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Generating unfavourable VaR scenarios with patchwork copulas

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Abstract: The central idea of the paper is to present a general simple patchwork construction principle for multivariate copulas that create unfavourable VaR (i.e. Value at Risk) scenarios while maintaining given marginal distributions. This is of particular interest for the construction of Internal Models in the insurance industry under Solvency II in the European Union.

Key words: copulas, patchwork copulas, Bernstein copulas, Monte Carlo methods

AMS Classification: 62H05, 62H12, 62H17, 11K45

1. Introduction

Reasonable VaR-estimates from original data or suitable scenarios for risk management within so-called Internal Models are – besides the banking sector under Basel III – of particular interest in the insurance industry under Solvency II (see, e.g., Cadoni (2014); Cruz (2009); Doff (2011,2014); McNeil et al. (2015); Arbenz et al. (2012) or Sandström (2011)). In this paper, we propose a simple stochastic Monte Carlo algorithm on patchwork copulas for the generation of VaR scenarios that are suitable for comparison purposes in Internal Models for the calculation of solvency capital requirements. Note that the European Union (2015) concerning the implementation of Solvency II in the EU requires the consideration of such scenarios in several Articles, in particular in Article 259 on Risk Management Systems saying that insurance and reinsurance undertakings shall, where appropriate, include performance of stress tests and scenario analyses with regard to all relevant risks faced by the undertaking, in their risk-management system. The results of such analyses also have to be reported in the ORSA (Own Risk and Solvency Assessment, see e.g. Ozdemir (2015)) as described in Article 306 of the Commission Delegated Regulation. The problem is, however, that the Commission Delegated Regulation does not make any clear statements on how such stress tests or scenario analyses have to be performed.

Article 1 of the Commission Delegated Regulation defines a “scenario analysis” as an analysis of the impact of a combination of adverse events. The Monte Carlo simulation algorithm developed in this paper allows for a mathematically rigorous description how such scenarios can be generated, being flexible enough to cover also extreme situations.
2. Unfavourable patchwork copulas

Patchwork copulas in the context of risk management have been treated in detail by Arbenz et al. (2012), Cottin and Pfeifer (2014), Pfeifer (2013), Pfeifer et al. (2016, 2017, 2019) and Hummel (2018), among others. In several of the cited papers the question of an unfavourable, i.e. superadditive VaR estimate for a portfolio of aggregated risks was in particular emphasized, see also Pfeifer and Ragulina (2018). However, the construction of worst VaR scenarios in this context is quite complicated; a numerical approach to a constructive solution is e.g. given by the rearrangement algorithm (see e.g. Arbenz et al. (2012), Embrechts et al. (2013) or Mainik (2015)). From a practical point of view, simpler and yet explicit constructions for unfavourable VaR estimates by appropriate copula constructions seem to be a useful alternative. In this paper, we describe how such a construction could be performed. We start with an explicit approach in two dimensions that is later extended to arbitrary dimensions.

**Theorem 1.** Let, for $d \geq 2$, $d \in \mathbb{N}$, $\mathbf{U} = (U_1, \ldots, U_d)$ and $\mathbf{V} = (V_1, \ldots, V_d)$ be $d$-dimensional random vectors over $[0,1]^d$ with continuous uniform margins (i.e., $\mathbf{U}$ and $\mathbf{V}$ represent $d$-dimensional copulas). Let further $I$ denote a binomially distributed random variable, independent of $\mathbf{U}$ and $\mathbf{V}$, with $P(I = 1) = p \in (0,1)$. Then the random vector $\mathbf{W}$ with components $W_i := I \cdot p \cdot U_i + (1-I) \cdot [p + (1-p) \cdot V_i]$ for $1 \leq i \leq d$ also has continuous uniform margins, i.e. $\mathbf{W}$ represents a $d$-dimensional copula (a kind of patchwork copula).

**Proof:** The density of $p \cdot U_i$ is given by $f_i(x) = \begin{cases} 1/p, & 0 \leq x \leq p \\ 0, & \text{otherwise} \end{cases}$ and the density of $p + (1-p) \cdot V_i$ by $g_i(x) = \begin{cases} 1/(1-p), & p \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ which implies that the density of $W_i$ is given by the mixture density $p \cdot f_i(x) + (1-p) \cdot g_i(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$.

Suppose now that a portfolio of $d$ insurance risks is considered where a mutual probabilistic dependence structure is assumed, described by $\mathbf{U}$. If the $d$ (for simplicity assumed continuous) marginal risk distribution functions are denoted by $F_1, \ldots, F_d$ and by $Q_1, \ldots, Q_d$ their pseudo-inverses (quantile functions), then both random vectors $(Q_1(U_1), \ldots, Q_d(U_d))$ and $(Q_1(W_1), \ldots, Q_d(W_d))$ represent a risk vector $\mathbf{X} = (X_1, \ldots, X_d)$ with the given marginal distributions. However, w.r.t. to risk aggregation, $\mathbf{X} := (Q_1(W_1), \ldots, Q_d(W_d))$ creates in general
an unfavourable VaR scenario for $S = \sum_{i=1}^{d} X_i$, even if $p$ is close to 1 and therefore $U$ and $W$ differ only marginally. The following graph shows the corresponding support of $W$ in two dimensions.

![Graph showing the support of W in two dimensions](image)

Fig. 1

In the sequel put $p := 1 - \beta$ for $0 < \beta < 1$. Then $W = I \cdot (1 - \beta) \cdot U + (1 - I) \cdot (1 - \beta + \beta \cdot V)$. 

We start with some preliminary Lemmata.

**Lemma 1.** Let $W_1 := (1 - \beta) \cdot U$, $W_2 := 1 - \beta + \beta \cdot V$, $Z_{ii} := Q_i(W_i)$ and $Z_{2i} := Q_i(W_{2i})$, $i = 1, 2$. Then there hold

$$F_{Z_{ii}}(x, \beta) = \begin{cases} \frac{F_i(x)}{1 - \beta}, & 0 \leq x \leq Q_i(1 - \beta) \\ 1, & x \geq Q_i(1 - \beta) \end{cases}$$

and

$$F_{Z_{2i}}(x, \beta) = \begin{cases} \frac{F_i(x)}{1 - \beta}, & 0 \leq x \leq Q_i(1 - \beta) \\ 1, & x \geq Q_i(1 - \beta) \end{cases}$$

**Proof.** We have

$$F_{Z_{ii}}(x, \beta) = P\left(Q_i((1 - \beta) \cdot U_i) \leq x\right) = P((1 - \beta) \cdot U_i \leq F_i(x))$$

$$= P\left(U_i \leq \frac{F_i(x)}{1 - \beta}\right) = \frac{F_i(x)}{1 - \beta}, \quad 0 \leq x \leq Q_i(1 - \beta)$$

and
Lemma 2. Assume that $f$ and $g$ are Lebesgue densities of independent random variables $X$ and $Y$, concentrated on the same finite interval $[0, M]$ with $M > 0$. Then $S := X + Y$ has the density $h_1$ given by

$$h_1(x) = \begin{cases} \int_0^x f(x-y)g(y)\,dy, & 0 \leq x \leq M \\ \int_{x-M}^M f(x-y)g(y)\,dy, & M \leq x \leq 2M. \end{cases}$$

If $f$ and $g$ are concentrated on the same infinite interval $[M, \infty)$ with $M \geq 0$, then $S := X + Y$ has the density $h_2$ given by

$$h_2(x) = \int_M^{x-M} f(x-y)g(y)\,dy, \quad x \geq 2M.$$

In particular, if $F$ and $G$ are the corresponding cdf’s pertaining to $f$ and $g$, resp., then in either case, $\frac{d}{dx} F \ast G(x) \bigg|_{x=2M} = 0$, where $\ast$ means convolution.

Proof. In the finite interval case, we have, by the usual convolution formula,

$$h_1(x) = \int_{\max(0, x-M)}^{\min(x, M)} f(x-y)g(y)\,dy = \int_{\max(0, x-M)}^{\min(x, M)} f(x-y)g(y)\,dy.$$ 

Now for $0 \leq x \leq M$, we have $\max(0, x-M) = 0$, $\min(x, M) = x$, from which the upper formula in brackets above follows. For $M \leq x \leq 2M$, we have $\max(0, x-M) = x-M$, $\min(x, M) = M$, from which the lower formula in brackets above follows.

The proof for the infinite interval case is analogous, observing that for $x \geq 2M$, we have

$$h_2(x) = \int_{\max(M, x-M)}^{\min(M, x)} f(x-y)g(y)\,dy = \int_{\max(M, x-M)}^{\min(M, x)} f(x-y)g(y)\,dy.$$
Further, under the conditions made, we have, in either case,

\[
\frac{d}{dx} F \ast G(x) \bigg|_{x=2M} = h_1(2M) = h_2(2M) = \int_M^f (x-y) g(y) dy = 0,
\]

as stated. ●

**Lemma 3.** Assume all \( F_i \equiv F \) being equal with quantile function \( Q \), and that \( U \) and \( V \) have independent components each. Denote

\[
F(x, \beta) := \begin{cases} 
\frac{F(x)}{1-\beta}, & x \leq Q(1-\beta) \\
1, & x \geq Q(1-\beta)
\end{cases} \quad \text{and} \quad \overline{F}(x, \beta) := \frac{F(x + Q(1-\beta)) + \beta - 1}{\beta}, \quad x \geq 0.
\]

Let further denote \( X_i := Q(W_i) \) and \( S := \sum_{i=1}^d X_i \). Then we can conclude that

\[
F_s(x, \beta) = \begin{cases} 
(1-\beta) F^{d^s}(x, \beta), & x \leq dQ(1-\beta) \\
(1-\beta) + \beta \overline{F}^{d^s}(x - dQ(1-\beta), \beta), & x > dQ(1-\beta),
\end{cases}
\]

where \( \ast \) again means convolution. If \( F \) has a density \( f \), then correspondingly

\[
f(x, \beta) := \begin{cases} 
\frac{f(x)}{1-\beta}, & x \leq Q(1-\beta) \\
0, & x \geq Q(1-\beta)
\end{cases} \quad \text{and} \quad \overline{f}(x, \beta) := \frac{f(x + Q(1-\beta))}{\beta}, \quad x \geq 0
\]

and

\[
f_s(x, \beta) = \begin{cases} 
(1-\beta) f^{d^s}(x, \beta), & x \leq dQ(1-\beta) \\
(1-\beta) + \beta \overline{f}^{d^s}(x - dQ(1-\beta), \beta), & x > dQ(1-\beta),
\end{cases}
\]

**Proof.** Let \( \xi_i \) and \( \zeta_i \) be independent random variables with the cdf’s \( F(\cdot, \beta) \) and \( \overline{F}(\cdot, \beta) \), resp. Then \( I \cdot \xi_i + (1-I) \cdot (Q(1-\beta) + \zeta_i) \) is a stochastic representation of \( X_i, i = 1, \cdots, d \), where again \( I \) is a binomial random variable with \( P(I = 1) = 1-\beta \) and \( P(I = 0) = \beta \), independent of \( (U, V) \), according to Lemma 1. Hence
\[ I \cdot \sum_{i=1}^{d} \xi_i + (1-I) \cdot \sum_{i=1}^{d} (Q(1-\beta) + \zeta_i) = I \cdot \sum_{i=1}^{d} \xi_i + (1-I) \cdot \left\{ dQ(1-\beta) + \sum_{i=1}^{d} \zeta_i \right\} \]

is a stochastic representation of \( S \). Note that the cdf of \( \sum_{i=1}^{d} \xi_i \) is \( F_{d*}(\cdot, \beta) \) and that of \( \sum_{i=1}^{d} \zeta_i \) is \( \overline{F}_{d*}(\cdot, \beta) \), from which the assertion follows. \( \bullet \)

The following examples show the effect of a risk aggregation with an unfavourable VaR scenario for two dimensions in detail.

**Example 1** (exponential distributions). Assume that \( F_1 = F_2 = \begin{cases} 0, & x < 0 \\ 1-e^{-x}, & x \geq 0 \end{cases} \). Then

\[
F_{x_1}(x, \beta) = \frac{1-e^{-x}}{1-\beta}, 0 \leq x \leq -\ln(\beta) \quad \text{and} \quad F_{x_2}(x, \beta) = \frac{\beta e^{-x}}{\beta} = 1-e^{-x-\ln(\beta)}, x \geq -\ln(\beta), i = 1, 2.
\]

For the corresponding densities, we obtain by differentiation

\[
f_{x_1}(x, \beta) = \begin{cases} \frac{e^{-x}}{1-\beta}, & 0 \leq x \leq -\ln(\beta) \\ 0, & x > -\ln(\beta) \end{cases} \quad \text{and} \quad f_{x_2}(x, \beta) = \begin{cases} 0, & x < -\ln(\beta) \\ e^{-x-\ln(\beta)}, & x \geq -\ln(\beta), \quad i = 1, 2 \end{cases}
\]

and

\[
f(x, \beta) = \begin{cases} \frac{e^{-x}}{1-\beta}, & 0 \leq x \leq -\ln(\beta) \\ 0, & x > -\ln(\beta) \end{cases} \quad \text{and} \quad \overline{f}(x, \beta) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0. \end{cases}
\]

By Lemma 3, we obtain the following density \( f_S \) of the aggregated risk \( S \):

\[
f_S(x, \beta) = \begin{cases} \frac{xe^{-x}}{1-\beta}, & 0 \leq x \leq -\ln(\beta) \\ \frac{(2 \ln(\beta)-x)e^{-x}}{1-\beta}, & -\ln(\beta) \leq x \leq -2\ln(\beta) \\ \frac{(x+2\ln(\beta))e^{-x}}{\beta}, & x \geq -2\ln(\beta) \end{cases}
\]
with the corresponding cdf $F_S$:

$$F_S(x, \beta) = \begin{cases} 
\frac{1-(1+x)e^{-x}}{1-\beta}, & 0 \leq x \leq -\ln(\beta) \\
\frac{1-2\beta+2e^{-x}\ln(\beta)+(1+x)e^{-x}}{1-\beta}, & -\ln(\beta) \leq x \leq -2\ln(\beta) \\
\frac{\beta-2e^{-x}\ln(\beta)-(1+x)e^{-x}}{\beta}, & x \geq -2\ln(\beta).
\end{cases}$$

Here $g$ is the density of $T := Q_1(U_1) + Q_2(U_2)$ (independent summands, Gamma distribution).

plots of $f_S(x, \beta)$ for $\beta = 0.1$ (red) and $g(x)$ (blue)

Fig. 2

plots of $F_S(x, \beta)$ (red), $G(x)$ (blue), and $H(x, \beta)$ (khaki), for $\beta = 0.005$

Fig. 3
Here $G$ is the cdf for $T := Q_1(U_1) + Q_2(U_2)$ (independent summands, Gamma distribution) and $H$ the cdf for $S$ under the worst VaR scenario, i.e. the distribution of $V$ corresponds to the lower Fréchet bound or countermanotonicity copula (see e.g. Embrechts et al. (2013) or Pfeifer (2013)). In this case we have

$$H(x, \beta) = \begin{cases} 
F_S(x), & x \leq -2 \ln(\beta) \\
1 - \beta, & -2 \ln(\beta) \leq x \leq -2 \ln(\beta / 2) \\
1 - \beta + \sqrt{\beta^2 - 4e^{-x}}, & x \geq -2 \ln(\beta / 2).
\end{cases}$$

Note that with the Solvency II standard $\alpha = 0.005$, we get here, for $\beta = \alpha$, $\text{VaR}_\alpha(S) = 10.5914 > \text{VaR}_\alpha(T) = 7.4301$. For the worst VaR scenario, however, we get $w\text{VaR}_\alpha(S^*) = 11.9829 > 10.5966 = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$. Note that actually the worst VaR is obtained as a limit of $\text{VaR}_{\alpha+\varepsilon}(S^*)$ for $\varepsilon \downarrow 0$ due to the right continuity of cdf’s. Seemingly $\text{VaR}_\alpha(S) = 10.5914 < 10.5966 = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$ which means that even with the construction for $S$ with $\beta = \alpha$, we still have a (quite small) diversification effect, but not in the worst VaR scenario. This changes, however, if we look at $\text{VaR}_\alpha(S) = 10.9630$ when we replace $\beta$ by $\alpha + \varepsilon$ in the definition of $W$ for e.g. $\varepsilon = 0.001$.

The following graph shows the cdf’s for several choices of $\varepsilon$.

![Graph showing the cdf's for several choices of \( \varepsilon \).](image)

plots of $F_S(x, 0.005 + \varepsilon)$ for $\varepsilon = 0.001$ (blue), $\varepsilon = 0.002$ (red), $\varepsilon = 0.003$ (khaki) and $H(x, 0.005)$ (black)

Fig. 4

The following graph shows the values of $Q_S(0.995, \beta) = F^{-1}_S(0.995, \beta)$ in the range $0.0062 \leq \beta \leq 0.0076$. 

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A numerical calculation shows that for $\alpha = 0.005$ the worst $\text{VaR}_\alpha(S) = 10.98292909$ is attained for $\beta = 0.00679331$, i.e. $\varepsilon = 0.00179331$.

![Graph showing $Q_{\alpha}(0.995, \beta)$](image)

**Fig. 5**

**Example 2** (uniform distributions). Assume that $F_1 = F_2 = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1. \end{cases}$ Then

$$F_{x_1}(x, \beta) = \frac{x}{1-\beta}, \quad 0 \leq x \leq 1-\beta \quad \text{and} \quad F_{x_2}(x, \beta) = \frac{x+\beta-1}{\beta}, \quad x \geq 1-\beta, \ i = 1, 2.$$

By Lemma 2, we obtain the following density $f_S$ of the aggregated risk $S$:

$$f_S(x, \beta) = \begin{cases} \frac{x}{1-\beta}, & x \leq 1-\beta \\ \frac{2-2\beta-x}{1-\beta}, & 1-\beta \leq x \leq 2-2\beta \\ \frac{x-2+2\beta}{\beta}, & 2-2\beta \leq x \leq 2-\beta \\ \frac{2-x}{\beta}, & 2-\beta \leq x \leq 2 \end{cases}$$

with the corresponding cdf $F_S$. 

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\[ F_S(x,\beta) = \begin{cases} 
\frac{x^2}{2(1-\beta)}, & x \leq 1-\beta \\
\frac{4x(1-\beta) - x^2 - 2(1-\beta)^2}{2(1-\beta)}, & 1-\beta \leq x \leq 2 - 2\beta \\
\frac{2\beta}{4(1-\beta)(1-x) + x^2 - 2\beta + 2\beta^2}, & 2 - 2\beta \leq x \leq 2 - \beta \\
\frac{2\beta - 4(1-x) - x^2}{2\beta}, & 2 - \beta \leq x \leq 2.
\end{cases} \]

plots of \( f_S(x,\beta) \) for \( \beta = 0.1 \) (red) and \( g(x) \) (blue)

Fig. 6

Here \( g \) is the density of \( T := Q(U_1) + Q(U_2) \) (independent summands, triangle distribution).

plots of \( F_S(x,\beta) \) (red), \( G(x) \) (blue), and \( H(x,\beta) \) (khaki) for \( \beta = 0.005 \)

Fig. 7

Here \( G \) is the cdf for \( T := Q(U_1) + Q(U_2) \) (independent summands, triangle distribution) and \( H \) the cdf for \( S \) under the worst VaR scenario, i.e. the distribution of \( V \) corresponds to the upper Fréchet bound (see e.g. Embrechts et al. (2013) or Pfeifer (2013)). In this case we have
Note that with the Solvency II standard $\alpha = 0.005$, we have here, for $\beta = \alpha$, $\text{VaR}_\alpha (S) = 1.99 = \text{VaR}_\alpha (T)$. For the worst VaR scenario, however, we get here $w\text{VaR}_\alpha (S^*) = 1.995 > 1.99 = \text{VaR}_\alpha (X_1) + \text{VaR}_\alpha (X_2)$. Note that actually the worst VaR is obtained as a limit of $\text{VaR}_{\alpha + \varepsilon} (S^*)$ for $\varepsilon \downarrow 0$ due to the right continuity of cdf’s. Seemingly $\text{VaR}_\alpha (S) = 1.99 = \text{VaR}_\alpha (X_1) + \text{VaR}_\alpha (X_2)$ which means that with the construction for $S$ we have no true diversification effect, likewise in the worst VaR scenario. This changes, however, if we look at $\text{VaR}_\alpha (S) = 1.991$ when we replace $\beta$ by $\alpha + \varepsilon$ in the definition of $W$ for e.g. $\varepsilon = 0.001$.

The following graph shows the cdf’s for several choices of $\varepsilon$.

![Graph showing cdf's for different choices of $\varepsilon$.](image)

plots of $F_S (x, 0.005 + \varepsilon)$ for $\varepsilon = 0.001$ (blue), $\varepsilon = 0.002$ (red), $\varepsilon = 0.003$ (khaki) and $H(x, 0.005)$ (black)

Fig. 8

The following graph shows the values of $Q_S (0.995, \beta) = F_S^{-1}(0.995, \beta)$ in the range $0.0054 \leq \beta \leq 0.007$. 

\[
H(x, \beta) = \begin{cases} 
  F_S(x) & x \leq 2 - 2\beta \\
  1 - \beta, & 2 - 2\beta \leq x < 2 - \beta \\
  1, & x \geq 2 - \beta.
\end{cases}
\]
A numerical calculation shows that for $\alpha = 0.005$ the worst $\text{VaR}_\alpha(S) = 1.991464466$ is attained for $\beta = 0.006035$, i.e. $\varepsilon = 0.001035$.

Note that in this example a closed-form representation for $Q_s(u, \beta)$ is given by

$$Q_s(u, \beta) = 2 - 2\beta + \sqrt{2}\beta(\beta + u - 1), 1 - \beta \leq u \leq 1 - \frac{\beta}{2}.$$ This implies

$$Q_s(1 - \alpha, \beta) = 2 - 2\beta + \sqrt{2}\beta(\beta - \alpha), \alpha \leq \beta \leq 2\alpha$$

with its maximum being attained for $\beta_o = \frac{1 + \sqrt{2}}{2}\alpha$ with value

$$Q_s(1 - \alpha, \beta_o) = 2 - \left(1 + \frac{\sqrt{2}}{2}\right)\alpha.$$ Note that the worst VaR here is $\text{wVaR}_\alpha(S^*) = 2 - \alpha$.

**Example 3** (Pareto distributions). Assume that $F_1 = F_2 = \begin{cases} 0, & x \leq 0 \\ \frac{x}{1 + x}, & x > 0. \end{cases}$ Then

$$F_{z_i}(x, \beta) = \frac{x}{(1 - \beta)(1 + x)}, \quad 0 \leq x \leq \frac{1}{\beta} - 1 \quad \text{and} \quad F_{z_i}(x, \beta) = 1 - \frac{1}{\beta(1 + x)}, \quad x \geq \frac{1}{\beta} - 1, \quad i = 1, 2.$$ For the corresponding densities, we obtain by differentiation

$$f_{z_i}(x, \beta) = \begin{cases} \frac{1}{(1 - \beta)(1 + x)^2}, & 0 \leq x \leq \frac{1}{\beta} - 1 \\ 0, & x > \frac{1}{\beta} - 1 \end{cases} \quad \text{and} \quad f_{z_2}(x, \beta) = \begin{cases} 0, & x < \frac{1}{\beta} - 1 \\ \frac{1}{\beta(1 + x)^2}, & x \geq \frac{1}{\beta} - 1 \end{cases}$$
and

$$f(x, \beta) = \begin{cases} \frac{1}{(1-\beta)(1+x)^2}, & 0 \leq x \leq \frac{1}{\beta} - 1 \\ 0, & x > \frac{1}{\beta} - 1 \end{cases}$$

and

$$\bar{f}(x, \beta) = \begin{cases} 0, & x < 0 \\ \frac{\beta}{(1+\beta x)^2}, & x \geq 0. \end{cases}$$

In order to calculate the density $f_s$ of the aggregated risk $S$, we need a suitable partial fraction representation of $f(x-y)f(y)$ and $\bar{f}(x-y)f(y)$. Note that in general, we have

$$\frac{1}{(1+x-y)(1+y)} = \frac{1}{2+x} \left[ \frac{1}{1+x-y} + \frac{1}{1+y} \right]$$

and

$$\frac{1}{(1+x-y)^2(1+y)^2} = \frac{1}{(2+x)^2} \left[ \frac{1}{(1+x-y)} + \frac{1}{(1+y)} \right]^2$$

$$= \frac{1}{(2+x)^2} \left[ \frac{1}{(1+x-y)^2} + \frac{1}{(1+y)^2} + \frac{2}{2+x} \left( \frac{1}{1+x-y} + \frac{1}{1+y} \right) \right]$$

from which we obtain, by Lemma 3,

$$F_s(x, \beta) = \begin{cases} \frac{x^2 + 2x - 2 \ln(1+x)}{(2+x)^2(1-\beta)}, & 0 \leq x \leq \frac{1}{\beta} - 1 \\ \frac{(1-2\beta)x^2 + (4-6\beta)x - 4\beta + 4 + 2 \ln(\beta x + 2\beta - 1)}{(2+x)^2(1-\beta)}, & \frac{1}{\beta} - 1 \leq x \leq 2 \left( \frac{1}{\beta} - 1 \right) \\ \frac{x^2 - 2x + \frac{2}{\beta} \ln(\beta x + 2\beta - 1)}{(2+x)^2}, & x \geq 2 \left( \frac{1}{\beta} - 1 \right) \end{cases}$$

The density $f_s(x)$ follows by differentiation.
plots of $f_S(x, \beta)$ for $\beta = 0.1$ (red) and $g(x)$ (blue)

Fig. 10

Here $g$ is the density of $T := Q_1(U_1) + Q_2(U_2)$ (independent summands).

plots of $F_S(x, \beta)$ (red), $G(x)$ (blue), and $H(x, \beta)$ (khaki) for $\beta = 0.005$

Fig. 11

Here $G$ is the cdf for $T := Q_1(U_1) + Q_2(U_2)$ (independent summands) and $H$ the cdf for $S$ under the worst VaR scenario, i.e. the distribution of $V$ corresponds again to the upper Fréchet bound. In this case we have

$$H(x, \beta) = \begin{cases} 
F_S(x, \beta), & x \leq \frac{2}{\beta} - 2 \\
1 - \beta, & \frac{2}{\beta} - 2 \leq x \leq \frac{4}{\beta} - 2 \\
1 - \beta + \sqrt{\beta^2 - \frac{4\beta}{2 + x}}, & x \geq \frac{4}{\beta} - 2.
\end{cases}$$
Note that with the Solvency II standard $\alpha = 0.005$, we have here, for $\beta = \alpha$, $\text{VaR}_\alpha(S) = 397.3168 < \text{VaR}_\alpha(T) = 403.9161$. For the worst VaR scenario, however, we get $w\text{VaR}_\alpha\left(S^*\right) = 798 > 398 = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$. Note that actually the worst VaR is obtained as a limit of $\text{VaR}_{\alpha+\varepsilon}\left(S^*\right)$ for $\varepsilon \downarrow 0$ due to the right continuity of cdf’s. Seemingly $\text{VaR}_\alpha(S) = 397.32 < 398 = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$ which means that even with the construction for $S$ we still have a (quite small) diversification effect, but not in the worst VaR scenario. This changes, however, if we look at $\text{VaR}_\alpha(S) = 488.2116$ when we replace $\beta$ by $\beta + \varepsilon$ in the definition of $W$ for e.g. $\varepsilon = 0.001$.

The following graph shows the cdf’s for several choices of $\varepsilon$.

![Graph of cdf's for several choices of $\varepsilon$.](image)

plots of $F_s(x, 0.005 + \varepsilon)$ for $\varepsilon = 0.001$ (blue), $\varepsilon = 0.002$ (red), $\varepsilon = 0.003$ (khaki) and $H(x, 0.005)$ (black)

Fig. 12

The following graph shows the values of $Q_s(0.995, \beta) = F_s^{-1}(0.995, \beta)$ in the range $0.007 \leq \beta \leq 0.012$. 

15
A numerical calculation shows that for $\alpha = 0.005$ the worst VaR, $\text{VaR}_\alpha(S) = 509.3798950$ is attained for $\beta = 0.0088963$, i.e. $\varepsilon = 0.0038963$.

These examples show that it is generally possible to obtain near worst VaR scenarios by a suitable choice of $\beta = \alpha + \varepsilon$ in the definition of $W$.

We continue with a particular construction of $W$ which allows in general for an unfavourable VaR scenario.

**Theorem 2.** For $d \in \mathbb{N}, d > 1$ let $I_d = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ denote the $d$-dimensional unit matrix and $E_d = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1 \end{bmatrix}$ the $d \times d$ matrix with all entries equal to unity. Then

$$
\Sigma_d = (1-r)I_d + rE_d = \begin{bmatrix} 1 & r & \cdots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & 1 \end{bmatrix}
$$

is a correlation matrix iff $-\frac{1}{d-1} \leq r \leq 1$. In the general case, the eigenvalues $\lambda_i$ of $\Sigma_d$ are given by $\lambda_i = 1 + (d-1)r$ and
$\lambda_i = 1 - r, i = 2, \ldots, d$. An orthonormal basis $T_1, \ldots, T_d$ of corresponding eigenvectors is given by $T_i = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $T_j = \begin{pmatrix} t_{ij} \\ \vdots \\ t_{jj} \end{pmatrix}$ for $2 \leq j \leq d$ where $t_{ij} = \begin{cases} -\frac{1}{\sqrt{j(j-1)}}, & 1 \leq i < j \\ \frac{j-1}{\sqrt{j}}, & j = i \\ 0, & i > j. \end{cases}$

Hence $\Sigma_d$ possesses the spectral decomposition $\Sigma_d = AA^\top$ with $A = T \sqrt{\Delta}$ where $T = [T_1, \ldots, T_d]$ and $\Delta = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d \end{bmatrix}$.

To prove Theorem 2, we need the following Lemma.

**Lemma 4.** For all $d \geq 2$, we have

a) $\sum_{k=2}^{d} \frac{1}{k(k-1)} = \frac{d-1}{d}$,

and for $1 \leq i \leq d$, we have

b) $\frac{i-1}{i} + \sum_{k=1}^{d-1} \frac{1}{(i+k)(i+k-1)} = \frac{d-1}{d}$.

**Proof.** Part a) of Lemma 4 can be easily proved by induction. For $d = 2$, this is obvious. Now assume that the statement is true for some $d \geq 2$. Then we have

$\sum_{k=2}^{d} \frac{1}{k(k-1)} = \sum_{k=2}^{d-1} \frac{1}{k(k-1)} + \frac{1}{d(d+1)} = \frac{d-1}{d} + \frac{1}{d(d+1)} = \frac{d}{d+1}$, hence the statement is also true for $d+1$, which proves a).

Part b) of Lemma 4 follows immediately from part a) since $\frac{i-1}{i} = \sum_{k=2}^{i} \frac{1}{k(k-1)}$ and

$\sum_{k=1}^{d-1} \frac{1}{(i+k)(i+k-1)} = \sum_{k=1}^{d} \frac{1}{k(k-1)}$.

**Proof of Theorem 2.** We first show that $TT^\top = I_d = T^\top T$. Let $TT^\top = [b_{ij}]_{i,j=1,\ldots,d}$. For $1 \leq i \leq d$ we obtain, by part b) of Lemma 4, $b_{ii} = \frac{1}{d} + \frac{i-1}{i} + \sum_{k=1}^{d-i} \frac{1}{(i+k)(i+k-1)} = 1$. For
1 \leq i, j \leq d \text{ with } i \neq j \text{ we get, with } i \vee j := \max(i, j), \text{ following part b) of Lemma 4,}

\begin{align*}
b_{ij} &= \frac{1}{d} - \frac{1}{i \vee j} + \sum_{k=\lceil i \vee j \rceil + 1}^{d} \frac{1}{k(k-1)} \frac{1}{i \vee j} + \sum_{k=1}^{d-i \vee j} \frac{1}{(k+i \vee j)(k+i \vee j-1)} \\
&= \frac{1}{d} - \frac{1}{i \vee j} + \frac{d-1}{d} - \frac{i \vee j-1}{i \vee j} = 1 - 1 = 0.
\end{align*}

This proves \( \mathbf{T}^\top \mathbf{T} = \mathbf{I}_d \). On the other hand, let \( \mathbf{T}^\top \mathbf{T} = [c_{ij}]_{i,j=1}^{\cdots,d} \). It is obvious that

\( c_{ii} = \frac{1}{i} \cdot 1 = 1 \) and for all \( 2 \leq i \leq d \), \( c_{ii} = \frac{1}{i(i-1)}(i-1) + \frac{i-1}{i} = 1. \)

Next, for all \( 2 \leq j \leq d \), we obtain \( c_{ij} = \frac{1}{\sqrt{d}} \left( -\frac{1}{\sqrt{(j+1)}}(j-1) + \sqrt{\frac{j-1}{j}} \right) = 0 \), and for all \( 2 \leq i \leq d \), we get \( c_{ii} = \frac{1}{\sqrt{d}} \left( -\frac{1}{\sqrt{(i-1)}}(i-1) + \sqrt{\frac{i-1}{i}} \right) = 0. \) Finally, for \( 2 \leq i, j \leq d \) with \( i \neq j \), we get \( c_{ij} = -\frac{1}{\sqrt{(i \vee j)(i \vee j-1)}} \left( -\frac{1}{\sqrt{(i \vee j)(i \vee j-1)}} \left( i \vee j-1 \right) + \sqrt{\frac{i \vee j-1}{i \vee j}} \right) = 0. \)

This proves \( \mathbf{T}^\top \mathbf{T} = \mathbf{I}_d. \)

Now let \( \lambda_1 = 1 + (d-1)r, \lambda_i = 1 - r, i = 2, \cdots, d \) and \( \Delta_j = \begin{bmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_d - t \end{bmatrix} \).

A standard computation yields, for \( t \in \mathbb{R} \),

\[
\mathbf{T} \Delta_j = \begin{bmatrix}
\frac{1+(d-1)r-t}{\sqrt{d}} & -\frac{1-r-t}{\sqrt{2(2-1)}} & -\frac{1-r-t}{\sqrt{3(3-1)}} & \cdots & -\frac{1-r-t}{\sqrt{(d-1)(d-2)}} & -\frac{1-r-t}{\sqrt{d(d-1)}} \\
\frac{1+(d-1)r-t}{\sqrt{d}} & \sqrt{\frac{2-1}{2}}(1-r-t) & -\frac{1-r-t}{\sqrt{3(3-1)}} & \cdots & -\frac{1-r-t}{\sqrt{(d-1)(d-2)}} & -\frac{1-r-t}{\sqrt{d(d-1)}} \\
\frac{1+(d-1)r-t}{\sqrt{d}} & 0 & \sqrt{\frac{3-1}{3}}(1-r-t) & \cdots & -\frac{1-r-t}{\sqrt{(d-1)(d-2)}} & -\frac{1-r-t}{\sqrt{d(d-1)}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1+(d-1)r-t}{\sqrt{d}} & 0 & 0 & \sqrt{\frac{d-2}{d-1}}(1-r-t) & -\frac{1-r-t}{\sqrt{d(d-1)}} \\
\frac{1+(d-1)r-t}{\sqrt{d}} & 0 & 0 & 0 & \sqrt{\frac{(d-1)(d-2)}{d}}(1-r-t) \\
\end{bmatrix}
\]

Let \( \mathbf{T} \Delta_j \mathbf{T}^\top = [d_{ij}]_{i,j=1}^{\cdots,d} \). From part a) of Lemma 4 it follows that

\[
d_{ii} = \frac{1+(d-1)r-t}{d} + (1-r-t) \sum_{k=2}^{d} \frac{1}{k(k-1)} = \frac{1+(d-1)r-t}{d} + (1-r-t) \cdot \frac{d-1}{d} = 1-t,
\]
and for $2 \leq i \leq d$, part b) of Lemma 4 gives

$$d_i = \frac{1+(d-1)r-t}{d}+(1-r-t)\left(\frac{i-1}{i}+\sum_{k=1}^{d-i} \frac{1}{(i+k)(i+k-1)}\right)$$

$$= \frac{1+(d-1)r-t}{d}+(1-r-t)\frac{d-1}{d} = 1-t.$$

Next, for $2 \leq i, j \leq d$ with $i \neq j$ we obtain from part b) of Lemma 4,

$$d_{ij} = \frac{1+(d-1)r-t}{d} - \frac{1-r-t}{i \lor j} + (1-r-t)\left(\sum_{k=1}^{d \lor i, j} \frac{1}{(i \lor j + k)(i \lor j + k - 1)}\right)$$

$$= \frac{1+(d-1)r-t}{d} - \frac{1-r-t}{i \lor j} + (1-r-t)\left(\frac{d-1}{d} - \frac{i \lor j - 1}{i \lor j}\right) = r.$$

This in turn means $T \Delta_i T^\top = \begin{bmatrix} 1-t & r & \cdots & r \\ r & 1-t & r & \cdots \\ \vdots & r & \ddots & r \\ r & \cdots & r & 1-t \end{bmatrix} = \Sigma_d - tI_d$. Consequently, the characteristic polynomial for $\Sigma_d$ is given by

$$\varphi_{\Sigma_d}(t) = \det(\Sigma_d - tI_d) = \det(T \Delta_i T^\top) = \det(T) \cdot \det(\Delta_i) \cdot \det(T^\top) = \det(T) \cdot \det(\Delta_i) \cdot \det(T^{-1})$$

$$= \det(\Delta_i) = \prod_{i=1}^{d} (\lambda_i - t).$$

Hence $\lambda_i, 1 \leq i \leq d$, are the eigenvalues of $\Sigma_d$. Therefore, $\Sigma_d$ is a correlation matrix, i.e. positive semidefinite iff $\lambda_i \geq 0$ for all $1 \leq i \leq d$, i.e. $-\frac{1}{d-1} \leq r \leq 1$.

Thus Theorem 2 is proved. •

In what follows we will call a Gaussian copula derived from the correlation matrix

$$\Sigma_d = \begin{bmatrix} 1 & r & \cdots & r \\ r & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & 1 \end{bmatrix}$$

with $r = -\frac{1}{d-1}$ a minimal correlation Gaussian copula.

3. A case study

The following example shows the effects of such an approach for the 19-dimensional data set discussed in Pfeifer et al. (2019). It contains insurance losses from a non-life portfolio of natural perils in $d = 19$ areas in central Europe over a time period of 20 years. The losses are given in Mio. monetary units.
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Tab. 2
A statistical analysis of the data shows a good fit to lognormal $\mathcal{LN}(\mu, \sigma)$-distributions for the losses per Area $k$, $k = 1, \ldots, 19$. The parameters $\mu_k$ and $\sigma_k$ for Area $k$ were hence estimated from the log data by calculating means and standard deviations.

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Tab. 4

As is to be expected, insurance losses in locally adjacent areas show a high degree of stochastic dependence, which can also be seen from the following correlation tables. For a better readability, only two decimal places are displayed.

correlations between log losses in adjacent areas

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Tab. 5

correlations between original losses in adjacent areas

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Tab. 6

21
The following graph shows estimated cdf’s on a basis of 100,000 Monte Carlo simulations for the aggregated loss with a Bernstein copula representing \( U \) and a minimal correlation Gaussian copula representing \( V \), for various values of \( p \). For comparison purposes, we have also added an estimated cdf for the aggregated loss for a Bernstein copula representing \( U \) and an upper Fréchet (or comonotonicity) copula representing \( V \).

![Plots of estimated cdfs in the tail](image)

**Fig. 14**

The following graphs correspond to a Bernstein copula \( U \) with a minimal correlation Gaussian copula \( V \): black: \( p = 1 \); blue: \( p = 0.99 \); red: \( p = 0.994 \)

The following graphs correspond to a Bernstein copula \( U \) with \( p = 0.994 \) but different copulas \( V \): green: upper Fréchet copula; grey: independence copula

The following table shows the estimated risk measures \( \text{VaR}_\alpha \) for \( \alpha = 0.005 \) (Solvency II-standard) for the various values of \( p \) and different types of \( V \).

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<td>min corr Gauss</td>
<td>upper Fréchet</td>
<td>independence</td>
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<tr>
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<td>5,272</td>
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<td>5,018</td>
<td>2,229</td>
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</table>

**Tab. 7**

As can clearly be seen, the patchwork construction with the minimal correlation Gaussian copula representing \( V \) with no tail dependence gives the largest VaR estimate here and is typically larger than the upper Fréchet copula which has a positive tail dependence. Note that the sum of individual VaR’s is given by 2,745 which means that using the Bernstein copula alone would lead to a diversified portfolio while all others do not.
Finally, it should be pointed out that the effects described here are independent of the particular copula chosen for $U$, i.e. the magnitude of the estimated VaR’s under the patchwork construction would remain roughly equal also under an elliptical, an Archimedean, a vine or an adapted Bernstein copula approach for $U$ (see e.g. Pfeifer and Ragulina (2020)).

References