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Oldenburg Discussion Papers in Economics

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V – 398–17

February 2017

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February 7, 2017

Abstract

Shall investments become more robust or more short-lived if unfavorable exogeneous conditions become more uncertain? What if the investments' design is irreversible for its whole life time? Such decision problems are frequently encountered, for example in infrastructure construction. We analyze this problem by combining an irreversible design decision when the investment starts with an irreversible decision to abandon an outdated investment. We formulate the second decision as a stopping problem of stochastic dynamic control, derive the value function, and the comparative statics for an optimal design. We find a decreasing optimal expected life-time and decreasing robustness for more rapidly changing conditions if the original life-time is not too large. For rising uncertainty, originally shorter-lived investments' life-times are expanded. For more long-lived investments, these effects may reverse. There can be a case for making investments less robust in the light of uncertain and ongoing change.

1 Introduction

This paper analyses a frequent and important decision problem for large-scale infrastructure investors and other investors alike, be they private or public.

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How shall we design irreversible investments with technology commitment if they are exposed to uncertain change? How should long-lived infrastructure be adapted to ongoing and uncertain trends? If uncertainty rises or trends become faster: shall we adopt designs that are robust to a broader range of conditions, or shall we reduce investments' life-times? We analyze these questions by determining the optimal life-time and design of an investment in the presence of a Brownian motion with drift. The decision problem combines a real option to abandon with an irreversible decision about a design parameter at the beginning of the investment's life-time.

There are various examples for decision problems of this kind. Investments in large construction projects, in chemical engineering, process technology or the steel industry require technology and design choices at the time of investment that cannot easily be revised later on. The construction industry makes up a share of about 5% to 11% of GDP globally, and megaprojects might account for 8% of GDP (depending on estimate, e.g. Crosthwaite, 2000; Flyvbjerg, 2014; World Economic Forum, 2016; OECD, 2016). The profitability of such investments depends on uncertain trends in demand, and might suffer from the risk of becoming technologically outdated. Consider, as another example, utilities investment in the light of demographic change, economic growth or regulatory uncertainty. New water, transport or electricity infrastructure components are typically associated with lumpy investment, high sunk costs, and a technical design that is irreversible for multiple decades (a retrofit of pipelines, rail track, airport runways, dams or electricity grids, for example, can be prohibitively costly, e.g. Turvey, 2000; Flyvbjerg, 2014; Ansar et al., 2014). On the other hand, the stream of benefits obtained from infrastructures typically depends on the size of the population or the scale of economic activity that is served. Over long time scales, benefit streams are (i) likely not constant and (ii) quite difficult to predict.

As a further example, take adaptation of infrastructures to climate change. This issue has got increasing political attention during the last years (e.g. OECD, 2008; UNEP, 2016). It is beneficial if infrastructure designs (e.g. the chosen type of concrete or steel and the technological specification of machinery) are well fitted to climatic conditions (e.g. temperature, precipitation, wind patterns), since infrastructure maintenance will be cheaper, its durability will be increased and service disruptions will be less likely (cf. IPCC, 2014). Extreme weather events have always threatened the reliability and quality of infrastructure. Global warming will affect the statistics of such events and other climatic parameters over the coming decades and centuries (IPCC, 2013). It is thus reasonable to construct infrastructures in modified ways. This may cost up to 100 b$ annually (or even more, depending on the estimate, Stern, 2007; OECD, 2008; UNEP, 2016), or alone 70 b$ annually for coastal protection (Hinkel et al., 2014). There are two additional challenges here (cf. Hallegatte, 2009): (i) If the climate is subject to ongoing change, the design of long-lived infrastructures need to fit to a broader range of climatic conditions. (ii) Projections about the rate of climate change are prone to different kinds of uncertainty (e.g. Weitzman, 2013; Heal and Millner, 2014; IPCC, 2014). In addition to the examples mentioned so far, there are further cases of the decision problem analyzed in this paper with less long-lived investments, like choosing a new computer between an expensive device that has high performance for a while, and a cheaper one that becomes outdated earlier.

These examples have in common that an irreversible technology commitment needs to be made once a new investment is undertaken. This design decision in the beginning shapes the effects of uncertain changes in exogenous conditions, and thus influences the ultimate life-time of the investment. There are trade-offs involved here. It might be one intuitive option to design the investment more robustly in order to remain profitable also under more unlikely or more future conditions. Although that might increase costs in the present, it will reduce losses in the future. A more robust design might then increase the investment's life-time. On the other hand, if more rapid change is expected, the optimal design might be less robust in anticipation of a shorter expected life-time. Finally, increasing uncertainty might lead to a higher value of the option to abandon the investment, so that the expected life-time rises, making a more robust design optimal. It is prima facie unclear how these two decision variables (robustness and time to abandon) interact, and this paper shows that this relation is indeed non-trivial.

Irreversible design and abandonment together with uncertainty lends to an analysis by real-options methods (cf. Dixit and Pindyck, 1994) that has – to our knowledge – not been published so far in the vast literature. Technology commitment is studied in the context of the investment-uncertainty relationship (e.g. Ramey and Ramey, 1995; Sarkar, 2000; Jovanovic, 2006). This literature takes more a macro-perspective and focuses on the option to invest instead of on the option to abandon. Some studies investigate flexibility instead of robustness by considering switching options between projects (e.g. Farzin et al., 1998; Decamps et al., 2006). The investment's life-time or time to abandon is analyzed, but not by including an irreversible design decision, by Farzin et al. (1998); Myers and Maid (2004); Dahlgren and Leung (2015). Some publications investigate environmental decision problems, but not with considering adaptation of investments (Pindyck, 2000, 2002). Some papers address related questions on adapting investments as we do (e.g. Fisher and Rubio, 1997; Callaway, 2004; Hallegatte, 2009; de Bruin and Ansink, 2011; Felgenhauer and Webster, 2014; vander Pol et al., 2014), but most of them do not explicitly focus on irreversible design and abandonment, or follow a less formal approach. Our paper contributes to this literature by investigating the interdependence of two irreversible decisions: design and abandonment. We precisely show how more uncertainty and a faster trend ambiguously affect optimal investment life-times, and derive conditions where it is optimal to adapt with more robust or less robust investments.

Section 2 presents our general theoretical model and the essential decision structure. Section 3 investigates a specific optimal stochastic dynamic control model to maximize an investment's expected net value. Analytical comparative statics results for different exogenous variables, in particular uncertainty, are presented in Section 4, together with some numerical experiments. Section 5 concludes.

2 General model setup and decision structure

We analyze the decision for a long-lived investment that operates within uncertain exogenous conditions x that influence the investment's stream of current benefits over time t, modeled by a geometric stochastic process

$$dx = \mu x\, dt + \sigma x\, dz, \tag{1}$$

with $x(0) = x_0 > 0$, trend parameter $\mu > 0$, standard deviation $\sigma > 0$, and (z_t) being a standardized Wiener Process, so that $x \geq 0$. We will call σ uncertainty in this paper. The decision's objective is to maximize

$$J(x_0, a, \mu, \sigma) = E[\int_0^{T^\star} \pi(x, a)e^{-rt}dt] - C(a), \tag{2}$$

with respect to the technical design vector a that describes the investment's properties, and with respect to the stopping time T^* where the investment is ultimately stopped. Here and in the following, $E[\cdot]$ is the expectation operator, and $\pi(x, a)$ denotes the current benefits, which depend on how the design fits to the conditions. Current benefits are discounted to present values at rate $r > 0$. We assume technology commitment, i.e. that the technical design is fixed over the complete investment's life time. The investment costs $C(a)$ depend on this design and incur at the start. This kind of irreversibility can be justified, for example, if the costs of a retrofitting the investment to new conditions is prohibitively costly. After the investment is constructed with the chosen design, the remaining decision is when to stop it's life time. We thus assume the following multi stage decision structure. First, the irreversible design a is chosen. Second, it is decided at each point in time whether to continue or stop operating the investment. Stopping at some time $T^\star$ is an irreversible decision.

This problem will be solved by backward induction, where the second stage is a standard stopping problem. At the time where the investment starts, we do not know the stopping time yet, but we can, in principle, determine the expected stopping time $E[T^\star]$, which depends on μ, σ and on a as chosen in the first stage. In the first stage, the design decision will depend, in turn, on the expected stopping time. We will actually determine, after some more general considerations, the optimal design $a^\star$ for an exemplary specification of π, and will determine its comparative statics, in particular with respect to (μ, σ). This yields whether the expected life-time with optimally design $T^{\star\star} = E[T^\star]$ with $a = a^\star$ is extended or shortened if the conditions are more rapidly changing, or more uncertain.

Some general implications can already be drawn from the decision structure without specifying the current benefits $\pi(x, a)$, supposed that it is at least specified in a well-posed way. In the second stage, stopping problems typically yield decisions rule with a cutoff-value $x^\star(a, \mu, \sigma)$. At the optimal stopping time $T^\star(x_0, a, \mu, \sigma)$ we have $x(T^\star) = x^\star(a, \mu, \sigma)$ (cf. Dixit and Pindyck, 1994). Note that $E[x(t)] = x_0 e^{\mu t}$, so that the expected stopping time $E[T^\star](x_0, a, \mu, \sigma)$ can be obtained by solving $x_0 e^{\mu t} = x^\star(a, \mu, \sigma)$ for t. Thus, the total differential yields

$$\frac{dE[T^\star]}{d\mu} = \frac{1}{\mu}\Big(\frac{\partial_\mu x^\star}{x^\star} - E[T^\star]\Big), \tag{3}$$

$$\frac{dE[T^\star]}{d\sigma} = \frac{1}{\mu}\frac{\partial_\sigma x^\star}{x^\star}, \tag{4}$$

$$\frac{dE[T^\star]}{da} = \frac{1}{\mu}\frac{\partial_a x^\star}{x^\star}. \tag{5}$$

Here and in the following, $\partial.$ denotes partial derivatives. This shows that a rising design parameter a shifts the expected stopping time in the same direction as the the cutoff-value, which is quite intuitive. Also the effect of rising uncertainty on the cutoff-value goes in the same direction as the effect on the expected stopping time. The effect of a faster trend is more complicated. If the expected stopping time is ceteris paribus smaller, it is more likely that cutoff-value and expected stopping time move in the same direction.

In the first stage, a is selected to maximize J. Note that the cutoff-value can be determined from a value function $h(x)$ that expresses the expected value of not stopping the investment (yet) for given conditions and assuming that the investment is stopped at the optimal time later (cf. Dixit and Pindyck, 1994). Thus, $J(x_0, a, \mu, \sigma) = h(x_0) - C(a)$, and the first-order and second-order conditions are

$$\partial_a J(x_0, a, \mu, \sigma) = \frac{d}{da} h(x_0) - C'(a) = 0, \tag{6}$$

$$\partial_{aa} J(x_0, a, \mu, \sigma) = \frac{d^2}{da^2} h(x_0) - C''(a) < 0. \tag{7}$$

Now suppose that Eq. (7) holds. Solving Eq. (6) for a yields the optimal design $a^\star(x_0, \mu, \sigma)$, and thus the optimally designed expected life-time $T^{\star\star}(x_0, \mu, \sigma) = E[T^\star](x_0, a^\star(x_0, \mu, \sigma), \mu, \sigma)$. These functions have the following comparative statics properties (with $\doteq$ denoting equivalence in signs) by making use of Eq. (3)-Eq. (5)):

$$\partial_\mu a^\star(x_0, \mu, \sigma) = -\frac{\partial_{a\mu} J}{\partial_{aa} J} \doteq \partial_{a\mu} J(x_0, a^\star, \mu, \sigma) = \frac{d^2}{da\mu} h(x_0), \tag{8}$$

$$\partial_\sigma a^\star(x_0, \mu, \sigma) = -\frac{\partial_{a\sigma} J}{\partial_{aa} J} \doteq \partial_{a\sigma} J(x_0, a^\star, \mu, \sigma) = \frac{d^2}{da\sigma} h(x_0), \tag{9}$$

$$\begin{aligned} \partial_\mu T^{\star\star}(x_0, \mu, \sigma) &= \frac{d}{d\mu} E[T^\star](x_0, a^\star(x_0, \mu, \sigma), \mu, \sigma) = \\ &= \partial_\mu E[T^\star] + \partial_a E[T^\star] \cdot \partial_\mu a^\star = \\ &= \frac{1}{\mu x^\star}\big(\partial_\mu x^\star + \partial_a x^\star \partial_\mu a^\star - E[T^\star] x^\star\big), \end{aligned} \tag{10}$$

$$\begin{aligned} \partial_\sigma T^{\star\star}(x_0, \mu, \sigma) &= \frac{d}{d\sigma} E[T^\star](x_0, a^\star(x_0, \mu, \sigma), \mu, \sigma) = \\ &= \partial_\sigma E[T^\star] + \partial_a E[T^\star] \cdot \partial_\sigma a^\star = \\ &= \frac{1}{\mu x^\star}(\partial_\sigma x^\star + \partial_a x^\star \partial_\sigma a^\star). \end{aligned} \tag{11}$$

The last two equations show that the effect of uncertainty and the trend on the optimal expected life-time can be decomposed into two effects. The first summand is the direct effect. One might expect, for example, if the conditions change with a faster trend (larger μ), that the expected stopping-time $E[T^\star]$ is shortened. The time where the old investment's design does not fit the conditions so well any more can be expected to come earlier. The further summands refer to an indirect effect. Obviously, the design affects the stopping time. We might expect, for example, that an investment that is more robust to changes in the conditions (higher a) will be stopped at a later time. On the other hand, more rapid change might lead to another optimal design, and it is not obvious at this stage whether it is optimal to make the investment more robust. If this would be the case, the direct and the indirect effect would point in the opposite direction. A similar argument can be made with respect to uncertainty. We expect ambiguous results depending on the detailed specification of the optimization problem.

3 A model with optimal stopping and robustness

This section analyses the model for an ideal type case with linear current benefits and one design parameter. This specification has the advantage that central results can be derived analytically, and that it covers a broad set of possible applications. More complex specifications might either require more numerical analysis, or a piecewise linear approximation with our model. We first solve the optimal stopping problem, determine the direct effects in Eq. (10), Eq. (11) by means of a comparative statics analysis. Subsequently, the optimal design and the full comparative statics are studied.

The conditions x and the design parameter a are assumed to determine the investment's current benefit according to $\pi(x, a) = \gamma - x/a$ with some $\gamma > 0$. Therefore, we model that rising x reduces the current benefit over time because the committed design needs to fit to increasingly worse conditions. The current benefit derived from the investment is always below a maximum γ, and diminishes to zero if x approaches γa. Designing an investment with a larger a implies that it generates positive benefits for a broader interval of conditions. We thus can conceive the design parameter as the investment's robustness. We further assume that a more robust design comes at constant robustness unit costs $c > 0$, so that $C = ca$.

3.1 Optimal stopping with arbitrary design

We now turn to the second stage of the decision problem

$$\max_{T} \; E\left[\int_0^T (\gamma - x(t)/a)\, e^{-rt} dt\right], \tag{12}$$

subject to Eq. (1). This is an autonomous optimal stopping problem in current-value formulation that ultimately determines $T^\star$. We can solve this problem by determining the value function $h(x)$ that is required to satisfy the Hamilton-Jacobi-Bellmann equation

$$-rh + (\gamma - \frac{x}{a}) + \mu x\, h' + \frac{1}{2}\sigma^2 x^2 h'' = 0. \tag{13}$$

The optimal stopping rule is to continue operation as long as $x(t) < x^\star$, the latter being the cutoff-value. At stopping time $T^\star$ the conditions reach $x^\star$, which is characterized by the standard value matching and smooth pasting conditions $h(x^\star) = 0, h'(x^\star) = 0$. We can show the following solution (see Appendix):

Proposition 1 *The optimal stopping problem Eq. (1), Eq. (12) is solved by the value function*

$$h(x) = \frac{\gamma}{r(\beta - 1)}(\frac{x}{x^\star})^\beta - \frac{1}{a(r-\mu)}\, x + \frac{\gamma}{r}, \tag{14}$$

with $\beta > 0$ being the positive root of the characteristic polynomial

$$\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0, \tag{15}$$

and the cutoff-value $x^\star = \omega\gamma a$, if $x^\star > x_0$. We define the reappearing term $\omega := \frac{r-\mu}{r}\frac{\beta}{\beta-1}$.

Since we consider a positive trend μ, the cutoff-value $x^\star$ should be larger than x_0. Otherwise, the investment will not be undertaken at all. The root

β can be explicitly written as

$$\beta = \frac{1}{2\sigma^2}\left((\sigma^2 - 2\mu) + ((\sigma^2 - 2\mu)^2 + 8r\sigma^2)^{1/2}\right) > 0.$$

It can be verified that

$$\begin{aligned} r > \mu \text{ if and only if } 1 < \beta < r/\mu, \\ r < \mu \text{ if and only if } r/\mu < \beta < 1, \end{aligned} \tag{16}$$

and always $\omega > 1$, so that $x^\star > 0$. It is important to recognize that $r < \mu$ is, in contrast to other optimal stopping problems in the literature, a reasonable case. Since current benefits are decreasing in x, there are no problems with a non-existing net value J. In our setting, both r and μ lead to less benefits in the future. The size of β in relation to unity distinguishes between cases where discounting or where changing conditions dominate in the long run. Furthermore, it follows that $x^\star > \gamma a > 0$, so that the current benefit $\pi(x^\star, a)$ is always negative when the investment is stopped at $t = T^\star$. This is due to the option value of postponing to stop the investment.

The option value and the interpretation of the value function Eq. (14) can be further illuminated by comparing with the solution that maximizes the net value in the absence of uncertainty, so that $x(t) = x_0 e^{\mu t}$ (see Appendix):

Proposition 2 *For $\sigma = 0$, the second stage decision problem is solved by the value function*

$$h^\circ(x) = \frac{\mu\gamma}{r(r-\mu)}\left(\frac{x}{x^\circ}\right)^{\frac{r}{\mu}} - \frac{1}{a(r-\mu)}\,x + \frac{\gamma}{r} \tag{17}$$

with cutoff-value $x^\circ = \gamma a > 0$, if $x^\circ > x_0$.

Obviously, when the investment is stopped at $x = x^\circ$, the current benefit π is exactly zero. There is no gain from further operating the investment, and also no option value. Since $x^\circ = \gamma a < x^\star$, uncertainty leads to stopping the investment at a later time. The role of β in Prop. 1 is taken over by r/μ in Prop. 2. Now consider the difference between the value functions for both

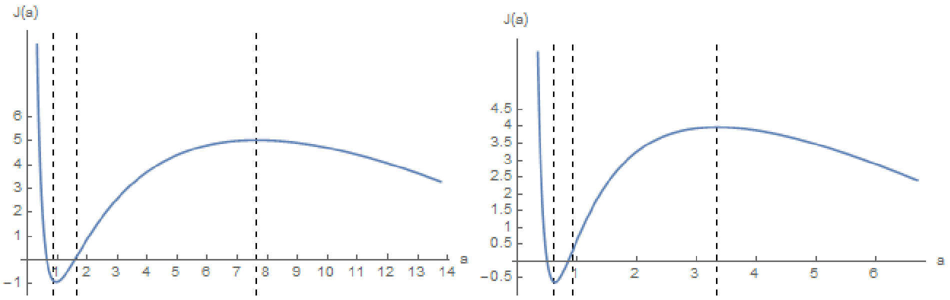

Figure 1: Two examples for net value J depending on design a. Dashed vertical lines for a_0, $\bar{a}$, $a^\star$ (left: $r = 0.02, \mu = 0.09, \sigma^2 = 0.02, c = \gamma = x_0 = 1, a_0 = 0.89, \bar{a} = 1.67, a^\star = 7.64$; right: $r = 0.09, \mu = 0.02, \sigma^2 = 0.06, c = \gamma = x_0 = 1, a_0 = 0.61, \bar{a} = 0.92, a^\star = 3.35$).

cases, i.e. the option value

$$\Theta(x) = h(x) - h^\circ(x) = \frac{\gamma}{r}(\frac{\mu}{\mu - r}(\frac{x}{x^\circ})^{\frac{r}{\mu}} + \frac{1}{\beta - 1}(\frac{x}{x^\star})^\beta). \tag{18}$$

The difference is only in the first term of the value functions h, h°. The second and the third terms in both value functions represent the value of the investment if it would never be stopped, while the first term represent the gain from stopping at the best time, either with our without uncertainty. Uncertainty has no effect on the value of a non-stopped investment, while a faster trend μ decreases these components of the value function. The option value, however, depends on both μ, σ in a non-linear way. We can conclude that $\Theta(x^\circ)$ is positive due to Eq. (16). The effect of the design a on the option value is completely captured by its influence on the cutoff-values $x^\star = \omega x^\circ$, so that the parameter ω captures the effects of uncertainty.

3.2 Optimal design

Now turn to the optimal design, that is robustness $a^\star$. After establishing the existence of an optimum, its level is further characterized. This is not straightforward as the net value J does not have a simple shape (see Fig. 1 for examples), and the optimum cannot be solved explicitly. Our analysis is confined to the relevant cases where $x^\star > x_0$. The first-order condition

Eq. (6) then evaluates to

$$c\frac{r-\mu}{x_0}a^2 = 1 - (\frac{x_0}{x^\star})^{\beta-1}.$$

Observe that this is not a quadratic in a, since $x^\star$ depends on a as well. Also the second-order condition is not easy to confirm. The proof of the following proposition (see Appendix) actually shows that $J = h(x_0) - ca$ has an inflection point at $a = \bar{a} := (\frac{2}{\beta+1})^{\frac{1}{1-\beta}} a_0 > a_0$, where we define, for convenience,

$$a_0 := \frac{x_0}{\omega\gamma} > 0. \tag{19}$$

Note from Prop. 1 that

$$\frac{x^*}{x_0} = \frac{a}{a_0}, \tag{20}$$

so that the cutoff-value relates to the initial conditions in the same way as the chosen design to a_0. In other words, a_0 denotes an extreme design choice so that $a > a_0$ needs to hold when the investment is not stopped immediately.

Proposition 3 *There exists a global inner maximum $a^\star > \bar{a}$ of J if and only if*

$$c < \frac{\gamma^2\omega^2}{(\mu-r)x_0}\left((\frac{2}{\beta+1})^{\frac{\beta+1}{\beta-1}} - (\frac{2}{\beta+1})^{\frac{2}{\beta-1}}\right). \tag{21}$$

This is, effectively, a kind of participation constraint. The right-hand side of Eq. (21) expresses an upper limit for the unit costs of robustness. If robustness would be more expensive, then the investment would yield a negative net value even if robustness is optimally chosen. This upper limit is actually the marginal value $\frac{d}{da}h(x_0)$ at the inflection point $\bar{a}$.

At $a = a_0$ the net value J is always negative. This follows from the value matching condition and the definition of a_0 according to $J(x_0, a_0, \mu, \sigma) = h(x_0) - ca_0 = h(x^\star) - ca_0 = -ca_0 < 0$. It further follows from the smooth pasting condition that $J_a(x_0, a_0, \mu, \sigma) < 0$. If $a^\star$ exists, then J has exactly one minimum and one maximum. The minimum lies between the extreme choice a_0 and the inflection point $\bar{a}$. The maximum lies to the right of the inflection point. The proof establishes that there must be a design $a^\star > a_0$ that fulfills the first and second order conditions. It might be possible, however, that a corner solution $a = a_0$ would yield a larger net value. The following propositions shows that this can be ruled out when the optimally designed investment yields a positive net value.

Proposition 4 *If Prop. 3 applies and $J(x_0, a^\star, \mu, \sigma) > 0$, then $a^\star$ is a unique global maximum.*

The following proposition provides an explicit expression for $a^\star$ from solving an implicit equation.

Proposition 5 *Assume that Prop. 3 holds and that $J(x_0, a^\star, \mu, \sigma) > 0$. Then, the optimal design can be expressed as*

$$a^\star = \left(\frac{\beta+1}{2}\right)^{\frac{z}{\beta-1}} a_0, \tag{22}$$

where $z > 1$ is the unique solution of

$$\left(\frac{2}{\beta+1}\right)^{\frac{z(1+\beta)}{\beta-1}} - \left(\frac{2}{\beta+1}\right)^{\frac{2z}{\beta-1}} = \frac{c}{\gamma}\frac{\mu-r}{\omega}a_0. \tag{23}$$

The maximum $a^\star$ is the optimal robustness in the presence of uncertainty and the trend, anticipating the expected stopping time at the beginning of the investment. The investment's net value $J(a)$ increases in a if robustness is low since the value for not stopping the investment outweighs the robustness costs. If robustness becomes too large, the marginal costs of robustness become too high. More intuition will be provided through the comparative statics and numerical examples in the following section.

4 Comparative statics

We want to know how the optimal expected life-time, the optimal design and the option value of an investment depend on various parameters, in particular μ, σ^2. We focus on the comparative statics of the stopping problem first, and the proceed with the analysis of optimal design. Does the robustness and the life time increase or decrease if there is a faster trend or more uncertainty?

4.1 Arbitrary design

The comparative statics of the stopping time with respect to uncertainty and robustness carry over from the effects on the cutoff-value $x^\star$ according to Eq. (4) - Eq. (5):

Proposition 6 *Let $x^\star$ and $E[T^\star]$ be the solution to the optimal stopping problem from Prop. 1. Then,*

$$\partial_\mu x^\star < 0,\ \partial_{\sigma^2} x^\star > 0,\ \partial_a x^\star = \gamma\omega > 0, \tag{24}$$

and

$$\partial_\mu E[T^\star] < 0,\ \partial_{\sigma^2} E[T^\star] > 0,\ \partial_a E[T^\star] > 0,\ \partial_{x_0} E[T^\star] < 0. \tag{25}$$

If the trend is faster, the current benefits deteriorate earlier. This ultimately leads to negative current benefits, and the gains from stopping the investment at a lower cut-off value (i.e. earlier) become larger. Similarly, if the conditions are less favorable at the time where the investment starts, the expected stopping time $E[T^\star]$ is earlier. More uncertainty means the more information appears over time. As more information eases the stopping decision, the premium for waiting to stop the investment raises. Thus, the cut-off value becomes larger, i.e. less favorable conditions are accepted in the end, and expected stopping is later. Intuitively, rising robustness makes the investment more beneficial in the light of changing conditions. Thus, a higher cut-off level and later stopping time is intuitive.

Eq. (25) also illuminates how the decision about robustness and expected stopping time are inter-related. The second stage decision links the first stage's robustness choice to the expected life-time. The first stage can thus also be conceived as choosing the optimal expected life-time in light of the indirect effect's costs and benefits of more long-lived investments, adjusted by the option value.

It thus helps to interpret the interlinked effects of a faster trend and rising uncertainty by further inspecting the option value Θ and how it depends on the design. First observe that $\partial_a x^\star > \gamma = \partial_a x^\circ > 0$ by Prop. 6 and Prop. 2. Thus, uncertainty enlarges the positive effect of robustness on the expected stopping time. The effect of robustness on the option value

$$\partial_a \Theta(x) = \frac{\gamma}{a(\mu - r)} (\frac{x}{x^\circ})^{\frac{r}{\mu}} \left((\frac{(r-\mu)\beta}{r(\beta-1)})^{1-\beta} (\frac{x}{x^\circ})^{\beta - \frac{r}{\mu}} - 1 \right), \tag{26}$$

however, is ambiguous (see Appendix):

Proposition 7 *Let $x^\star$ and x° be the solution to the optimal stopping problems from Prop. 1 and Prop. 2. Then, $\partial_a \Theta > 0$ if and only if*

$$\gamma a < x \omega^{\frac{\mu(\beta-1)}{r-\mu\beta}}. \tag{27}$$

Due to Eq. (16), the exponent is always positive. The inequality shows that increasing robustness raises the option value up to a maximum. For even more robustness, the value of the stopping option is decreasing again. It also shows that robustness raises the option value if the conditions become increasingly unfavorable.

4.2 Optimal expected life-time

Now turn to the comparative statics if the irreversible design a is optimally chosen. By applying the general Eq. (8) and Eq. (9) to our robustness model, the results are as follows (see Appendix):

Proposition 8 *Assume that Prop. 4 holds. Then, $\partial_c a^\star < 0$. For changing uncertainty, $\partial_{\sigma^2} a^\star > 0$ if and only if $T^{\star\star} < (\beta\mu)^{-1}$. For a changing trend, $\partial_\mu a^\star > 0$ if and only if $T^{\star\star} > \bar{T}$, where $\bar{T} > 0$ is the unique root of*

$$(\mu - r)(\frac{1}{\beta} - \mu\bar{T})\partial_\mu \beta + e^{\mu(\beta-1)\bar{T}} - \beta.$$

This highlights that the effect of uncertainty and the trend on the optimal design is not a simple one. While a faster trend might intuitively imply more robust design in some cases, it might not be worth it in other cases. Rising uncertainty increases the option value, thus making a more robust design ever more beneficial since it may be used for a longer time. Thus, the indirect effect amplifies the effect of the option value. However, uncertainty can also devaluate robustness, in particular for quite long-lived investments ($T^{\star\star} > (\mu\beta)^{-1}, \bar{T}$). Then, less uncertainty and a faster trend imply more robustness. For long-lived investments it might be particularly beneficial to increase robustness in light of a faster trend because the gained flexibility outweighs higher robustness costs. More uncertainty does not leads to increasing robustness for such investments, because the option value does not rise sufficiently (or even decreases) to justify additional robustness costs. These comparative statics of robustness to uncertainty are thus in line with the option value decreasing once a certain threshold is crossed (cf. Prop. 7). Conversely, if the optimal expected life-time is comparatively short, robustness would be increased if there is more uncertainty or a slower trend. For quite short-lived investments a faster trend leads to decreasing robustness because higher robustness costs do not balance the gains from robustness that are only achieved for a relative limited time. These short-lived investments, on the other hand, may substantially benefit from getting more information due to rising uncertainty such that it might be worth to increase robustness despite its additional costs. Other cases are possible for investments with intermediate life-times.

The threshold $(\beta\mu)^{-1}$ gives some indication about what time scales might make the difference between the cases. For the special case where the trend in the conditions μ roughly balances the discount rate r, the parameter β comes close to unity. Thus $(\beta\mu)^{-1} \approx 1/r$. For usual discount rates, the long-lived investments with the 'unconventional' comparative statics are then those with economic life times of more than 20 to 50 years. Even longer life-times are quite common, e.g., for buildings or transport infrastructure.

We now assess changes of the expected life-time if the investment's robustness is optimally chosen (see Appendix). This requires to add up the direct and indirect effects in Eq. (10) and Eq. (11).

Proposition 9 *Let Prop. 4 hold. Then,* $\partial_c T^{\star\star} < 0$*. If* $T^{\star\star} \leq (\beta\mu)^{-1}$*, then* $\partial_{\sigma^2} T^{\star\star} > 0$*, and if* $T^{\star\star} \leq \bar{T}$*, then* $\partial_\mu T^{\star\star} < 0$*.*

For $T^{\star\star} > \bar{T}$, we do not obtain an analytical result for the effect of the trend, and similarly for $T^{\star\star} > (\beta\mu)^{-1}$ with respect to uncertainty. Our numerical experiments (see below for examples) yet lend to the hypothesis that the optimal expected life time of particularly long-lived investments further decreases for a faster trend, and further increases for rising uncertainty. We summarize the main comparative statics results in Tab. 1. The effects of

Optimal expected life time	**Changing parameters**	**Effect on** Robustness $a^\star$	Life time $T^{\star\star}$
Short			
$T^{\star\star} < \bar{T}$	Trend μ	(−)	(−)
$T^{\star\star} < (\beta\mu)^{-1}$	Uncertainty σ^2	(+)	(+)
Long			
$T^{\star\star} > \bar{T}$	Trend μ	(+)	(?)
$T^{\star\star} > (\beta\mu)^{-1}$	Uncertainty σ^2	(−)	(?)

Table 1: Comparative statics results.

rising trend or uncertainty are ambiguous as expected in section 2. They depend on whether the chosen optimal life time of the investment is relatively short or long. If the chosen life time is relatively short, the direct and indirect effects on optimal life time, both due to trend and uncertainty, go in the same direction. Thus, the overall effect of rising trend is then negative while the one of rising uncertainty is then positive. If the chosen life time is relatively long, the direct and indirect effects go in opposite directions. Unfortunately, we cannot prove the overall effect analytically in this case.

When there is a faster trend there are gains from a lower expected cut-off value from the negative direct effect as well as gains from decreasing robustness and from the negative indirect effect, if the life-time is relatively short. If the life-time is relatively long, the gains from the now positive indirect effect might outweigh the gains from a still lower expected cut-off value such that the expected optimal life-time might rise.

When there is more uncertainty there are gains from a larger option value due to the positive direct and indirect effect, if the investment is relatively short-lived. If the life-time is relatively long, a less robust design might outweigh a larger option value, such that the expected optimal life-time is shortened.

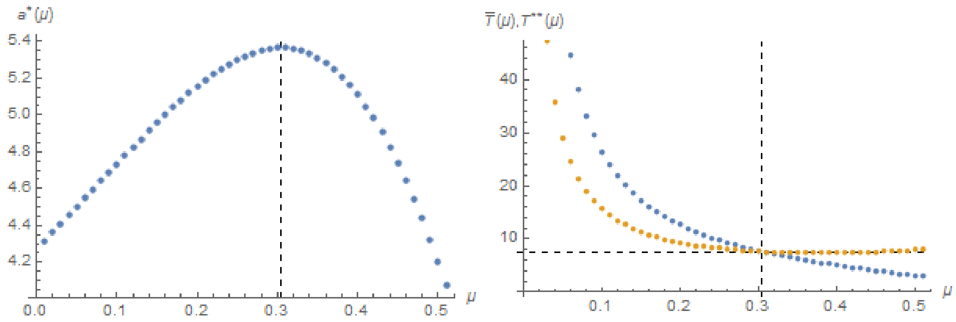

Figure 2: Example for optimal robustness (left) and optimal expected life time (right, blue) depending on the trend μ (with $r = 0.041, \sigma^2 = c = \gamma = 1, x_0 = 4$). At the dashed lines $T^{\star\star} = \bar{T}$, and $\bar{T}$ as a function of μ (right, yellow).

4.3 Numerical experiments

The analytical results show that the effect of uncertainty and trend on expected life-time of an optimally adapted investment can be ambiguous. For some cases, we can make clear analytical predictions, while in other cases the outcomes depend on the solutions of implicit equations that do not allow for a closed-form representation. We thus explore these cases by means of numerical solutions in order to illustrate our results and to improve our interpretation.

Fig. 2 shows optimal robustness and optimal expected life time depending on rising trend for a specific scenario. In accordance with Tab. 1, ranges with increasing and with decreasing robustness can be observed, separated by a trend μ where the optimal expected life time equals the threshold $\bar{T}$. Expected optimal life-time decreases if $T^{\star\star} < \bar{T}$ as has been shown in Prop. 9. The theory does not show the effect of a faster trend on investments with a long life-time. Here it is decreasing. In the example, if $\mu = 0.4$, then $T^{\star\star} = 5.06 < \bar{T} = 7.42$, so that optimal robustness as well as optimal expected life time decreases. If $\mu = 0.30$, then $\bar{T} = 7.53 > T^{\star\star} = 7.14$, so that optimal robustness increases.

If investments with a comparatively long (expected optimal) life-time are exposed to a faster trend in detrimental exogenous conditions, the additional costs of designing the investment in a more robust way pay off. This is

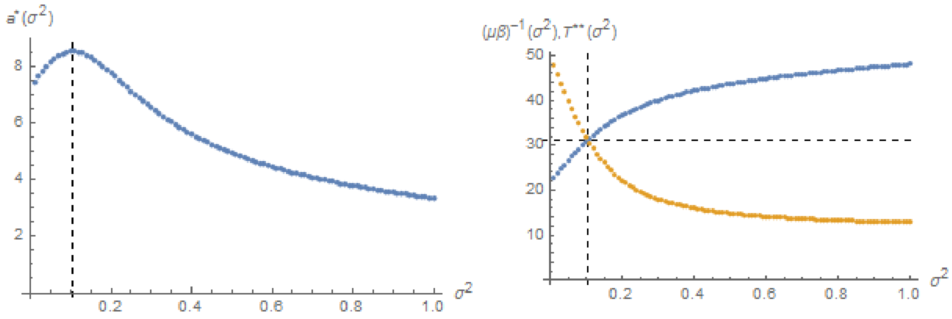

Figure 3: Example for optimal robustness (left) and optimal expected life time (right, blue) depending on uncertainty σ^2 (with $\mu = 0.09, r = 0.02, x_0 = c = \gamma = 1$). At the dashed lines $T^{\star\star} = (\beta\mu)^{-1}$, and $(\beta\mu)^{-1}$ as a function of σ^2 (right, yellow).

intuitive as the benefits from more robustness are obtained for a longer time. Yet, increasing robustness is not sufficient to compensate the faster trend in the conditions completely – the investment's life-time become shorter and shorter. So, if the trend becomes even faster, the life-time becomes so short that more robustness is no longer justified. The decision rule switches to less robust designs in light of heavily detrimental conditions. Ultimately, a reduced life-time is the necessary consequence.

Fig. 3 shows optimal robustness and optimal expected life time depending on uncertainty for a specific scenario. Again, the different cases according to Tab. 1 can be observed. For low uncertainty, robustness is increased, while the investment becomes less robust for high uncertainty. The expected optimal life-time increases for low uncertainty with $T^{\star\star} < (\mu\beta)^{-1}$ as has been shown in Prop. 9, and rises further on if uncertainty becomes so severe that the robustness is decreased. In the example, if $\sigma^2 = 0.05$, then $T^{\star\star} = 26.61 < (\mu\beta)^{-1} = 40$, and if $\sigma^2 = 0.2$, then $T^{\star\star} = 36.68 > (\mu\beta)^{-1} = 22.22$.

If an investment is designed for relative certain conditions, the expected optimal life-time is comparatively short. There are only limited reasons to keep an investment with negative current benefits running since the option value is low. If uncertainty rises, a more robust design become beneficial to even out random fluctuations. Then, the indirect effect of robustness on life-time adds to the increasing option value, so that life-times are further extended. At some point, however, more uncertainty leads to decreasing robustness. Although a longer (expected) life-time might be a good reason to invest more robustly, two effects counterbalance this. First, there are diminishing returns from robustness anyway. Second, as shown in Prop. 7, the option value begins to decrease at some level of robustness. The overall effects on life-time remain positive yet. The (negative) indirect effect of reduced robustness is overcompensated by the (positive) direct effect from uncertainty on the stopping times.

Interestingly, we were not able to find parameter sets for a case where a faster trend leads to longer optimal expected life-times, or where rising uncertainty implies shorter life-times. Our experiments showed both the cases of positive and negative effects of rising uncertainty and a faster trend on optimal robustness. Optimal life-time yet always seems to increase with uncertainty, and decreases for faster trends. The direct effects on life-time seem to dominate the indirect effects.

5 Conclusion

This paper started from the question whether the expected life-time of an investment with irreversible technical design should become shorter or more robust if detrimental exogeneous conditions change at a higher speed or with more uncertainty. We first analyzed the problem from a general perspective, and then focused on an application with geometric Brownian motion and a technical design parameter that can be interpreted as the investment's robustness.

The effects of trends and uncertainty on optimal investment design and life-time can generally be decomposed into direct and indirect effects. Directly, a faster trend leads to shorter life-times if the investment becomes unprofitable earlier. In line with real options theory, more uncertainty leads stopping the investment at a later time due to an increasing premium to wait. The indirect effect stems from the adjustment of the technical design to different trends and levels of uncertainty, mediated through the influence of design on life-time. These indirect effects generally introduce ambiguity. In the application, conditions for different cases can be identified, that particularly depend on whether the investment is comparatively short- or long-lived. In the short-lived cases, the direct effects dominate. One might say, if time does not matter so much, everything is intuitive. In contrast, for comparatively long-lived investments, more complex and prima facie counterintuitive designs can be optimal. More uncertainty then leads to less robust designs, while faster trends make more robust designs optimal.

This analysis, although addressing a quite common decision problem, thus shows some unexpected effects that have, to our knowledge, not been investigated in the theoretical literature so far. Obviously, there are some limitations to our analysis that lend to natural extensions. It showed up to be complicated to derive the general comparative statics about the optimal life-time of particularly long-lived investments. Results likely depend on the concrete numerical application. Geometric Brownian motion was deliberatively chosen as a case with substantial uncertainty. Further research could focus on alternative stochastic processes such as arithmetic Brownian motion, and other specifications of the current benefits. Our assumption of completely irreversible design is admittedly an extreme one, chosen to put the main effects to the surface. A more general model might include more flexibility by considering subsequent investment cycles. A further interesting extension would be to consider risk aversion, as this would balance the effects from uncertainty and robustness in another way. Most of such extensions would likely require simulation methods as it is common in the real options literature.

Although theoretical in nature, our results may be important for both private and public decisions about long-lived investments. Guidelines for wisely planning the design and life-time could avoid unnecessary excessive expenditures in infrastructure development, construction projects, the energy transition, and in dealing with the impacts of climate change. The general considerations made in this paper can be the basis for applied numerical computations. More generally, one take-home message for decision makers is that uncertainty about unfavorable conditions does not necessarily require more robust investment. Flexibility in terms of investments with shorter life-times can pay off.

Appendix

Proof. 1 (Optimal stopping with arbitrary design) We first show that the function Eq. (14) satisfies the Hamilton-Jacobi-Bellmann equation. This can be tested straightforwardly by differentiation. Note that

$$\mu x h' = \frac{\mu\gamma}{r}\frac{\beta}{\beta-1}\left(\frac{x}{x^\star}\right)^\beta - \frac{\mu x}{a(r-\mu)}, \tag{28}$$

$$\frac{1}{2}\sigma^2 x^2 h'' = \frac{\sigma^2\gamma}{2r}\beta\left(\frac{x}{x^\star}\right)^\beta, \tag{29}$$

so that the Hamilton-Jacobi-Bellmann equation evaluates to $\sigma^2\beta^2 + (2\mu - \sigma^2)\beta - 2r$, which vanishes according to the definition of β. Also the value matching and smooth pasting conditions $h(x^\star) = h'(x^\star) = 0$ are straightforward to verify. Moreover, Eq. (29) shows that $h'' > 0$, so that $x^\star$ is the global minimum of the value function. Thus, the investment would be immediately be stopped if $x_0 \leq x^\star$. ■

Proof. 2 (Decision without uncertainty) We first show that the function Eq. (17) satisfies the Hamilton-Jacobi-Bellmann equation when considering the additional restriction $\sigma = 0$. This can be tested straightforwardly by differentiation and substitution. Note that

$$\mu x h^{\circ\prime} = \frac{\gamma\mu}{r-\mu}(\frac{x}{x^\circ})^{\frac{r}{\mu}} - \frac{\mu}{a(r-\mu)}x, \tag{30}$$

$$-rh^\circ = -\frac{\gamma\mu}{r-\mu}(\frac{x}{x^\circ})^{\frac{r}{\mu}} + \frac{r}{a(r-\mu)}x - \gamma, \tag{31}$$

so that the Hamilton-Jacobi-Bellmann equation vanishes. Again the value matching and smooth pasting conditions $h(x^\circ) = h'(x^\circ) = 0$ are straightforward to verify. Note also that

$$h^{\circ\prime\prime} = \frac{\gamma}{r-\mu}(\frac{r}{\mu} - 1)(\frac{x}{x^\circ})^{\frac{r}{\mu}} x^{-2} > 0, \tag{32}$$

so that x° is the global minimum of this value function. Thus, the investment would immediately be stopped if $x_0 \leq x^\circ$. ■

Proof. 3 (Optimal design: existence of inner maximum) Solve

$$\max_a \; J(x_0, a, \mu, \sigma) = h(x_0) - ca. \tag{33}$$

By considering that a_0 is defined such that $\frac{x_0}{x^\star} = \frac{a_0}{a}$, the first-order condition can be written, after some re-arranging, as

$$\partial_a J = -\frac{\gamma\beta}{r(\beta-1)} a_0^\beta a^{-(1+\beta)} - \frac{x_0}{a^2(\mu - r)} - c = 0. \tag{34}$$

The second-order condition requires

$$\partial_{aa} J = \frac{\gamma\beta(\beta+1)}{r(\beta-1)} a_0^\beta a^{-(2+\beta)} + \frac{2x_0}{a^3(\mu - r)} < 0. \tag{35}$$

Yet, the second derivative $\partial_{aa}J$ vanishes at $a = \bar{a}$ (exactly once), since $\partial_{aa}J$ becomes negative if a exceeds $\bar{a}$. For $a < \bar{a}$, Eq. (35) cannot be satisfied.

We now show that the first and second-order condition will be satisfied at some point $a^\star > \bar{a}$ if Eq. (21) holds. First note that, by definition of a_0, the expected net value $J(x_0, a_0, \mu, \sigma) = h(x_0) - ca_0 = h(x^\star) - ca_0 < 0$. On the other hand, it can easily be seen from Eq. (34) that $\lim_{a \to \infty} J(x_0, a, \mu, \sigma) \to -\infty$. Thus, since J is continuous, there must be a global inner maximum if J increases at least at one point $a > a_0$. This is indeed the case. In particular, rearranging and evaluating Eq. (21) yields $\partial_a J(x_0, \bar{a}, \mu, \sigma) > 0$.

This implies, beyond existence, that the inner maximum $a^\star > \bar{a}$, since $a^\star$ have to be to the right of the inflection point, where $J(a)$ is concave. Moreover, $a^\star > a_0$, since $\bar{a} = a_0(\frac{2}{\beta+1})^{\frac{1}{1-\beta}} > a_0$ because $(\frac{2}{\beta+1})^{\frac{1}{1-\beta}} > 1$.

The additional assumption is also necessary. Suppose that $\partial_a J(x_0, \bar{a}, \mu, \sigma) < 0$. Then J will remain decreasing above $\bar{a}$ since there is no further inflection point. Since J is also monotonically decreasing for $a < \bar{a}$, we would only obtain a corner solution $a = 0 < a_0$, i.e. a situation where the investment would be stopped immediately. ■

Proof. 4 (Optimal design: existence of unique global maximum) When Prop. 3 holds, then $a^\star > \bar{a} > a_0$ is a global inner maximum of $J(a, x_0)$ in the domain, where $x^\star > x_0$ holds. If $J(a^\star, x_0) > 0$ then $J(a^\star)$ is also larger than $J(a_0)$ in any case because $J(a_0)$ is always negative. $a^\star$ would then be a global maximum. Furthermore, $a^\star$ is then the only maximum, since $J(a)$ has only one inflection point and only one domain where $J(a)$ is concave. ■

Proof. 5 (Optimal robustness) We first show that Eq. (22) yields the optimal design if Eq. (23) is fulfilled for some $z > 1$. In the second step, existence of z is shown.

We know from the previous propositions that some global maximum $a^\star > \bar{a}$ exists, and that it satisfies the first-order condition Eq. (34). Substitute $a^\star$ from Eq. (22) into Eq. (34) to obtain $\frac{\beta\gamma}{a_0 r(\beta-1)}(\frac{2}{\beta+1})^{z(-\frac{1+\beta}{1-\beta})} - \frac{x_0}{(\mu-r)a_0^2}(\frac{2}{\beta+1})^{z(-\frac{2}{1-\beta})} - c = 0$. Rearranging gives Eq. (23). Moreover, since $z > 1$, we have $((\frac{2}{\beta+1})^{\frac{1}{1-\beta}})^z > (\frac{2}{\beta+1})^{\frac{1}{1-\beta}}$, so that $a^\star > \bar{a}$.

Now turn to existence and uniqueness of z. If there would be no $z > 1$ solving Eq. (23), then there would be no $a > \bar{a}$ solving Eq. (34), which would contradict Prop. 4. If there would be more than one $z > 1$ solving Eq. (23), then there would also be more than one $a > \bar{a}$ solving Eq. (34). This would then contradict Prop. 4. ■

Proof. 6 (Comparative statics with arbitrary design) First turn to the derivatives of the stopping rule $x^\star$. Recall that $\omega > 1$, so that we directly obtain

$$\partial_a x^\star = \frac{\gamma(r-\mu)\beta}{r(\beta-1)} = \gamma\omega > 0.$$

For the following it is helpful to know the derivative of the positive root of characteristic polynomial Eq. (15) (see standard literature (e.g. Dixit and Pindyck, 1994), which is extended here to $\mu > r$). Totally differentiating the polynomial and considering $\sigma^2 > 0$ yields $\partial_\mu \beta < 0$. Furthermore, $\partial_{\sigma^2}\beta > 0$

iff $\mu > r$. We now obtain

$$\partial_{\sigma^2} x^\star = -\frac{a\gamma(r-\mu)}{r(\beta-1)^2}\partial_{\sigma^2}\beta > 0,$$

and

$$\partial_\mu x^\star = \frac{a\gamma}{r(\beta-1)^2}\alpha,$$

with $\alpha = -\partial_\mu \beta(r-\mu) - \beta(\beta-1)$.

We show in the following that $\partial_\mu x^\star < 0$. The sign of $\partial_\mu x^\star$ carries over to the sign of α. Substituting the explicit expressions for the root β and its partial derivative yields

$$\alpha = \frac{2\sigma^2\beta(-4r\sigma^2 - \sigma^4 - 4\mu^2 + (\sigma^2+2\mu)\sqrt{(\sigma^2-2\mu)^2+8r\sigma^2})}{4\sigma^4\sqrt{(\sigma^2-2\mu)^2+8r\sigma^2}}. \tag{36}$$

The denominator is always positive, and the outer bracket of the numerator can be rearranged to

$$(\sigma^2+2\mu)(\sqrt{(\sigma^2-2\mu)^2+8r\sigma^2} - (\sigma^2+2\mu)) + 4\sigma^2(\mu - r).$$

If can be verified by some equivalence transformations that this expression is negative iff $\frac{16\sigma^4(\mu-r)^2}{(\sigma^2+2\mu)^2} > 0$. The latter obviously holds. Thus, α and consequently $\partial_\mu x^\star$ are negative.

Finally, turn to the comparative statics for $E[T^\star]$. With respect to σ, a they can be determined directly from Eq. (4),Eq. (5). The derivative $\partial_\mu E[T^\star]$ becomes negative according to Eq. (3), because the expected stopping time is substracted from the negative $\partial_\mu x^\star$. Finally, we obtain $\partial_{x_0} E[T^\star] = -\frac{1}{\mu x_0} < 0$ by considering $E[x(t)] = x_0 e^{\mu t}$. ■

Proof. 7 (Comparative statics of option value) The derivative Eq. (26) can be written as $\partial_a \Theta(x) = \alpha_1\alpha_2$ with $\alpha_1 := \frac{\gamma}{\mu-r}(\frac{x}{x^\circ})^{\frac{r}{\mu}}\frac{1}{a}$, and $\alpha_2 := \omega^{1-\beta}(\frac{x}{x^\circ})^{\beta-\frac{r}{\mu}} - 1$.

If $r < \mu$, then $\alpha_1 > 0$. Due to Eq. (16), $\beta - \frac{r}{\mu} > 0$, so that $\alpha_2 > 0$ is equivalent to $\omega^{\frac{\mu(1-\beta)}{r-\mu\beta}} < \frac{x}{x^\circ}$.

If $r > \mu$, then $\alpha_1 < 0$. Due to Eq. (16), $\beta - \frac{r}{\mu} < 0$, so that $\alpha_2 < 0$ is equivalent to the same expression $\omega^{\frac{\mu(1-\beta)}{r-\mu\beta}} < \frac{x}{x^\circ}$.

Since $x^\circ = \gamma a$, the expression (for any of both cases) is equivalent to Eq. (27). ■

Proof. 8 (Comparative statics with optimal design) We know from Eq. (8) and Eq. (9) that $\partial_\mu a^\star \doteq \partial_{a\mu} J(x_0, a^\star, \mu, \sigma^2)$ and $\partial_{\sigma^2} a^\star \doteq \partial_{a\sigma^2} J(x_0, a^\star, \mu, \sigma^2)$. In addition,

$$\partial_c a^\star \doteq \partial_{ac} J(x_0, a^\star, \mu, \sigma^2, c) = -1 < 0. \tag{37}$$

Considering the first-order condition for optimal design and Eq. (19) for a_0 in section 3.2, the following holds:

$$\partial_{a\sigma^2} J = -\frac{\gamma}{r(\beta-1)} a^{-\beta-1} {a_0}^\beta \partial_{\sigma^2}\beta(\beta \ln(\frac{a_0}{a}) + 1), \tag{38}$$

$$\partial_{a\mu} J \;=\; -\frac{\gamma}{r(\beta-1)} a^{-\beta-1} {a_0}^\beta (\beta \partial_\mu \beta \ln(\frac{a_0}{a}) + \frac{\beta^2}{r-\mu} + \partial_\mu \beta) + \frac{x_0}{a^2(\mu-r)^2}. \tag{39}$$

In order to determine the sign of Eq. (38) and Eq. (39), we introduce a change of variables. Let $a^\star > \bar{a} > a_0$ be as defined in Eq. (19) and in Prop. 3. It will be helpful to express $a^\star$ as dependent of the optimal expected life-time, so we define $\tau := \frac{1}{\mu} \ln(\frac{a}{a_0})$. If the design is optimal, we have $\frac{x_0}{x^\star} = \frac{a_0}{a^\star}$ due to Eq. (20), so that $T^{\star\star} = \frac{1}{\mu} \ln(\frac{x^\star}{x_0}) = \tau$. Then, the optimal design can be expressed as $a^\star = a_0 e^{\mu\tau}$.

We now turn to Eq. (38). The factor before the bracket with the logarithm is always positive since $\partial_{\sigma^2}\beta > 0$ iff $\mu > r$ (and $\beta < 1$). If we change variables to τ in Eq. (38), the bracket is equivalent to $1 - \beta\mu\tau$. This is decreasing in τ with the zero at $(\beta\mu)^{-1}$. Thus, Eq. (38) is positive iff $\tau < (\beta\mu)^{-1}$. Since we need to evaluate at the optimal design $a^\star$, we can conclude that $T^{\star\star} < (\beta\mu)^{-1}$ if and only if $\partial_{a\mu} J(a^\star) > 0$.

Finally, turn to Eq. (39). The change of variables yields, after some transformations,

$$\partial_{a\mu} J(a^\star) \doteq (\mu - r)\partial_\mu \beta(\frac{1}{\beta} - \mu\tau) + e^{\mu(\beta-1)\tau} - \beta. \tag{40}$$

We now show that this expression has exactly one positive root for τ (which cannot be solved explicitly) , denoted by $\bar{T}$. First, the expression is continuous in τ. Second, observe that the second derivative with respect to τ is

$$\mu^2(\beta-1)^2 e^{\mu(\beta-1)\tau} > 0,$$

i.e. Eq. (40) is strictly convex in τ. Third, evaluate Eq. (40) at $\tau = 0$ to obtain

$$\partial_\mu \beta \frac{\mu - r}{\beta} + 1 - \beta.$$

By substituting the explicit expressions for $\partial_\mu \beta, \beta$, this is equivalent to

$$\frac{(\sigma^2 + 2\mu)(\sqrt{(\sigma^2 - 2\mu)^2 + 8r\sigma^2} - (\sigma^2 + 2\mu)) + 4\sigma^2(\mu - r)}{2\sigma^2\sqrt{(\sigma^2 - 2\mu)^2 + 8r\sigma^2}},$$

which is, in turn, equivalent to α from Eq. (36). It has been shown in the proof of Prop. 6 that $\alpha < 0$, so that Eq. (40) is always negative at $\tau = 0$. We can thus summarize that Eq. (40) is convexly increasing from some negative value, so it needs to vanish exactly once, and becomes positive for higher values of τ.

We have thus shown that the root $\bar{T} > 0$ is well-defined, and that Eq. (40) is negative iff $\tau < \bar{T}$. Thus, since we evaluated at $a = a^\star$, this implies that $T^{\star\star} > \bar{T}$ iff $\partial_{a\mu} J(a^\star) > 0$. ■

Proof. 9 (Comparative statics of optimal expected life-time) Consider Eq. (10) - Eq. (11) and the comparative statics results from Prop. 6 and Prop. 8. If $T^{\star\star} \leq (\beta\mu)^{-1}$ then

$$\partial_{\sigma^2} T^{\star\star}(x_0, \mu, \sigma^2) = \partial_{\sigma^2} E[T^\star] + \partial_a E[T^\star] \cdot \partial_{\sigma^2} a^\star > 0, \tag{41}$$

and if $T^{\star\star} \leq \bar{T}$ then

$$\partial_\mu T^{\star\star}(x_0, \mu, \sigma^2) = \partial_\mu E[T^\star] + \partial_a E[T^\star] \cdot \partial_\mu a^\star < 0. \tag{42}$$

Finally, $\partial_c T^{\star\star}(x_0, \mu, \sigma^2, c) = \partial_a E[T^\star] \cdot \partial_c a^\star < 0.$ ■

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