the $\{e,x\}$-places of the second net retain their x-capability. When $\{e,x\}$-places of both nets are combined together, these new places are consequently treated as $\{e,x\}$-places.

Under choice composition, the symmetry of the treatment of the nets implies that, as soon as a token has entered an $\{e,x\}$-place of one net, it can at once be considered as contained in an exit place of this net and consequently as being at the level of the exit places of the other net. Systematically, therefore, the $\{e,x\}$-places of each net have to be combined with both pure entry and pure output places of the other net. Thus, for instance, 

in choice with 

is the same as an iteration, a^*, of a.[1]

Under iteration, the pure entry and exit sets of a net are merged either directly, or via a place of some auxiliary net, when an outcome without combined $\{e,x\}$-places is preferred. The distinction between these two versions of iteration (and the reasons for introducing them) will be discussed in Section 5.2.

Under refinement, the intuition is similar to that corresponding to the choice, a difference being, however, that the (pure) entry, exit and side places of the refined transition now play the role of the (pure) entry, exit, and $\{e,x\}$-places of a net in the choice composition, respectively.

In step (2) of the general scheme, we will use place multiplication, and this will assume that the sets of places to be multiplied are ψ-compatible (recall Definition 4.4). In order to guarantee this property, however, we need to assume that the participating sets satisfy a stronger property than just being disjoint. One of the reasons is the same as discussed in Figure 7: net union may lead to connections due to the underlying basis; and this is what is desired in step (3) of the general scheme, but it is not desired in step (1). We call this stronger property basis-disjointness, and it is defined as follows.

[1]This perhaps somewhat unexpected effect stresses the power of $\{e,x\}$ places.

Definition 5.1. (Basis-disjoint MNB's) Let N_1 and N_2 be MNB's over the same ψ and bases $N_{B,1}$ and $N_{B,2}$, respectively. Assume $\widetilde{S_{B,i}} = \bigcup\{s \mid s \in S_i\}$, for $i=1,2$. N_1 and N_2 are called basis-disjoint if $\widetilde{S_{B,1}}$, $\widetilde{S_{B,2}}$, T_1, T_2, $\alpha(\widetilde{S_{B,1}})\setminus\psi$, $\alpha(\widetilde{S_{B,2}})\setminus\psi$ are mutually disjoint, and $\forall y_1 \in \widetilde{S_{B,1}} \cup T_1\ \forall y_2 \in \widetilde{S_{B,2}} \cup T_2$: $\iota_{B,1}(y_1, y_2) = \emptyset = \iota_{B,2}(y_1, y_2)$. ■ 5.1

Note that the above definition implies only the disjointness of exactly those sets of basic places from which the places of the concerned MNB are composed (i.e., $\widetilde{S_{B,1}}$ and $\widetilde{S_{B,2}}$), rather then the disjointness of the entire bases ($S_{B,1}$ resp. $S_{B,2}$). The same holds with respect to transition sets: only disjointness of T_1 and T_2 rather then $T_{B,1}$ and $T_{B,2}$. Moreover, in the following we will assume the existence of a universal common basis for all MNB. In this context, the above definition implies that basis-disjoint MNB are constructed from *disjoint and non-adjacent subnets* of the common basis.

We choose the common basis setup for the following reasons. First, it fits our intention to have an 'alphabet' for MNB construction. Second, it allows us to introduce a notion of MNB union without requiring an accompanying union of the bases. Third, we did not have to define a union of *non-disjoint* nets, not even for step (3) of the general scheme.

It is clear that if N_1 and N_2 are basis-disjoint then they are disjoint (in the standard sense of net disjointness, defined in Section 2). If N_1 and N_2 are disjoint but not basis-disjoint, one can apply place and/or value renaming until the results are weakly isomorphic to the original nets and basis-disjoint. Hence basis-disjointness is not a severe restriction.

Lemma 5.1 below reveals a further, very important consequence of basis-disjointness: the conformability of types of basis-disjoint MNB's; this property ensures that step (2) of the general scheme can be carried out.

Lemma 5.1. (Conformability of place types of basis-disjoint MNBs) *Let $N_i=(S_i, T_i, \iota_i)$, $i \in I$, be basis-disjoint MNB's over the same ψ. Then $\{\alpha(S_i) \mid i \in I\}$ is ψ-conformable.*

Proof: Straightforward from Definitions 3.1 and 5.1. ■ 5.1

The example shown in Figure 9 illustrates another reason why basis-disjointness – rather than disjointness – is imperative. Nets N_1 and N_2 are disjoint, but not basis-disjoint. Some composition, e.g., sequence, involves, in step (3), union with the outcome of the multiplication (step (2)) of the interface place sets $\{\{s_1\}\}$ and $\{\{s_2\}\}$. However, this results in duplicate nodes, since a place $\{s_1, s_2\}$, equal to that contained in the outcome of the multiplication, already exists in N_1. Moreover, these duplicates might have different statuses, leading to ambiguity and destroying consistency in performing compositions, in particular, when eliminating interface nodes. Every place arising through the place multiplication under the assumption of basis-disjointness, by contrast, inevitably contains places belonging to the base of *each* argument net, and, consequently, cannot coincide with any place of the these nets. Hence, the mutual disjointness of all

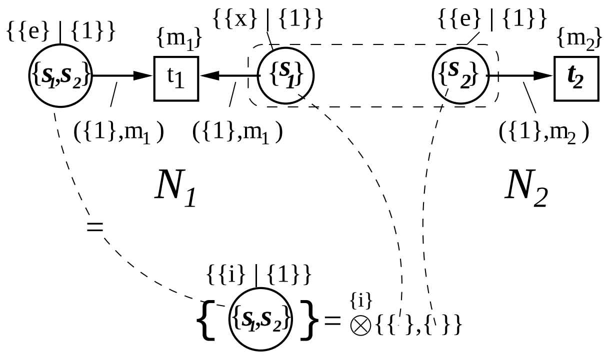


Figure 9. Motivating the necessity of basis-disjointness.

the arguments of any composition operator (as required by the always incorporated operator $\cup$) is guaranteed at all stages of the constructions.

The restriction operator (in the forthcoming definitions) eliminates all places belonging to the sets partaking in place multiplications. It may happen that some weak value might not be present in the types of the emerging places because in each product at least one place set did not contain any place with this value in the type, or strong values might not contribute to new types if in each product there was a set with empty strong part of the type. After the restriction of the original places such values will be lost. This may lead to not immediately expected behavioural effects; for example, in Figure 10, transition t_2 becomes live (Figure 10(c)).

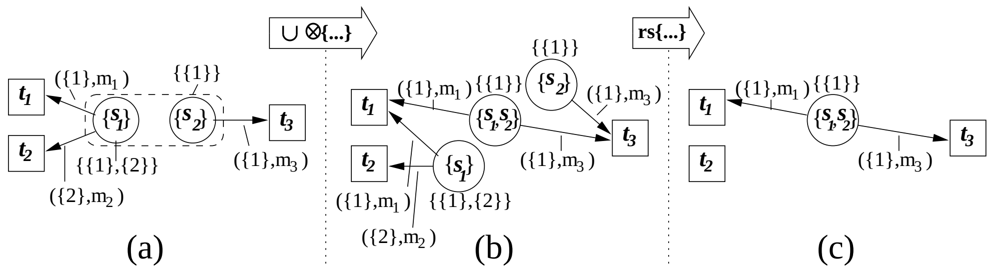


Figure 10. Illustrating a 'lost' value ($\psi = \{\{1\}\}$).

A similar effect has been discussed in [6]: Consider the sequential composition of a net N_1 without $\{x\}$ place followed by a net N_2 with an $\{e\}$ place. Should the latter be retained, possibly with an $\{i\}$ label, in the resulting net $N_1; N_2$? We have adopted the rule that it will not be retained, on the formal grounds that a cartesian product with one $\emptyset$ factor is itself the empty set. Empty participating sets can easily be prevented by requiring that the types of the participating (in place multiplication) sets have identical weak parts, and their strong parts are either all empty or all nonempty. The ex-restrictedness property of [2] (which says that every net has at least one $\{e\}$ place and at least one $\{x\}$ place) serves a similar purpose.

5.1. Parallel, sequential and choice compositions

Parallel composition of two basis-disjoint nets is synonymous to the union of MNB's; the former serves as an independent, the latter as an auxiliary, operator.

Definition 5.2. (Parallel composition)
Let N_1 and N_2 be two disjoint MNB's over the same ψ. Then $N_1 \parallel N_2 \stackrel{\text{def}}{=} N_1 \cup N_2$. ■ 5.2

Parallel composition is idempotent, commutative, associative, and coherent with respect to unfolding (cf. Proposition 4.1).

Definition 5.3. (Sequential composition)
Let N_1 and N_2 be two basis-disjoint MNB's over the same ψ. Then,

$$\begin{aligned} N_1; N_2 \stackrel{\text{def}}{=} & \; (\, (N_1 \cup N_2) \cup \overset{\{i\}}{\otimes} \{N_1^\circ, {}^\circ N_2\} \cup \overset{\{e\}}{\otimes} \{S_1^{\{e,x\}}, {}^\circ N_2\} \\ & \cup \overset{\{x\}}{\otimes} \{N_1^\circ, S_2^{\{e,x\}}\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \,) \ \mathbf{rs} \ (N_1^\bullet \cup {}^\bullet N_2). \end{aligned}$$

■ 5.3

This definition can be read informally as follows: take the union of the two participating nets N_1 and N_2; then multiply the pure $\{x\}$ places of N_1 with the pure $\{e\}$ places of N_2 and make them internal; also multiply the $\{e,x\}$ places of N_1 with the pure $\{e\}$ places of N_2 and make them $\{e\}$; also multiply the pure $\{x\}$ places of N_1 with the $\{e,x\}$ places of N_2 and make them $\{x\}$; also multiply the $\{e,x\}$ places of N_1 with the $\{e,x\}$ places of N_2 and make them $\{e,x\}$; finally get rid of the original $\{x\}$ interface of N_1 and the original $\{e\}$ interface of N_2.

If there are no $\{e,x\}$ places in either net, then only the first of the four disjuncts remains in this definition, and the formula can be simplified as follows:

$$N_1; N_2 \; = \; (\, (N_1 \cup N_2) \cup \overset{\{i\}}{\otimes} \{N_1^\circ, {}^\circ N_2\} \,) \ \mathbf{rs} \ (N_1^\bullet \cup {}^\bullet N_2).$$

Theorem 5.1. (Properties of sequential composition) *Let N_1, N_2, and N_3 be mutually basis-disjoint MNB's over the same ψ. Then the following hold:*

(1) *(Associativity)* $(N_1; N_2); N_3 = N_1; (N_2; N_3)$.
(2) *(Coherence w.r.t. unfolding)* $\mathcal{U}(N_1; N_2) \cong \mathcal{U}(N_1); \mathcal{U}(N_2)$. ■ 5.1

This is the first of a series of similar results. For their proofs, we refer to the appendix of this paper, and to [5]. Note that the coherence with respect to unfolding involves isomorphism rather than identity. This is due to the fact that it relies on Proposition 4.2(3).

In order to appreciate the behavioural properties of the above definition of sequential composition, consider the example in Figure 11, where only place statuses and transition labels

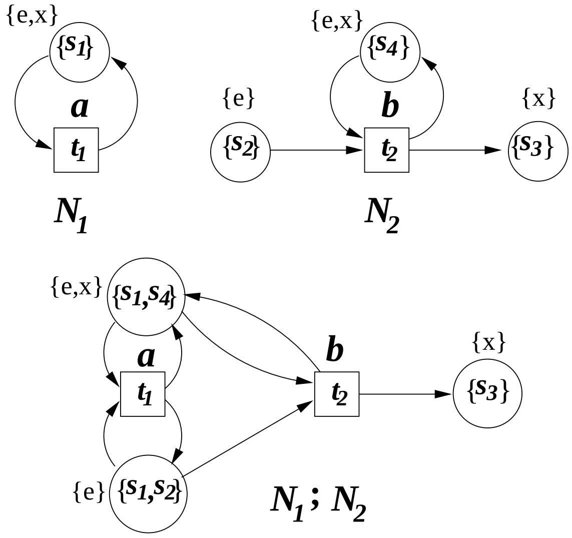


Figure 11. Motivating place connections under sequential composition.

are shown; the types of all places are assumed to be the same singleton set, e.g., $\{\{0\}\}$ (this corresponds to ordinary place/transition nets).

The rationale behind the above definition in its $\{e,x\}$-places-related part, in addition to being systematically motivated in Section 5, can be supported by behavioural reasons. Since the initial marking of N_1 is also final, and thus can instantaneously be re-interpreted as the initial marking of N_2 (tokens on $\{s_2\}$ and $\{s_4\}$), it can be expected that, under the $\{0\}$–marking of the entry places (i.e., those having status $\{e\}$ or $\{e,x\}$), the firing sequences in the sequential composition of nets N_1 and N_2 are a^*b. In the right-hand side of the figure, net $N_1;N_2$ is shown, obtained corresponding to Definition 5.3. It can be checked that $N_1;N_2$, with the initial marking consisting of tokens $\{0\}$ on $\{s_1,s_2\}$ and $\{s_1,s_4\}$, produces the same firing sequences.

Definition 5.4. (Choice composition) Let N_1 and N_2 be basis-disjoint MNB's over the same ψ, and let $\{{}^\circ N_1, N_1^\circ\}$ and $\{{}^\circ N_2, N_2^\circ\}$ be ψ-compatible. Then,

$$N_1 \,\Box\, N_2 \stackrel{\mathrm{def}}{=} (\ (N_1 \cup N_2) \cup \overset{\{e\}}{\otimes} \{{}^\circ N_1, {}^\circ N_2\} \cup \overset{\{x\}}{\otimes} \{N_1^\circ, N_2^\circ\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}$$
$$\cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, {}^\circ N_2, N_2^\circ\} \cup \overset{\{e,x\}}{\otimes} \{S_2^{\{e,x\}}, {}^\circ N_1, N_1^\circ\}\ )\ \mathbf{rs}\ ({}^\bullet N_1 \cup N_1^\bullet \cup {}^\bullet N_2 \cup N_2^\bullet).$$

■ 5.4

This definition can be read in a similar way as the previous one, and it can also be simplified in the absence of $\{e,x\}$ places. The condition that $\{{}^\circ N_1, N_1^\circ\}$ (respectively, $\{{}^\circ N_2, N_2^\circ\}$) be ψ-compatible does not follow from the basis-disjointness of the two nets, but it is important to make place multiplication applicable if there are $\{e,x\}$-places in N_2 (N_1, respectively). A similar condition does not occur in the definition of sequential composition, because only places from different nets are multiplied with each other.

Theorem 5.2. (Properties of choice composition) *Let N_1, N_2, and N_3 be mutually basis-disjoint MNB's over the same ψ such that $\{{}^\circ N_1, N_1^\circ\}$, $\{{}^\circ N_2, N_2^\circ\}$ and $\{{}^\circ N_3, N_3^\circ\}$ are ψ-compatible. Then the following hold:*

(1) *(Commutativity)* $N_1 \square N_2 = N_2 \square N_1$.
(2) *(Associativity)* $(N_1 \square N_2) \square N_3 = N_1 \square (N_2 \square N_3)$.
(3) *(Coherence w.r.t. unfolding)* $\mathcal{U}(N_1 \square N_2) \cong \mathcal{U}(N_1) \square \mathcal{U}(N_2)$. ■ 5.2

Commutativity already follows from the commutativity of restriction (Proposition 4.3(3)). For the other proofs, see [5].

5.2. Iteration

We define and motivate two versions of iteration, iteration-I and iteration-II. Iteration-I (Definition 5.5) corresponds to directly linking the pure input and pure output places of a net; the new places obtain status $\{e, x\}$, and the places which originally had $\{e, x\}$-status remain unchanged. Thus the interface places of the outcome of iteration-I are all of $\{e, x\}$-type. This definition is primarily inspired by an intention to gain an idempotence property (Theorem 5.3(1)), but also by the wish to obtain a 'small' translation without extra transitions.

Definition 5.5. (Iteration-I) Let N be an MNB over ψ, and $\{{}^\circ N, N^\circ\}$ be ψ-compatible. Then

$$*N \stackrel{\text{def}}{=} \left(N \cup \overset{\{e,x\}}{\otimes} \{{}^\circ N, N^\circ\} \right) \ \mathbf{rs}\ ({}^\circ N \cup N^\circ). \qquad ■\ 5.5$$

Theorem 5.3. (Properties of iteration-I) *Let N be an MNB. Then:*

(1) *(Idempotence)* $*(*N) = *N$.
(2) *(Coherence w.r.t. unfolding)* $\mathcal{U}(*N) \cong *(\mathcal{U}(N))$. ■ 5.3

(1) follows from the fact that all interface places of $*N$ have status $\{e, x\}$ and no further changes can be undertaken by further iterations. For the proof of (2), see [5].

Iteration-I may lead to a behavioural problem in case of a sequential composition of two nets corresponding to iterations a^* and b^*, i.e., the sequence $a^*; b^*$. Suppose that a^* is modelled by N_1 in Figure 11, and similarly b^* by another such net whose transition is labelled b. This corresponds to applying iteration-I in both cases. Then our operator for sequential composition yields a net with two transitions and a single place, with status $\{e, x\}$, which is a side place for both of transitions, i.e., a net with behaviour $(a|b)^*$ (written as a regular expression). That is, in the result, a behaviour such as $bbaaa\ldots$ is possible, which is not usually considered to be a behaviour of $a^*; b^*$.

Iteration-II (Definition 5.7) avoids such an effect by 'guarding' its argument by silent transitions. It is aimed at the case in which the resulting net has to have only pure entry and exit interfaces.

For this purpose we introduce an auxiliary ('pending') net (Definition 5.6) whose entry and exit places adopt the function of the interface places of the resulting net, while the places originating with the participation of the former interface places of the (iterated) net obtain internal status $\{i\}$.

Definition 5.6. (Pending interface net for iteration-II) MNB $N'{=}(S',T',\iota')$ over N_B, ψ is called a pending interface net for iteration-II if

$$
\begin{array}{rcl}
S' & = & \{\{s'\},\{s_0\},\{s''\}\},\\
T' & = & \{t',t''\},\\
\iota' & : & \left\{\begin{array}{rcl}
\iota'(\{s'\}) & = & (\{e\} \mid \xi'),\\
\iota'(\{s''\}) & = & (\{x\} \mid \xi'),\\
\iota'(\{s_0\}) & = & (\{i\} \mid \xi'),\\
\iota'_B(t') = \iota'(t'') & = & (\emptyset \mid MOD'),\\
\iota'(\{s'\},t') = \iota'(\{s_0\},t'') & = & \sum\limits_{v_i\in\xi'} (v_i, m_{v_i}) \quad (=\Theta_1),\\
\iota'(t',\{s_0\}) = \iota'(t'',\{s''\}) & = & \sum\limits_{v_i\in\xi'} (m_{v_i}, v_i) \quad (=\Theta_2).
\end{array}\right.
\end{array}
$$

where $MOD \supseteq MOD' = \{m_{v_i} \mid v_i \in \xi'\}$. ■ 5.6

N' is so constructed that each mode in it transfers exactly a single value.

Definition 5.7. (Iteration-II) Let $N{=}(S,T,\iota)$ and $N'{=}(S',T',\iota')$ be MNB's over N_B, ψ. Let $\{{}^\circ N, N^\circ, S^{\{e,x\}}\}$ be ψ-compatible, and $\xi{=} \underset{\psi}{\bowtie} \{\alpha({}^\circ N), \alpha(N^\circ)\} \cup \alpha(S^{\{e,x\}})$. Let N' be a pending interface net as in Definition 5.6, with

$$\xi' = \left\{\begin{array}{ll} \xi & \text{if } \xi\setminus\psi{=}\emptyset,\\ (\xi\cap\psi)\cup\{v'\}, & \text{where } (v'\in VAL\setminus\psi)\wedge(\{\xi,\{v'\}\} \text{ is } \psi\text{-conformable}), \text{ otherwise.}\end{array}\right.$$

Then, $\circledast N \overset{\text{def}}{=} (\ (N \cup N') \cup \overset{\{i\}}{\otimes} \{\ {}^\circ N, N^\circ, \{\{s_0\}\}\ \} \cup \overset{\{i\}}{\otimes} \{\ S^{\{e,x\}}, \{\{s_0\}\}\ \}\ )\ \mathbf{rs}\ ({}^\bullet N \cup N^\bullet \cup \{s_0\}\ )$.

■ 5.7

The auxiliary strong value v' in the above definition is used for the strong values in the to-be-iterated net; it participates in creating each new strong value. Without v', under the place multiplication with $\{s_0\}$ the strong parts of the types of the interface places of the original net would get lost. Note that, according to Definitions 5.6 and 5.7, transitions t' and t'' (originating from the pending net) obtain in $\circledast N$ different individual modes for each weak value and a single, common mode for all the strong values.

Figures 12 and 13 illustrate the result of applying the two iteration operators to the same two nets: without $\{e,x\}$-places (Figure 12) and with $\{e,x\}$-places (Figure 13). Components '5' in the strong values in nets from Figures 12,13 have resulted from combinations with an auxiliary value $\{5\}$ which represented v' in the corresponding pending nets.

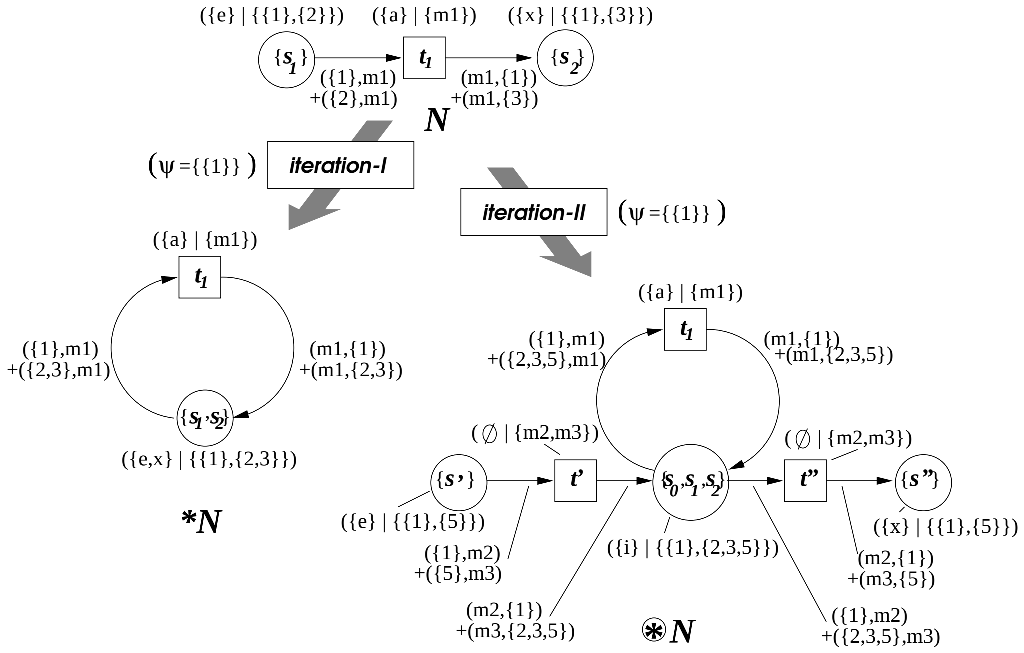


Figure 12. Iteration-I and iteration-II of a net without $\{e, x\}$-places.

Theorem 5.4. (Property of iteration-II) *Let N be an MNB. Then*
(Coherence of w.r.t. unfolding) $\mathcal{U}(\circledast N) \cong \circledast(\mathcal{U}(N))$. ■ 5.4

For the proof, see [5].

Returning to the previous example $a^*; b^*$, suppose now that a^* is modelled using iteration-II. This yields a net with an a-labelled transition and a side place s to it, as before. However, s now has status $\{i\}$, and a silent (i.e., $\emptyset$-labelled) transition leads from an $\{e\}$-labelled place to s, and another silent transition leads from s to an $\{x\}$-labelled place. Suppose that b^* is modelled similarly. Then the sequential composition yields a net having behaviour $\emptyset a^* \emptyset\emptyset b^* \emptyset$ (i.e., the usually assumed behaviour of $a^*; b^*$ if silent transitions are neglected).

This example, $a^*; b^*$, cannot be interpreted as a deficiency in the definition of sequential composition, or of iteration. Rather, the effect is inherent in the M-net framework, since it can be shown that there is no M-net without silent transitions whose behaviour is exactly that of $a^*; b^*$. This corresponds to the fact that there is no finite automaton without silent moves and with only one initial state and one final state that accepts the language a^*b^*.

In [2], this kind of situation has been forbidden by requiring that in a sequential composition, either the first net may not have output arcs from $\{x\}$ places, or the second net may not have input arcs into $\{e\}$ places, or both, and a detailed discussion is also given.

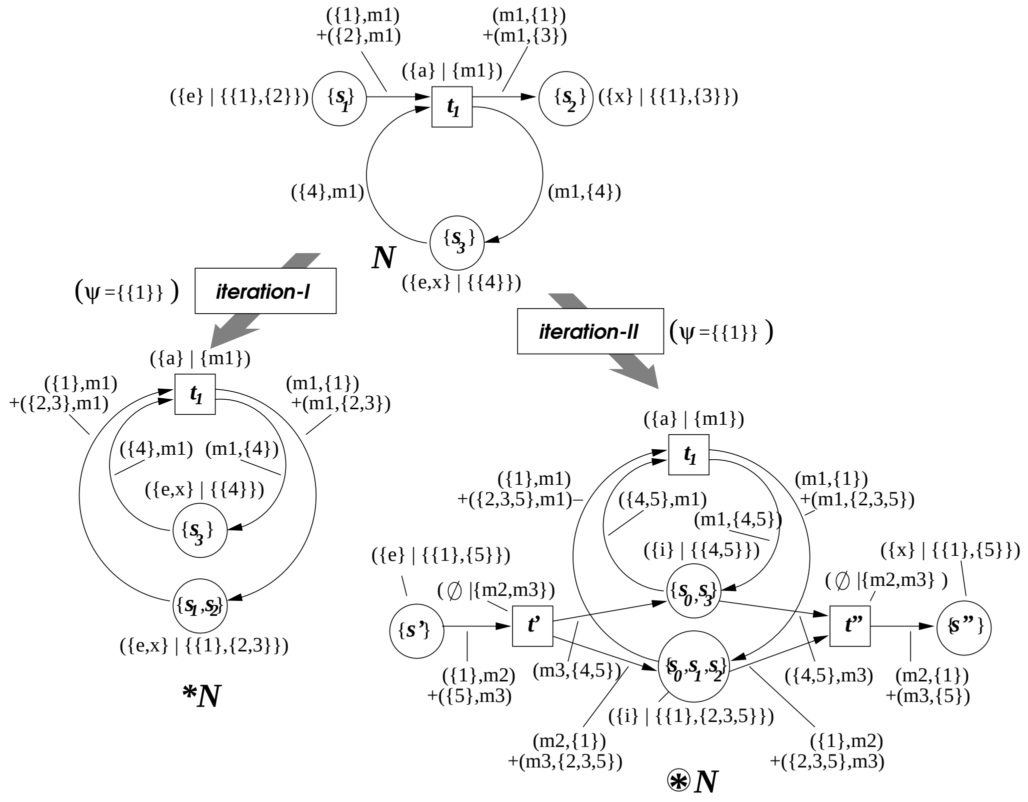


Figure 13. Iteration-I and iteration-II of a net with $\{e, x\}$-places.

5.3. Transition refinement

In the definition of transition refinement, we permit side places for the to-be-refined transition. However we do not allow $\{e, x\}$-places in the refining net, because otherwise the commutativity of the refinement in the general case could be destroyed. Under refinement, the statuses of new places are determined by the statuses of the contributing places from the main net. Under this convention, we do not put any τ over $\otimes$ in the following definition.

We also assume that the transition to be refined has a single mode only. This restriction is inessential. Because of Theorem 5.5 and the possibility of splitting a transition with many modes into a series of transitions with single modes, the definition can be generalised directly to multi-mode transitions.

Definition 5.8. (Single-mode transition refinement) Let N and N_0 be two basis-disjoint MNB's over the same ψ, $t_0 \in T$ with $|\alpha(t_0)| = 1$, ${}^\bullet N_0 \cap N_0^\bullet = \emptyset$, and let $\{{}^\bullet N_0, N_0^\bullet\}$ be ψ-compatible. The net obtained by refining transition t_0 in N with N_0 is defined as:

$$N[t_0 \leftarrow N_0] \overset{\text{def}}{=} (\,(\,(N \textbf{ rs } t_0) \cup N_0\,) \cup \otimes\{{}^\circ t_0, {}^\bullet N_0\} \cup \otimes\{t_0^\circ, N_0^\bullet\} \cup \otimes\{S_{t_0}^{(side)}, {}^\bullet N_0, N_0^\bullet\}\,) \textbf{ rs } ({}^\bullet t_0 \cup t_0^\bullet \cup {}^\bullet N_0 \cup N_0^\bullet). \qquad \blacksquare\ 5.8$$

If there are no side places next to transition t_0, then only two disjuncts remain in the above definition. Two examples of transition refinement are shown in Figures 14 and 15. They are of mixed - weak and strong - composition type and represent two different cases from the structural point of view: in the first one, t_0 does not have side places; in the second one, t_0 has a side place, and, in addition, an entry place of N_0 is at the same time a side place of some transition in N_0.

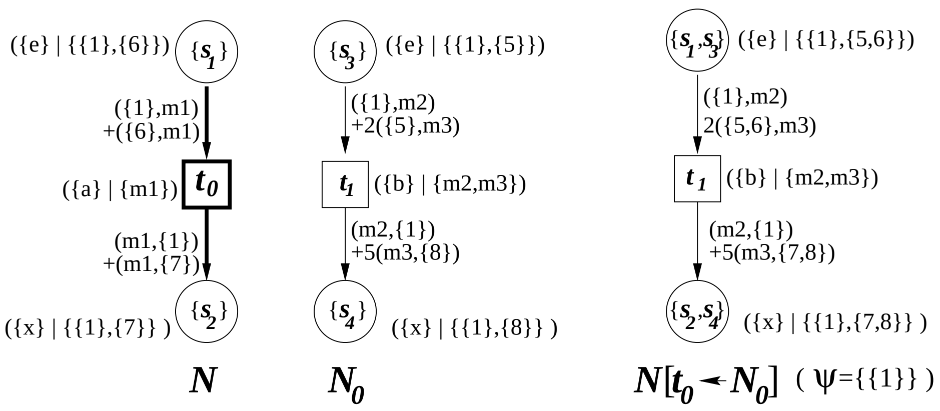


Figure 14. $N[t_0 \leftarrow N_0]$, t_0 without side places.

Theorem 5.5. (Properties of transition refinement) *Let N, N_1, and N_2 be mutually basis-disjoint MNB's. Then the following hold:*

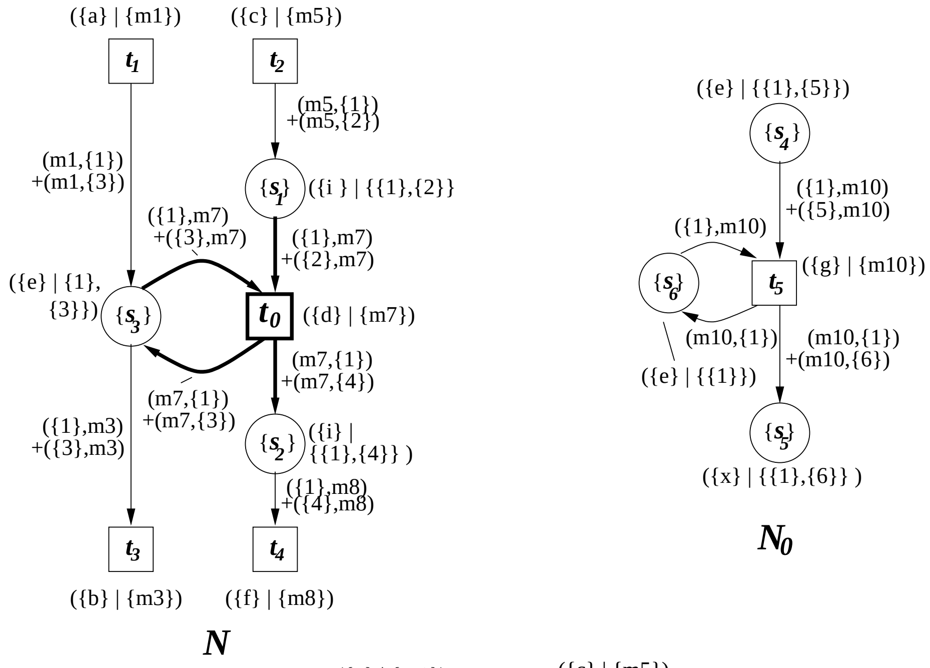
(1) *(Commutativity-I) If $t_1, t_2 \in T$ then* $(N[t_1 \leftarrow N_1])[t_2 \leftarrow N_2] \;=\; (N[t_2 \leftarrow N_2])[t_1 \leftarrow N_1]$.

(2) *(Commutativity-II) If $t_1 \in T$, $t_2 \in T_1$ and* $\left(S_{t_1}^{(side)} = \emptyset\right) \vee \left(({}^\bullet t_2 \cup t_2^\bullet) \cap ({}^\circ N_1 \cup N_1^\circ) = \emptyset\right)$, *then*

$$(N[t_1 \leftarrow N_1])[t_2 \leftarrow N_2] \;=\; N[t_1 \leftarrow (N_1[t_2 \leftarrow N_2])].$$

(3) *(Coherence w.r.t. unfolding)* $\mathcal{U}(N[t_1 \leftarrow N_1]) \;\cong\; \mathcal{U}(N)[t_1 \leftarrow \mathcal{U}(N_1)]$. $\blacksquare$ 5.5

For the proof, see the appendix and [5]. Note that (1) and (2) hold as identities.



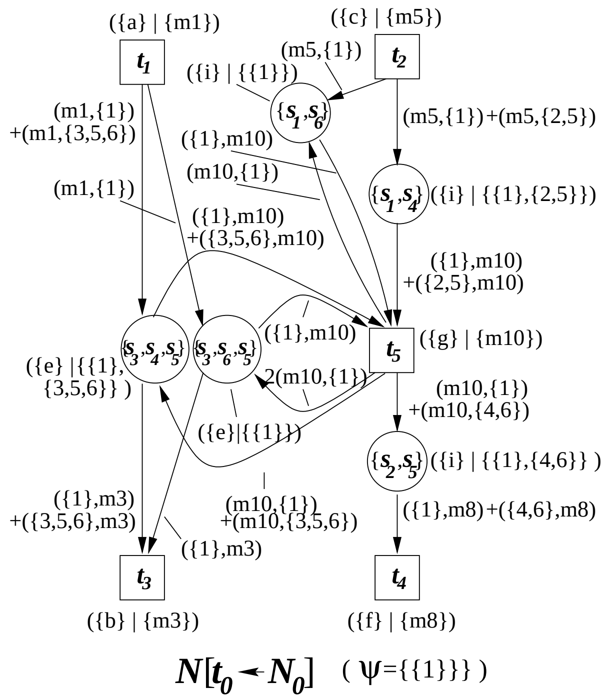


Figure 15. $N[t_0 \leftarrow N_0]$, t_0 with side places.

6. Concluding remarks

We have introduced a set of generic schemes for high-level net composition operations. A uniform approach has allowed us to obtain operations which preserve nice algebraic properties while remaining coherent with respect to net unfolding. Equipped with a parameter value set, the proposed schemes enable us to implement different - strong, weak, and mixed - composition disciplines. These definitions have been justified by examples, by proving the desired algebraic (structural) laws (and also by the fact that in the purely strong case, and neglecting $\{e, x\}$ interfaces, they reduce to better known constructions).

Our approach differs from other research in similar direction, e.g., [6, 7], in particular by the following: (a) combined entry/exit places are allowed, (b) values are sets rather than more complicated labelled trees, (c) associativity and commutativity of the operations hold as equalities, and the coherence with respect to unfolding holds as isomorphism up to place renaming.

A formal behavioural justification of our operations (along the line hinted at in discussing Figure 11) has not been the objective of this paper, but this is a task that is planned to be done in future research. In [10] and [13], Grahlmann and Thielke provide such a formal justification for various sub-theories (without weak values, without intersecting types and without $\{e, x\}$ places).

Other future research includes elaborating place-oriented refinement operations; checking the use of infinite sets to describe recursion based on refinement in the usual style [2, 7]; developing a formal definition of a universal, place- and transition-oriented, refinement scheme; extending it to a 'net-for-subnet substitution' case; and embedding linear algebraic elements as inherent parts into composition operations. We also think that the restriction imposed in Definition 5.8 (that the refining net has no $\{e, x\}$ places) can eventually be removed; however, a direct generalisation fails. We are presently working on ways to overcome this limitation.

Appendix

In this appendix, we give three typical proofs. All other proofs are omitted, including the heftiest one, which is the proof of Theorem 5.5(2). The reader is referred to [5] for missing proofs.

First, we turn to the *associativity of sequential composition* (Theorem 5.1(1)). We have to prove:

$$(N_1; N_2); N_3 = N_1; (N_2; N_3). \tag{1}$$

We start by transforming the left-hand side of equation (1). From the statuses assigned to places in accordance with Definition 5.3 follow the expressions for the $\{e, x\}$- and the pure exit places of $N_1; N_2$:

$$\left.\begin{array}{rcl} (N_1;N_2)^\circ & = & N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \\ S_{N_1;N_2}^{\{e,x\}} & = & \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \end{array}\right\} \quad (2)$$

In addition, we will use the following property of the interface places, which follows directly from their definition (Section 2): if N is an M-net then:

$$^\bullet N \cup N^\bullet = {}^\bullet N \cup N^\circ = {}^\circ N \cup N^\bullet = {}^\circ N \cup N^\circ \cup S^{\{e,x\}} \quad (3)$$

Now we can write:
$(N_1; N_2); N_3$

$$\begin{array}{rl}
= & \text{(by applying Definition 5.3 twice, and using equations (2))} \\
& \Big(\Big(\Big(\Big((N_1 \cup N_2) \\
& \quad \cup(\overset{\{i\}}{\otimes} \{N_1^\circ, {}^\circ N_2\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \cup \overset{\{e\}}{\otimes} \{S_1^{\{e,x\}}, {}^\circ N_2\} \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \\
& \quad\quad)\Big) \ \mathbf{rs}\ (N_1^\bullet \cup {}^\bullet N_2)\Big) \cup N_3\Big) \\
& \cup(\overset{\{i\}}{\otimes} \{(N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\}), {}^\circ N_3\} \cup \overset{\{e,x\}}{\otimes} \{ \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}, S_3^{\{e,x\}}\} \\
& \quad \cup \underset{\psi}{\overset{\{e\}}{\otimes}} \{ \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}, {}^\circ N_3\} \cup \overset{\{x\}}{\otimes} \{S_3^{\{e,x\}}, (N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\})\} \\
&)\Big) \ \mathbf{rs}\ (N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \cup {}^\bullet N_3) \\
= & \text{(by Propositions 4.1(3), 4.3(4,5))} \\
& \Big((N_1 \cup N_2 \cup N_3) \\
& \cup(\overset{\{i\}}{\otimes} \{N_1^\circ, {}^\circ N_2\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \cup \overset{\{e\}}{\otimes} \{S_1^{\{e,x\}}, {}^\circ N_2\} \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \\
& \quad \cup \overset{\{i\}}{\otimes} \{(N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\}), {}^\circ N_3\} \cup \overset{\{e,x\}}{\otimes} \{ \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}, S_3^{\{e,x\}}\} \\
& \quad \cup \overset{\{e\}}{\otimes} \{ \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}, {}^\circ N_3\} \cup \overset{\{x\}}{\otimes} \{S_3^{\{e,x\}}, (N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\})\} \\
& \quad)\Big) \ \mathbf{rs}\ (N_1^\bullet \cup {}^\bullet N_2 \cup N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \cup {}^\bullet N_3) \\
= & \text{(by Proposition 4.2(2))} \\
& \Big((N_1 \cup N_2 \cup N_3) \\
& \cup(\overset{\{i\}}{\otimes} \{N_1^\circ, {}^\circ N_2\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \cup \overset{\{e\}}{\otimes} \{S_1^{\{e,x\}}, {}^\circ N_2\} \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \\
& \quad \cup \overset{\{i\}}{\otimes} \{N_2^\circ, {}^\circ N_3\} \cup \overset{\{i\}}{\otimes} \{ \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\}, {}^\circ N_3\}
\end{array}$$

$$
\begin{array}{ll}
\cup & \overset{\{e,x\}}{\otimes} \{ \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}, S_3^{\{e,x\}}\} \cup \overset{\{e\}}{\otimes} \{ \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\}, {}^\circ N_3\} \\
\cup & \overset{\{x\}}{\otimes} \{S_3^{\{e,x\}}, N_2^\circ\} \cup \overset{\{x\}}{\otimes} \{S_3^{\{e,x\}}, \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\}\} \\
\big)\Big) & \mathbf{rs}\ (N_1^\bullet \cup {}^\bullet N_2 \cup N_2^\circ \cup \overset{\{x\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ\} \cup \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}\} \cup {}^\bullet N_3)
\end{array}
$$

$$
\begin{array}{lll}
= & \text{(by Propositions 4.2(1) and 4.3(4,5), and by (3))} & \\
& \Big((N_1 \cup N_2 \cup N_3) & \\
& \cup(& \overset{\{i\}}{\otimes} \{N_1^\circ, {}^\circ N_2\} \cup \overset{\{e\}}{\otimes} \{S_1^{\{e,x\}}, {}^\circ N_2\} \\
& \cup & \overset{\{i\}}{\otimes} \{N_2^\circ, {}^\circ N_3\} \cup \overset{\{i\}}{\otimes} \{S_2^{\{e,x\}}, N_1^\circ, {}^\circ N_3\} \\
& \cup & \overset{\{e,x\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}, S_3^{\{e,x\}}\} \cup \overset{\{e\}}{\otimes} \{S_1^{\{e,x\}}, S_2^{\{e,x\}}, {}^\circ N_3\} \\
& \cup & \overset{\{x\}}{\otimes} \{S_3^{\{e,x\}}, N_2^\circ\} \cup \overset{\{x\}}{\otimes} \{S_3^{\{e,x\}}, S_2^{\{e,x\}}, N_1^\circ\} \\
& \big)\Big) & \mathbf{rs}\ (N_1^\bullet \cup {}^\bullet N_2 \cup N_2^\bullet \cup {}^\bullet N_3).
\end{array}
$$

By proceeding in an analogous way, it is possible transform the right-hand side of equation (1), and it may be checked that both sides are equal (see [5]).

Next, we turn to the proof of *commutativity-I of transition refinement* (Theorem 5.5(1)). We have to prove $(N[t_1 \leftarrow N_1])[t_2 \leftarrow N_2] = (N[t_2 \leftarrow N_2])[t_1 \leftarrow N_1]$. Each interface place of transition t_1 may at the same time be an interface place of t_2. For example, s_2 in Figure 16(a) is a pure input place for both t_1 and t_2; s_4 is a side place for t_1 and a pure output place for t_2, etc. In accordance with the possible combinations of a place membership in the interface place sets of t_1 and t_2, we can represent these sets as partitions (Figure 16(b)):

$$
\left.
\begin{array}{rcl}
{}^\circ t_1 & = & A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \\
S_{t_1}^{(side)} & = & B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
t_1^\circ & = & C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF}
\end{array}
\right\} \tag{4}
$$

where $H_{AD} = {}^\circ t_1 \cap {}^\circ t_2$, $H_{AE} = {}^\circ t_1 \cap S_{t_2}^{(side)}$, $H_{AF} = {}^\circ t_1 \cap t_2^\circ$, $A_0 = {}^\circ t_1 \setminus (H_{AD} \cup H_{AE} \cup H_{AF})$, $H_{BD} = S_{t_1}^{(side)} \cap {}^\circ t_2$, $H_{BE} = S_{t_1}^{(side)} \cap S_{t_2}^{(side)}$, $H_{BF} = S_{t_1}^{(side)} \cap t_2^\circ$, $B_0 = S_{t_1}^{(side)} \setminus (H_{BD} \cup H_{BE} \cup H_{BF})$, $H_{CD} = t_1^\circ \cap {}^\circ t_2$, $H_{CE} = t_1^\circ \cap S_{t_2}^{(side)}$, $H_{CF} = t_1^\circ \cap t_2^\circ$, $C_0 = t_1^\circ \setminus (H_{CD} \cup H_{CE} \cup H_{CF})$, $D_0 = {}^\circ t_2 \setminus (H_{AD} \cup H_{BD} \cup H_{CD})$, $E_0 = S_{t_2}^{(side)} \setminus (H_{AE} \cup H_{BE} \cup H_{CE})$, $F_0 = t_2^\circ \setminus (H_{AF} \cup H_{BF} \cup H_{CF})$.

Corresponding to the above partition, we can write, for example, for the net from Figure 16(a): $H_{AD} = \{s_2\}$, $H_{AE} = \emptyset$, $H_{AF} = \emptyset$, $A_0 = \{s_1\}$, $H_{BD} = \emptyset$, $H_{BE} = \emptyset$, $H_{BF} = \{s_4\}$, $B_0 = \emptyset$, $H_{CD} = \emptyset$, $H_{CE} = \emptyset$, $H_{CF} = \{s_3\}$, $C_0 = \{s_5\}$, $D_0 = E_0 = F_0 = \emptyset$.

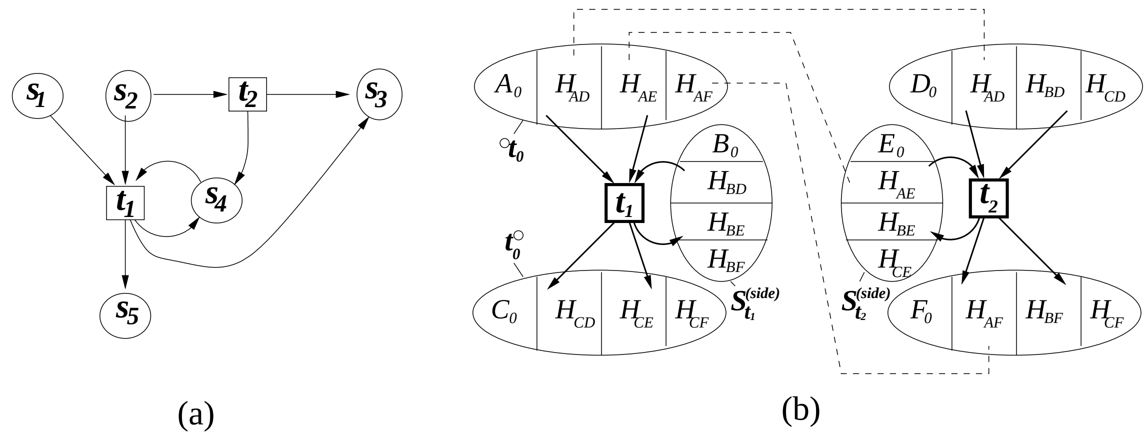


Figure 16. Interface sets of the to-be-refined transitions.

We introduce the following notations for the (pure) input and output places of nets N_1 and N_2:

$$\left.\begin{array}{ll} R_1 = {}^\circ N_1 \ (= {}^\bullet N_1), & R_2 = N_1^\circ \ (= N_1^\bullet), \\ Q_1 = {}^\circ N_2 \ (= {}^\bullet N_2), & Q_2 = N_2^\circ \ (= N_2^\bullet) \end{array}\right\} \tag{5}$$

(equalities such as $N_1^\circ = N_1^\bullet$, etc., in (5) are due to the absence of $\{e, x\}$-places in the refining net, as is required by Definition 5.8).

In the proof, we will sometimes use the following notation for place sets occurring as arguments of a place multiplication operator: if a set is represented as a union of its subsets then these will be vertically listed in brackets, i.e.:

$$\begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_n \end{pmatrix}, \quad \text{which corresponds to} \quad S_1 \cup S_2 \cup \dots \cup S_n. \tag{6}$$

$$\begin{array}{l} N[t_1 \leftarrow N_1] \\ \quad = \ \text{(by Definition 5.8, equations (4), and using (5))} \\ \qquad \Big(((N \ \mathbf{rs} \ t_1) \cup N_1) \\ \qquad\quad \cup(\ \ \otimes\{ \begin{pmatrix} A_0 \\ H_{AD} \\ H_{AE} \\ H_{AF} \end{pmatrix}, R_1\} \cup \otimes\{ \begin{pmatrix} C_0 \\ H_{CD} \\ H_{CE} \\ H_{CF} \end{pmatrix}, R_2\} \cup \otimes\{ \begin{pmatrix} B_0 \\ H_{BD} \\ H_{BE} \\ H_{BF} \end{pmatrix}, R_1, R_2\} \ ) \\ \qquad \Big) \ \ \mathbf{rs} \ (A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\ \qquad\qquad\quad \cup\, C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2) \end{array}$$

$$
\begin{aligned}
= \quad & \text{(by Proposition 4.2(2))} \\
& \Big(((N \ \mathbf{rs}\ t_1) \cup N_1) \\
& \cup(\quad \otimes\{A_0, R_1\} \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{AF}, R_1\} \\
& \quad \cup \quad \otimes\{C_0, R_2\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{CF}, R_2\} \\
& \quad \cup \quad \otimes\{B_0, R_1, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \\
& \ ) \\
& \Big) \quad \mathbf{rs}\ (A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
& \qquad\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2)\ .
\end{aligned}
$$

After the refinement of t_1 the interface place sets of transition t_2 changed: the interface places which were shared with t_1, having been combined with interface places of N_1, became 'parts' of the new places; these new places have thus inherited the incidences to t_2. The characteristics of the new places as interfaces of t_2 (i.e., entry, exit) can easily be recognised from the second subscript of H-arguments in the place multiplication operator: for example, $H_{\dots D}$-components determine 'entry'-interfaces. Taking this into account, we can write for the interface place sets of t_2 after the refinement of t_1:

$$
\left.
\begin{aligned}
({}^\circ t_2)' &= D_0 \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \\
(t_2^\circ)' &= F_0 \cup \otimes\{H_{AF}, R_1\} \cup \otimes\{H_{CF}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \\
(S_{t_2}^{(side)})' &= E_0 \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\}\ .
\end{aligned}
\right\} \tag{7}
$$

$(N[t_1 \leftarrow N_1])[t_2 \leftarrow N_2]$

$$
\begin{aligned}
= \quad & \text{(by Definition 5.8)} \\
& \Big(((N \ \mathbf{rs}\ t_2) \cup N_2) \\
& \quad \cup (\ \otimes\{(({}^\circ t_2)', {}^\circ N_2\} \ \cup\ \otimes\{((t_2^\circ)', N_2^\circ\} \cup \otimes\{(S_{t_2}^{(side)})', {}^\circ N_2 . N_2^\circ\}\ ) \\
& \Big)\ \mathbf{rs}\ \{{}^\bullet t_2 \cup t_2^\bullet \cup {}^\circ N_0 \cup N_0^\circ\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \text{(by Definition 5.8, equations (7), and using (5))} \\
& \Big(\Big(\Big(\Big(\Big(((N \ \mathbf{rs}\ t_1) \cup N_1) \\
& \qquad\qquad \cup (\quad \otimes\{A_0, R_1\} \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{AF}, R_1\} \\
& \qquad\qquad\quad \cup \otimes\{C_0, R_2\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{CF}, R_2\} \\
& \qquad\qquad\quad \cup \otimes\{B_0, R_1, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \\
& \qquad\qquad\quad) \\
& \qquad\qquad \Big)\ \mathbf{rs}\ (A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
& \qquad\qquad\qquad\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2) \\
& \quad \Big)\ \mathbf{rs}\ t_2 \Big) \cup N_2 \Big)
\end{aligned}
$$

$$
\begin{array}{l}
\quad\cup(\quad \otimes\{ \begin{pmatrix} D_0 \\ \otimes\{H_{AD}, R_1\} \\ \otimes\{H_{CD}, R_2\} \\ \otimes\{H_{BD}, R_1, R_2\} \end{pmatrix}, Q_1\} \ \cup\ \otimes\{ \begin{pmatrix} F_0 \\ \otimes\{H_{AF}, R_1\} \\ \otimes\{H_{CF}, R_2\} \\ \otimes\{H_{BF}, R_1, R_2\} \end{pmatrix}, Q_2\} \\
\qquad\quad \cup\otimes\{ \begin{pmatrix} E_0 \\ \otimes\{H_{AE}, R_1\} \\ \otimes\{H_{CE}, R_2\} \\ \otimes\{H_{BE}, R_1, R_2\} \end{pmatrix}, Q_1, Q_2\}\ ) \\
\Big)\ \mathbf{rs}\ \Big(D_0 \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup Q_1 \\
\qquad\qquad \cup F_0 \cup \otimes\{H_{AF}, R_1\} \cup \otimes\{H_{CF}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \cup Q_2 \\
\qquad\qquad \cup E_0 \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\}\Big)
\end{array}
$$

$=$ (by Propositions 4.1(3) and 4.3(4, 5))

$$
\begin{array}{l}
\Big(((((N\ \mathbf{rs}\ t_1) \cup N_1)\ \mathbf{rs}\ t_2) \cup N_2) \\
\quad\cup(\quad \otimes\{A_0, R_1\} \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{AF}, R_1\} \\
\qquad \cup \otimes\{C_0, R_2\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{CF}, R_2\} \\
\qquad \cup \otimes\{B_0, R_1, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \\
\qquad \cup\otimes\{ \begin{pmatrix} D_0 \\ \otimes\{H_{AD}, R_1\} \\ \otimes\{H_{CD}, R_2\} \\ \otimes\{H_{BD}, R_1, R_2\} \end{pmatrix}, Q_1\} \ \cup\ \otimes\{ \begin{pmatrix} F_0 \\ \otimes\{H_{AF}, R_1\} \\ \otimes\{H_{CF}, R_2\} \\ \otimes\{H_{BF}, R_1, R_2\} \end{pmatrix}, Q_2\} \\
\qquad \cup\otimes\{ \begin{pmatrix} E_0 \\ \otimes\{H_{AE}, R_1\} \\ \otimes\{H_{CE}, R_2\} \\ \otimes\{H_{BE}, R_1, R_2\} \end{pmatrix}, Q_1, Q_2\} \\
\quad) \\
\Big)\quad \mathbf{rs}\ \Big(A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
\qquad\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2 \\
\qquad\qquad \cup D_0 \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup Q_1 \\
\qquad\qquad \cup F_0 \cup \otimes\{H_{AF}, R_1\} \cup \otimes\{H_{CF}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \cup Q_2 \\
\qquad\qquad \cup E_0 \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\}\Big)
\end{array}
$$

$=$ (by Proposition 4.2(2))

$$
\begin{array}{l}
\Big(((((N\ \mathbf{rs}\ t_1) \cup N_1)\ \mathbf{rs}\ t_2) \cup N_2) \\
\quad\cup(\quad \otimes\{A_0, R_1\} \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{AF}, R_1\} \\
\qquad \cup \otimes\{C_0, R_2\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{CF}, R_2\} \\
\qquad \cup \otimes\{B_0, R_1, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup \otimes\{H_{BE}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\}
\end{array}
$$

$$
\begin{array}{l}
\qquad\qquad \cup \otimes \{D_0, Q_1\} \cup \otimes\{\otimes\{H_{AD}, R_1\}, Q_1\} \\
\qquad\qquad \cup \otimes \{\otimes\{H_{CD}, R_2\}, Q_1\} \cup \otimes\{\otimes\{H_{BD}, R_1, R_2\}, Q_1\} \\
\qquad\qquad \cup \otimes \{F_0, Q_2\} \cup \otimes\{\otimes\{H_{AF}, R_1\}, Q_2\} \\
\qquad\qquad \cup \otimes \{\otimes\{H_{CF}, R_2\}, Q_2\} \cup \otimes\{\otimes\{H_{BF}, R_1, R_2\}, Q_2\} \\
\qquad\qquad \cup \otimes \{E_0, Q_1, Q_2\} \cup \otimes\{\otimes\{H_{AE}, R_1\}, Q_1, Q_2\} \\
\qquad\qquad \cup \otimes \{\otimes\{H_{CE}, R_2\}, Q_1, Q_2\} \cup \otimes\{\otimes\{H_{BE}, R_1, R_2\}, Q_1, Q_2\} \\
\qquad)\\
\Big) \quad \mathbf{rs}\ \Big(A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
\qquad\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2 \\
\qquad\qquad \cup D_0 \cup \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{CD}, R_2\} \cup \otimes\{H_{BD}, R_1, R_2\} \cup Q_1 \\
\qquad\qquad \cup F_0 \cup \otimes\{H_{AF}, R_1\} \cup \otimes\{H_{CF}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \cup Q_2 \\
\qquad\qquad \cup E_0 \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{BE}, R_1, R_2\}\Big)
\end{array}
$$

$=$ (by Propositions 4.1(3) and 4.3(4, 5))

$$
\begin{array}{l}
\Big(\Big(((((((N\ \mathbf{rs}\ t_1) \cup N_1)\ \mathbf{rs}\ t_2) \cup N_2) \\
\qquad\qquad \cup(\quad \otimes \{H_{AD}, R_1\} \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{AF}, R_1\} \\
\qquad\qquad\quad \cup \otimes \{H_{CD}, R_2\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{CF}, R_2\} \\
\qquad\qquad\quad \cup \otimes \{H_{BD}, R_1, R_2\} \cup \otimes\{H_{BE}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \\
\qquad\qquad)\\
\quad)\ \mathbf{rs}\ (\quad \otimes\{H_{AD}, R_1\} \cup \otimes\{H_{AE}, R_1\} \cup \otimes\{H_{AF}, R_1\} \\
\qquad\qquad \cup \otimes \{H_{CD}, R_2\} \cup \otimes\{H_{CE}, R_2\} \cup \otimes\{H_{CF}, R_2\} \\
\qquad\qquad \cup \otimes \{H_{BD}, R_1, R_2\} \cup \otimes\{H_{BE}, R_2\} \cup \otimes\{H_{BF}, R_1, R_2\} \\
\qquad)\\
\quad \Big) \cup (\quad \otimes\{A_0, R_1\} \cup \otimes\{C_0, R_2\} \cup \otimes\{B_0, R_1, R_2\} \\
\qquad\qquad \cup \otimes \{D_0, Q_1\} \cup \otimes\{\otimes\{H_{AD}, R_1\}, Q_1\} \\
\qquad\qquad \cup \otimes \{\otimes\{H_{CD}, R_2\}, Q_1\} \cup \otimes\{\otimes\{H_{BD}, R_1, R_2\}, Q_1\} \\
\qquad\qquad \cup \otimes \{F_0, Q_2\} \cup \otimes\{\otimes\{H_{AF}, R_1\}, Q_2\} \\
\qquad\qquad \cup \otimes \{\otimes\{H_{CF}, R_2\}, Q_2\} \cup \otimes\{\otimes\{H_{BF}, R_1, R_2\}, Q_2\} \\
\qquad\qquad \cup \otimes \{E_0, Q_1, Q_2\} \cup \otimes\{\otimes\{H_{AE}, R_1\}, Q_1, Q_2\} \\
\qquad\qquad \cup \otimes \{\otimes\{H_{CE}, R_2\}, Q_1, Q_2\} \cup \otimes\{\otimes\{H_{BE}, R_1, R_2\}, Q_1, Q_2\} \\
\qquad\qquad \cup \otimes \{A_0, R_1\} \cup \otimes\{C_0, R_2\} \cup \otimes\{B_0, R_1, R_2\} \\
\qquad)\\
\Big)\ \mathbf{rs}\ (A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
\qquad\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2 \\
\qquad\qquad \cup D_0 \cup Q_1 \cup F_0 \cup Q_2 \cup E_0 \ )
\end{array}
$$

$=$ (by Proposition 4.3(4, 5))

$$
\begin{aligned}
&\Big(((((N \ \mathbf{rs}\ t_1) \cup N_1) \ \mathbf{rs}\ t_2) \cup N_2) \\
&\quad \cup(\quad \otimes\{A_0, R_1\} \cup \otimes\{C_0, R_2\} \cup \otimes\{B_0, R_1, R_2\} \\
&\qquad \cup \otimes \{D_0, Q_1\} \cup \otimes\{\otimes\{H_{AD}, R_1\}, Q_1\} \\
&\qquad \cup \otimes \{\otimes\{H_{CD}, R_2\}, Q_1\} \cup \otimes\{\otimes\{H_{BD}, R_1, R_2\}, Q_1\} \\
&\qquad \cup \otimes \{F_0, Q_2\} \cup \otimes\{\otimes\{H_{AF}, R_1\}, Q_2\} \\
&\qquad \cup \otimes \{\otimes\{H_{CF}, R_2\}, Q_2\} \cup \otimes\{\otimes\{H_{BF}, R_1, R_2\}, Q_2\} \\
&\qquad \cup \otimes \{E_0, Q_1, Q_2\} \cup \otimes\{\otimes\{H_{AE}, R_1\}, Q_1, Q_2\} \\
&\qquad \cup \otimes \{\otimes\{H_{CE}, R_2\}, Q_1, Q_2\} \cup \otimes\{\otimes\{H_{BE}, R_1, R_2\}, Q_1, Q_2\} \\
&\quad) \\
&\Big) \ \mathbf{rs}\ (A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
&\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2 \\
&\qquad \cup D_0 \cup Q_1 \cup F_0 \cup Q_2 \cup E_0)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \text{(by Proposition 4.2(1))} \\
& \left.\begin{aligned}
&\Big(((((N \ \mathbf{rs}\ t_1) \cup N_1) \ \mathbf{rs}\ t_2) \cup N_2) \\
&\quad \cup\Big(\quad \otimes\{A_0, R_1\} \cup \otimes\{C_0, R_2\} \cup \otimes\{B_0, R_1, R_2\} \\
&\qquad \cup \otimes \{D_0, Q_1\} \cup \otimes\{H_{AD}, R_1, Q_1\} \\
&\qquad \cup \otimes \{H_{CD}, R_2, Q_1\} \cup \otimes\{H_{BD}, R_1, R_2, Q_1\} \\
&\qquad \cup \otimes \{F_0, Q_2\} \cup \otimes\{H_{AF}, R_1, Q_2\} \\
&\qquad \cup \otimes \{H_{CF}, R_2, Q_2\} \cup \otimes\{H_{BF}, R_1, R_2, Q_2\} \\
&\qquad \cup \otimes \{E_0, Q_1, Q_2\} \cup \otimes\{H_{AE}, R_1, Q_1, Q_2\} \\
&\qquad \cup \otimes \{H_{CE}, R_2, Q_1, Q_2\} \cup \otimes\{H_{BE}, R_1, R_2, Q_1, Q_2\} \\
&\quad) \\
&\Big) \ \mathbf{rs}\ \Big(A_0 \cup H_{AD} \cup H_{AE} \cup H_{AF} \cup B_0 \cup H_{BD} \cup H_{BE} \cup H_{BF} \\
&\qquad \cup C_0 \cup H_{CD} \cup H_{CE} \cup H_{CF} \cup R_1 \cup R_2 \\
&\qquad \cup D_0 \cup Q_1 \cup F_0 \cup Q_2 \cup E_0\Big) \ .
\end{aligned}\right\} \qquad (8)
\end{aligned}
$$

Calculating the opposite direction is similar, and the results are the same [5].

Third, we show the proof of the *coherence of refinement w.r.t. unfolding* (Theorem 5.5(3)).

Corresponding to Definition 2.3, the statuses of places are inherited under unfolding, i.e., for $N = (S, T, \iota)$:

$$
\left.\begin{aligned}
&\mathcal{U}({}^\bullet N) = {}^\bullet(\mathcal{U}(N)), \quad \mathcal{U}(N^\bullet) = (\mathcal{U}(N))^\bullet, \qquad \mathcal{U}(S_N^{\{e,x\}}) = S_{\mathcal{U}(N)}^{\{e,x\}}; \\
&\mathcal{U}({}^\circ N) = {}^\circ(\mathcal{U}(N)), \quad \mathcal{U}(N^\circ) = (\mathcal{U}(N))^\circ \ .
\end{aligned}\right\} \qquad (9)
$$

Analogously, mutual structural relations of the nodes (such as 'being an input node', etc.) are also inherited under unfolding (which follows from Definition 2.3): if $N = (S, T, \iota)$ is an MNB, $y \in S \cup T$, and $t \in T$ then:

$$
\left.\begin{aligned}
&\mathcal{U}({}^\bullet y) = {}^\bullet(\mathcal{U}(y)), \quad \mathcal{U}(y^\bullet) = (\mathcal{U}(y))^\bullet, \qquad \mathcal{U}(S_t^{(side)}) = S_{\mathcal{U}(t)}^{(side)}. \\
&\mathcal{U}({}^\circ y) = {}^\circ(\mathcal{U}(y)), \quad \mathcal{U}(y^\circ) = (\mathcal{U}(y))^\circ \ .
\end{aligned}\right\} \qquad (10)
$$

From Definition 2.3 it follows that every transition with a single mode remains unchanged under unfolding, i.e., if $N = (S, T, \iota)$ is an MNB and $t \in T$ then:

$$\text{If } |\alpha(t)| \leq 1 \text{ then } \mathcal{U}(t) = t \quad (= t_{\mathcal{U}(N)}). \tag{11}$$

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{U}(N[m_0 \leftarrow N_0])} \\
= & \text{(by Definition 5.8)} \\
 & \mathcal{U}\Big(\Big(((N \ \mathbf{rs}\ t_0) \cup N_0) \\
 & \quad \cup (\ \otimes \{{}^\circ t_0,\ {}^\bullet N_0\} \cup \otimes \{t_0^\circ,\ N_0^\bullet\} \cup \otimes \{S_{t_0}^{(side)},\ {}^\bullet N_0,\ N_0^\bullet\} \ ) \\
 & \Big) \ \mathbf{rs}\ ({}^\bullet t_0 \cup t_0^\bullet \cup {}^\bullet N_0 \cup N_0^\bullet)\Big) \\
\cong & \text{(by Proposition 4.3(6))} \\
 & \mathcal{U}\Big(((N \ \mathbf{rs}\ t_0) \cup N_0) \\
 & \quad \cup (\ \otimes \{{}^\circ t_0,\ {}^\bullet N_0\} \cup \otimes \{t_0^\circ,\ N_0^\bullet\} \cup \otimes \{S_{t_0}^{(side)},\ {}^\bullet N_0,\ N_0^\bullet\}) \\
 & \Big) \ \mathbf{rs}\ \mathcal{U}(\ {}^\bullet t_0 \cup t_0^\bullet \cup {}^\bullet N_0 \cup N_0^\bullet\ ) \\
= & \text{(by Proposition 4.1(4))} \\
 & \Big(\mathcal{U}((N \ \mathbf{rs}\ t_0) \cup N_0) \\
 & \quad \cup \mathcal{U}(\ \otimes \{{}^\circ t_0,\ {}^\circ N_0\} \cup \otimes \{t_0^\circ,\ N_0^\circ\} \cup \otimes \{S_{t_0}^{(side)},\ {}^\circ N_0,\ N_0^\circ\}) \\
 & \Big) \ \mathbf{rs}\ \mathcal{U}(\ {}^\bullet t_0 \cup t_0^\bullet \cup {}^\bullet N_0 \cup N_0^\bullet\ ) \\
= & \text{(by Proposition 4.1(4))} \\
 & \Big((\mathcal{U}(N \ \mathbf{rs}\ t_0) \cup \mathcal{U}(N_0)) \\
 & \quad \cup (\ \mathcal{U}(\otimes \{{}^\circ t_0,\ {}^\circ N_0\}) \cup \mathcal{U}(\otimes \{t_0^\circ,\ N_0^\circ\}) \cup \mathcal{U}(\otimes \{S_{t_0}^{(side)},\ {}^\circ N_0,\ N_0^\circ\})) \\
 & \Big) \ \mathbf{rs}\ (\mathcal{U}({}^\bullet t_0) \cup \mathcal{U}(t_0^\bullet) \cup \mathcal{U}({}^\bullet N_0) \cup \mathcal{U}(N_0^\bullet)) \\
\cong & \text{(by Propositions 4.3(6) and 4.2(3))} \\
 & \Big(\Big((\mathcal{U}(N) \ \mathbf{rs}\ \mathcal{U}(t_0)) \cup \mathcal{U}(N_0)\Big) \\
 & \quad \cup (\ \otimes \{\mathcal{U}({}^\circ t_0),\ \mathcal{U}({}^\circ N_0)\} \cup \otimes \{\mathcal{U}(t_0^\circ),\ \mathcal{U}(N_0^\circ)\} \cup \otimes \{\mathcal{U}(S_{t_0}^{(side)}), \mathcal{U}({}^\circ N_0), \mathcal{U}(N_0^\circ)\}) \\
 & \Big) \ \mathbf{rs}\ (\mathcal{U}({}^\bullet t_0) \cup \mathcal{U}(t_0^\bullet) \cup \mathcal{U}({}^\bullet N_0) \cup \mathcal{U}(N_0^\bullet)) \\
= & \text{(by equations (9), (10), and (11))} \\
 & \Big(\Big((\mathcal{U}(N) \ \mathbf{rs}\ t_0) \cup \mathcal{U}(N_0)\Big) \\
 & \quad \cup (\quad \otimes \{{}^\circ(t_{0,\mathcal{U}(N)}),\ {}^\circ(\mathcal{U}(N_0))\} \cup \otimes \{(t_{0,\mathcal{U}(N)})^\circ,\ (\mathcal{U}(N_0))^\circ\} \\
 & \qquad \cup \otimes \{S_{(t_{0,\mathcal{U}(N)})}^{(side)}, {}^\circ(\mathcal{U}(N_0)), (\mathcal{U}(N_0))^\circ\} \\
 &)\Big) \ \mathbf{rs}\ ({}^\bullet(t_0)_{\mathcal{U}(N)} \cup (t_0)_{\mathcal{U}(N)}^\bullet \cup {}^\bullet(\mathcal{U}(N_0)) \cup (\mathcal{U}(N_0))^\bullet)
\end{array}$$

$$= \text{(by Definition 5.8)}$$
$$\mathcal{U}(N)[t_{0,\mathcal{U}(N)} \leftarrow \mathcal{U}(N_0)] \ .$$

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