Global Analysis 2 (SS 2014)

§ 0. Guiding example of the lecture

Goal: We want to study

- (pseudo-) differential operators, e.g. Δ Laplacian, on mfds.
- solutions to (pseudo-) diff. equations, eg $\Delta u = f$
- prove Hodge-de-Rham theorem, study spectral properties.

Consider the following example of a diff. operator: ($\mathbb{1} \equiv \mathrm{id}$)

$$D = \mathbb{1} - \Delta : C_0^\infty(\mathbb{R}^n) \longrightarrow C_0^\infty(\mathbb{R}^n); \quad \Delta = \sum_{i=1}^n \partial_{x_i}^2$$

For $u \in C_0^\infty(\mathbb{R}^n)$ we may define the Fourier transform:

$$\hat{u}(\xi) \equiv (\mathcal{F}_{x\to\xi} u)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x)\, d^n x$$

The Fourier transform of u is $\hat{u} \in C^\infty(\mathbb{R}^n)$, not compactly supported anymore, but vanishing rapidly at infinity. We may define the inverse Fourier-transform:

$$u(x) = \left(\mathcal{F}^{-1}_{\xi\to x}\hat{u}\right)(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\hat{u}(\xi)\, d^n\xi$$

Crucial: D acts as follows under the Fourier transform:

$$Du(x) = \left(\mathbb{1} - \sum_{i=1}^n \partial_{x_i}^2\right)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\hat{u}(\xi)\, d^n\xi$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\left(\mathbb{1} + |\xi|^2\right)\hat{u}(\xi)\, d^n\xi.$$

differentiate under the integral (possible by std calculus thms)

$|\xi|^2 = \xi_1^2 + \dots + \xi_n^2.$

Here we used the following computation: (for each $j = 1, \dots, n$)

$$\partial_{x_j}^2 \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{u}(\xi)\, d^n\xi$$

$$= \int_{\mathbb{R}^n} \left(\partial_{x_j}^2 e^{i(x_1\xi_1 + \dots + x_n\xi_n)}\right) \hat{u}(\xi)\, d^n\xi$$

$$= \int_{\mathbb{R}^n} (i\xi_j)^2 e^{i\langle x,\xi\rangle} \hat{u}(\xi)\, d^n\xi$$

$$= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (-\xi_j^2) \cdot \hat{u}(\xi)\, d^n\xi .$$

Altogether we indeed arrive at the following expression:

$$Du(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (\mathbb{1} + |\xi|^2) \cdot \hat{u}(\xi)\, d^n\xi$$

Observation: A differential operator D acts under the Fourier transform as a multiplication by its

$$\boxed{\text{"symbol" } \sigma(D) = \mathbb{1} + |\xi|^2}$$

Now we can solve $Du = f$ for some $f \in C_0^\infty(\mathbb{R}^n)$ as follows:

$$Du = f$$

$$\Leftrightarrow (1 + |\xi|^2)\hat{u}(\xi) = \hat{f}(\xi)$$

$$\Leftrightarrow \hat{u}(\xi) = \frac{1}{1 + |\xi|^2} \hat{f}(\xi)$$

$$\Leftrightarrow u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \frac{1}{(1 + |\xi|^2)} \hat{f}(\xi)\, d^n\xi =: (Pf)(x)$$

where P is our first example of a "Fourier integral pseudo-differential operator". (ΨDO's)

Crucial reason for solvability of $Du = f$:
symbol $\sigma(D)(\xi)$ is <u>invertible</u> for all $\xi \in \mathbb{R}^n$.

$$\boxed{\sigma(P)(\xi) = \sigma(D)(\xi)^{-1}}$$

The core of the Global Analysis 2 is the

- introduction and analysis of classes of symbols $\sigma(P)(\xi)$
- definition of Fourier-integral ΨDO's with symbols $\sigma(P)$.
- Mapping properties and equations with "elliptic" ΨDO's (i.e. those whose symbol is invertible in some sense)

§1. Differential operators

Definition 1.1 A differential operator of order $k \in \mathbb{N}_0$ on $\mathcal{U} \subset \mathbb{R}^n$ open, is an expression

$$\mathcal{D} = \sum_{|\alpha| \le k} a_\alpha \mathcal{D}^\alpha, \quad \mathcal{D}^\alpha := \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

with $a_\alpha \in C^\infty(\mathcal{U})$ (or more generally $\in C^\infty(\mathcal{U}, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$).

We want to generalize this notion to manifolds:

Definition 1.2 Let M^m be a smooth mfd, $E, F \longrightarrow M$ (complex) vector bdles over M. A differential operator of order $k \in \mathbb{N}_0$ between sections of E, F is a linear map

$$\mathcal{D} : \Gamma(E) \longrightarrow \Gamma(F) \quad \text{such that}$$

1) $\mathcal{D}$ is local, i.e $\operatorname{supp}(\mathcal{D}s) \subset \operatorname{supp}(s)$ for $s \in \Gamma(E)$.

2) For $\mathcal{U} \subset M$ an open trivializing neighborhood for E, F, with local trivializations

$$\Phi : E|_{\mathcal{U}} \longrightarrow \mathcal{U} \times \mathbb{C}^r$$
$$\Psi : F|_{\mathcal{U}} \longrightarrow \mathcal{U} \times \mathbb{C}^s$$

the following diagram commutes:

$$\begin{tikzcd}
\Gamma_0(E|_{\mathcal{U}}) \arrow[r, "\mathcal{D}"] & \Gamma_0(F|_{\mathcal{U}}) \\
C_0^\infty(\mathcal{U}, \mathbb{C}^r) \arrow[u, "\Phi^*"] \arrow[r, "\widetilde{\mathcal{D}}"'] \arrow[ur, phantom, "\circlearrowright"] & C_0^\infty(\mathcal{U}, \mathbb{C}^s) \arrow[u, "\Psi^*"']
\end{tikzcd}$$

where $(\Phi^* f)(p) = \Phi^{-1}(p, f(p))$, $f \in C_0^\infty(\mathcal{U}, \mathbb{C}^r)$
$(\Psi^* f)(p) = \Psi^{-1}(p, f(p))$, $f \in C_0^\infty(\mathcal{U}, \mathbb{C}^s)$

and $\widetilde{\mathcal{D}} \in \mathrm{Diff}^k(\mathcal{U}, \mathbb{C}^r, \mathbb{C}^s)$ is a diff. operator of order k in the sense of previous Def. 1.1.

<u>Notation:</u> $\mathcal{D} \in \mathrm{Diff}^k(E, F)$. ($\widetilde{\mathcal{D}}$ local expression for $\mathcal{D}$)

<u>Remark:</u> Condition (1) is satisfied if in each local neighborhood $\mathcal{U} \subset M$, $\mathcal{D}(\Gamma_0(E|_{\mathcal{U}})) \subset \Gamma_0(F|_{\mathcal{U}})$. Ü

Examples 1.3

- $\mathrm{Diff}^0(E,F) = \mathrm{Hom}(E,F)$, i.e. differential operators of zero'th order are exactly the bundle homomorphisms.
- The exterior derivative $d_p: \Omega^p(M) \to \Omega^{p+1}(M)$ is a diff. op. of first order, $d_p \in \mathcal{D}iff^1(\Lambda^p T^*M, \Lambda^{p+1} T^*M)$.
- If (M,g) is Rieamannian and oriented, then

  → $d^t_{p-1} \in \mathrm{Diff}^1(\Lambda^p T^*M, \Lambda^{p-1} T^*M)$

  → $\Delta = d^t_p d_p + d_{p-1} d^t_{p-1} \in \mathrm{Diff}^2(\Lambda^p T^*M, \Lambda^p T^*M)$
- Composition of $D_1 \in \mathcal{D}iff^\alpha(E,F)$, $D_2 \in \mathcal{D}iff^\beta(F,G)$

  $$D_2 \circ D_1 \in \mathcal{D}iff^{\alpha+\beta}(E,G)$$

  and $\mathrm{Diff}(E) := \bigcup_{k \geq 0} \mathrm{Diff}^k(E,E)$ with composition is a $\mathbb{Z}_+$-graded $\mathbb{C}$-algebra!

- Each diff. op. D defines a local expression $\tilde{D}$. Conversely, we can construct a global diff. op out of local expressions:

Let $(\mathcal{U}_j)$ be open cover of M, (φ_j) subordinate smooth partition of unity; $D_j \in \mathrm{Diff}^k(\mathcal{U}_j, E, F)$. Consider (ψ_j), $\psi_j \in C^\infty(M)$ with supp $\psi_j \subset \mathcal{U}_j$ and $\psi_j \equiv 1$ on a nbd of supp φ_j. Put for $s \in \Gamma(E)$

$$Ds := \sum_j \psi_j D_j(\varphi_j s/_{\mathcal{U}_j}), \quad D \in \mathrm{Diff}^k(E,F).$$

Symbol of a differential operator

Recall $\sigma(\mathbb{1}-\Delta)(\xi)=\mathbb{1}+|\xi|^2$ in the introductory example.
For general $D\in \mathrm{Diff}^k(E,F)$ with a local expression

$$D_{\mathcal{U}}=\sum_{|\alpha|\leq k} a_\alpha(x)\, D^\alpha, \quad D^\alpha := \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots \partial x_n^{\alpha_n}}$$

we may define the

- <u>full symbol</u>: $\sigma(D_{\mathcal{U}})(x,\xi) := \sum_{|\alpha|\leq k} a_\alpha(x)\,\xi^\alpha$
  where $\xi^\alpha := \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$.
- <u>principal (leading) symbol</u>: $\boxed{\sigma^k(D_{\mathcal{U}})(x,\xi) := \sum_{|\alpha|=k} a_\alpha(x)\,\xi^\alpha.}$

Transformation of the full symbol under coordinate changes is intricate. However the leading symbol admits an invariant global description:

<u>Definition 1.4</u> Let $D\in \mathrm{Diff}^k(E,F)$. Consider $p\in M$, $\xi\in T_p^*M$ and $e\in E_p$. Choose $f\in C_0^\infty(M)$, $s\in \Gamma_0(E)$ s.t.

$$f(p)=0, \quad df(p)=\xi, \quad s(p)=e$$

Then we define (a posteriori independent of choices f,s)

$$\sigma_D^k(p,\xi)\,[e] := \frac{i^k}{k!}\, D(f^k\cdot s)(p)$$

$\sigma_D^k(p,\xi)\in \mathrm{Hom}(E_p,F_p)$ is called the "principal symbol of order k" of D at $(p,\xi)\in T_p^*M$.

Hence $\sigma_D^k \in \Gamma(\pi^*\mathrm{Hom}(E,F))$ a smooth section. $(\pi: T^*M\to M)$.

Remarks

1) Construction is indeed independent of a particular choice of f, s:
Let $\tilde{s}(p) = s(p)$; $\tilde{f}(p) = 0$, $d\tilde{f}(p) = \xi$. Then derivatives of

$$\tilde{f}^k \cdot \tilde{s} - f^k \cdot s = (\tilde{f}^k - f^k) \cdot s - \tilde{f}^k \cdot (\tilde{s} - s)$$

up to order k (included) vanish at $p \in M$. Hence

$$D(f^k \cdot s)(p) = D(\tilde{f}^k \cdot \tilde{s})(p).$$

2) σ_D^k is homogeneous in the fibre variable $(\lambda \in \mathbb{R})$

$$\boxed{\sigma_D^k(p, \lambda\xi) = \lambda^k \sigma_D^k(p, \xi)}$$

replace f by λf, since $d(\lambda f) = \lambda \xi$
and $(\lambda f)^k = \lambda^k f^k$ in the definition.

3) In local coordinates we get

$$\sigma_D^k(x, \xi \in \mathbb{R}^n)[e] = \frac{i^k}{k!} \sum_{|\alpha| \leq k} a_\alpha(x) \left(\frac{1}{i}\right)^{|\alpha|} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} (f^k \cdot s)(x)$$

$(f(x) = 0)$
$$= \frac{i^k}{k!} \sum_{|\alpha| \leq k} a_\alpha(x) \left(\frac{1}{i}\right)^{|\alpha|} (\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f^k(x)) \cdot \underbrace{s(x)}_{=e}$$

(if $|\alpha| < k$ then at least one factor in $f^k = f \cdots f$ remains undifferentiated and vanishes at x)

$$= \cancel{\frac{i^k}{k!}} \sum_{|\alpha| = k} a_\alpha(x) \cancel{\left(\frac{1}{i}\right)^{|\alpha|}} \cdot k \, (\partial_{x_1}^{\alpha_1} f) \cdots (\partial_{x_n}^{\alpha_n} f)(x) \cdot e$$

$$= \frac{i^k}{k!} \sum_{|\alpha| = k} a_\alpha(x) \, \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdot e$$

$$\Rightarrow \sigma_D^k(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \, \xi^\alpha \text{ as desired.}$$

Alternative definitions of symbols (to be discussed in (ii))

1) For any $f \in C^\infty(M)$ with $df(p) = \xi$ (we do not assume $f(p)=0$) and $s \in \Gamma(E)$ with $s(p) = e$:

$$\boxed{\sigma_D^k(p,\xi)[e] = \frac{(-i)^k}{k!}\left[(\mathrm{ad} f)^k(D)\right]\varphi(p)}$$

where $\mathrm{ad} f\,(D)[\varphi] = (f \cdot D - D f)[\varphi]$.
$$= f \cdot D\varphi - D(f\cdot\varphi).$$

2) Under the same choices:

$$\boxed{\sigma_D^k(p, df(p)) = \lim_{t\to\infty} t^{-k}\left(e^{-itf} \cdot D e^{itf}\right)(\cdot)\Big|_p}$$

Definition 1.5 (Elliptic differential operators)

$D \in \mathrm{Diff}^k(E,F)$ is said to be "*elliptic*" if

$\forall (p,\xi) \in T^*M,\ \xi \neq 0:\ \sigma_D^k(p,\xi)$ is an isom-m.

Examples 1.6

1) $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$, $d_k \in \mathcal{D}\mathrm{iff}^1(\Lambda^k T^*M, \Lambda^{k+1} T^*M)$

$$\underset{\substack{\| \\ df(p)}}{\sigma_d^1(p,\xi)}\ \underset{\substack{\| \\ \omega(p)}}{[e]} = i\,d(f\cdot\omega)(p)$$

$$= i\,\underbrace{df}_{=\xi} \wedge \omega\Big|_p + i\,\cancel{\underbrace{f(p)}_{=0} d\omega\Big|_p}$$

$$= i\,\xi \wedge \omega(p)$$

$$\boxed{\sigma_d^1(p,\xi) = i\cdot \mathrm{ext}(\xi)}$$

not an isom-m since dimensions of the sourse and the target spaces are different.

2) Let (M,g) be Riemannian, oriented and consider

$$d_k^t : \Omega^{k+1}(M) \to \Omega^k(M), \quad d_k^t \in \mathrm{Diff}^1(\Lambda^{k+1}T^*M, \Lambda^k T^*M)$$

Note beforehand: $(d^t(f\omega),\tau)_k = (f\omega, d\tau)$

$$= (\omega, f\,d\tau) = (\omega, d(f\tau) - df\wedge\tau)$$

$f\in C^\infty(M)$

$$= (f\cdot d^t\omega, \tau) - (\omega, df\wedge\tau)$$

$$= (f\,d^t\omega - \mathrm{int}(\mathrm{grad} f)\,\omega\,;\,\tau)$$

$$\Rightarrow \sigma_{d^t}^1(\underset{\underset{df(p)}{\parallel}}{p,\xi})\underset{\underset{\omega(p)}{\parallel}}{[e]} = -i\cdot \mathrm{int}(\underbrace{\mathrm{grad} f(p)}_{=f^\#})\,\omega$$

$$\boxed{\sigma_{d^t}^1(p,\xi) = -i\cdot \mathrm{int}(\xi^\#)}$$ not an isom-m.

3) Gauß–Bonnet operator $D = d + d^t : \Omega^*(M) = \bigoplus_{k=0}^m \Omega^k(M) \circlearrowleft$

$$\sigma_D^1(p,\xi) = i\cdot[\mathrm{ext}(\xi) - \mathrm{int}(\xi^\#)]$$

$$\sigma_D^1(p,\xi)^2 = \mathrm{ext}(\xi)\,\mathrm{int}(\xi^\#) + \mathrm{int}(\xi^\#)\,\mathrm{ext}(\xi)$$

$$= |\xi|^2\cdot \mathrm{Id}$$ (this was actually a question in the written GA 1 exam)

in particular D, $\underset{\underset{\Delta}{\parallel}}{D^2}$ are elliptic.

Composition of operators and symbols

Proposition 1.7 Let $D_j \in \mathrm{Diff}^{k_j}(E,F)$, $j=1,2$ and $D \in \mathrm{Diff}^{k}(F,G)$.

1) $\sigma^{\ell}_{D_1} + \sigma^{\ell}_{D_2} = \sigma^{\ell}_{D_1+D_2}$ where $\ell = \max(k_1,k_2)$

In particular, D_1+D_2 elliptic if D_1 ell and $k_1 > k_2$ (ie lower order terms don't matter for ellipticity)

2) $\sigma^{k+k_1}_{D\circ D_1}(p,\xi) = \sigma^{k}_{D}(p,\xi)\circ \sigma^{k_1}_{D_1}(p,\xi)$

Proof: One may check the statements in local coordinates. (1) is obvious. For (2) we compute

$$D = \sum_{|\alpha|\le k} a_\alpha(x)\, D^\alpha, \quad D_1 = \sum_{|\beta|\le k_1} b_\beta(x)\, D^\beta$$

$$D\circ D_1 = \sum_{\substack{|\alpha|=k\\|\beta|=k_1}} a_\alpha(x)\, b_\beta(x)\, D^{\alpha+\beta} + \text{lower orders}$$

$\boxtimes$

Note that for the full symbol one finds

$$\sigma_{D\circ D_1}(x,\xi) = \sum_{|\alpha|\le k} \frac{i^{|\alpha|}}{\alpha!}\left[D^\alpha_\xi \sigma_D(x,\xi)\right]\left[D^\alpha_x \sigma_{D_1}(x,\xi)\right].$$

Proposition 1.8 Assume (M,g) is Riemannian and oriented. Assume E,F admit Hermitian metrics. Then we may define scalar products on $\Gamma_0(E), \Gamma_0(F)$ by

$$(s,t) := \int_M \langle s(p), t(p)\rangle_{E_p}\, d\mathrm{vol}_M(p), \quad s,t\in\Gamma_0(E).$$

Let $D \in \mathrm{Diff}^k(E,F)$. Then there exists unique $D^t \in \mathrm{Diff}^k(F,E)$ st.

$$(D\varphi,\psi) = (\varphi, D^t\psi), \quad \varphi\in\Gamma_0(E),\ \psi\in\Gamma_0(F)$$

Moreover one has $\sigma^k_{D^t}(x,\xi) = \sigma^k_D(x,\xi)^*$

where $\sigma_D^\kappa(x,\xi)^*$ is defined wrt Hermitian metrics on E_p, F_p.

D^t is called the <u>formal adjoint</u> of D

<u>Proof:</u> <u>Uniqueness</u>: Assume D_1^t, D_2^t are both formal adjoints. Then for any φ, ψ as above: $\langle \varphi, (D_1^t - D_2^t)\psi\rangle = 0$

$$\Rightarrow D_1^t\psi = D_2^t\psi.$$

<u>Existence</u>: Consider $\mathcal{U}\subset M$ local trivializing nbd and choose a local orthonormal frame on $E|_\mathcal{U}$, $F|_\mathcal{U}$. Then

$\Gamma_0(E|_\mathcal{U}) = C^\infty(\mathcal{U}, \mathbb{C}^r)$ with scalar product:

$$(\varphi,\psi) = \int_\mathcal{U} \underbrace{\overline{\varphi(x)}^t}_{=\varphi(x)^*} \psi(x)\sqrt{g(x)}\,dx \text{ ; Similar for } F|_\mathcal{U}.$$

$D|_\mathcal{U} = \sum_{|\alpha|\le\kappa} a_\alpha(x) D^\alpha$. Then with Stokes we find for $\varphi \in \Gamma_0(E|_\mathcal{U})$, $\psi\in\Gamma_0(F|_\mathcal{U})$:

$$\begin{aligned}(D\varphi,\psi) &= \sum_{|\alpha|\le\kappa}\int_\mathcal{U} (D_x^\alpha\varphi(x))^* a_\alpha(x)^*\psi(x)\sqrt{g(x)}\,dx \\ &= \sum_{|\alpha|\le\kappa}\int_\mathcal{U} \varphi(x)^* D_x^\alpha\{a_\alpha(x)^*\psi(x)\sqrt{g(x)}\}\,dx \\ &=: (\varphi, D_\mathcal{U}^t\psi)\end{aligned}$$

where we now have a local expression for the adjoint

$$D_\mathcal{U}^t = \frac{1}{\sqrt{g(x)}}\sum_{|\alpha|\le\kappa} D_x^\alpha\left(\sqrt{g(x)}\,a_\alpha(x)^*(\cdot)\right).$$

This can be assembled to a global definition of D^t:

$M = \bigcup \mathcal{U}_i$; $D_i^t := (D|_{\mathcal{U}_i})^t$

(ρ_i) subordinate smooth partition of unity

Choose $\tilde{\rho}_i \equiv 1$ in a nbd of $\operatorname{supp}\rho_i$; $\operatorname{supp}\tilde{\rho}_i \subset \mathcal{U}_i$

$$\boxed{D^t := \sum_i \tilde{\rho}_i \, D_i^t(\rho_i \,\cdot\,)}$$

Indeed:
$$\begin{aligned}(D\varphi, \psi) &= \sum_i (\rho_i D\varphi, \psi) \\ &= \sum_i (\rho_i D \tilde{\rho}_i \varphi, \psi) \\ &= \sum_i (D|_{\mathcal{U}_i} \tilde{\rho}_i \varphi, \rho_i \psi) \\ &= \sum_i (\varphi, \tilde{\rho}_i D_i^t(\rho_i \psi))\end{aligned}$$

$\varphi \in \Gamma(E)$, $\psi \in \Gamma(F)$ global sections with cpt support now.

It remains to identify $\sigma^\kappa_{D^t}(x,\xi)$. We use the alternative formula:

$$\sigma^\kappa_D(p, df(p)) = \frac{(-i)^\kappa}{\kappa!} (\operatorname{ad} f)^\kappa D \cdot /_p$$

Note
$$\begin{aligned}((\operatorname{ad} f) D\varphi, \psi) &= (f D\varphi - D f\varphi, \psi) \\ &= (\varphi, D^t f\psi - f D^t \psi) \\ &= (\varphi, (-1)(\operatorname{ad} f) D^t \psi)\end{aligned}$$

$$\Rightarrow \{(\operatorname{ad} f)^\kappa D\}^t = (-1)^\kappa (\operatorname{ad} f)^\kappa D^t$$

$$\begin{aligned}\Rightarrow \sigma_{D^t}(p, df(p)) &= \left\{ \frac{(-i)^\kappa}{\kappa!} (\operatorname{ad} f)^\kappa D^t \right\}(p) \\ &= \left\{ \frac{(-i)^\kappa}{\kappa!} (\operatorname{ad} f)^\kappa D \right\}^t (p) \\ &= \sigma_D(p, df(p))^*. \qquad \blacksquare\end{aligned}$$

§ 2. Fourier transform and Sobolev-spaces in $\mathbb{R}^n$

<u>Definition 2.1</u> $f \in L^1(\mathbb{R}^n)$

$$\mathcal{F}f(\xi) := \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\,dx$$

"<u>Fourier-transform of f</u>"

<u>Facts about Fourier-transform:</u>

- $\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow C(\mathbb{R}^n)$, ie Fourier transform of an integrable function is continuous.

- $D_{\xi_j} \mathcal{F}f(\xi) = -\mathcal{F}(x_j f)(\xi)$ if $f, x_j f \in L^1(\mathbb{R}^n)$

$\frac{1}{i}\partial_{\xi_j}$

- $\xi_j \hat{f}(\xi) = (\widehat{D_{x_j} f})(\xi)$, if $f, D_{x_j} f \in L^1(\mathbb{R}^n)$

- $\widehat{f * g} = \hat{f} \cdot \hat{g}$; $\widehat{f \cdot g} = (2\pi)^{-n} \hat{f} * \hat{g}$

where $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\,d^n y$.

- $\mathcal{F}\left(e^{-\|\cdot\|^2/2}\right)(\xi) = (2\pi)^{n/2} e^{-\|\xi\|^2/2}$

- Inverse Fourier transform: if $f, \hat{f}$ are both $L^1(\mathbb{R}^n)$ we may define:

$$(\mathcal{F}^{-1}\hat{f})(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi)\,d^n\xi$$

$$\overset{!}{=} f(x).$$

- $(2\pi)^{-n/2}\mathcal{F} : L^1 \cap L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is isometry wrt L^2-norm/ scalar product

Mapping properties of the Fourier transform

Proposition 2.2 (Plancherel-theorem)
There exists unique isom-m $T: L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ st

- $\| Tf \|_{L^2} = \| f \|_{L^2}$
- $Tf = \mathcal{F}f$ for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$
- $T^{-1}g = \mathcal{F}^{-1}g$ for $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Proof: Assuming that $(2\pi)^{-n/2}\mathcal{F} : L^1 \cap L^2 \to L^2$ is an isometry, statement follows easily. Set $T = (2\pi)^{-n/2}\mathcal{F}$ on $L^1 \cap L^2$. Since $C_0^\infty(\mathbb{R}^n) \subset L^1 \cap L^2$ lies densely in $L^2(\mathbb{R}^n)$, so does $L^1 \cap L^2$ and hence each $f \in L^2$ may be approximated by $f_n \subset L^1 \cap L^2$ in $\| \cdot \|_{L^2}$-norm. Since T is isometric, Tf_n is Cauchy in L^2 and we define

$$Tf := \lim_{n \to \infty} Tf_n .$$ $\square$

Definition 2.3 (Schwartz-spaces)

$$\mathcal{S}(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{Z}_+^n : x^\alpha D_x^\beta f \text{ bded} \}$$

"Schwartz-space of rapidly decaying fns"

Remarks 2.4

- Each $f \in \mathcal{S}(\mathbb{R}^n)$ decays faster than polynomial grows:

$$|f(x)| = |(1+\|x\|^2)^N f(x) \cdot (1+\|x\|^2)^{-N}|$$
$$\leq \underbrace{\sup_x \left[|(1+\|x\|^2)^N f(x)| \right]}_{= \text{const.}} \cdot (1+\|x\|^2)^{-N}$$

- $\boxed{\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)}$ continuous and bijective.

In particular, for any $f \in C_0^\infty(\mathbb{R}^n)$, $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$.

Proof-idea: Under Fourier transform x^α multiplication transfers into D_ξ^α-differentiation, similarly D_x^β. ☒

Sobolev-Spaces For each $s \in \mathbb{R}$ we define a scalar product on $\mathcal{S}(\mathbb{R}^n) \ni f, g$: ($đ\xi = (2\pi)^{-n/2} d\xi$)

$$(f,g)_s := \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \cdot \hat{g}(\xi) \cdot (1+\|\xi\|^2)^s \, đ\xi$$

$$= (2\pi)^{-n/2} \left((1+\|\xi\|^2)^{s/2} \hat{f}, (1+\|\xi\|^2)^{s/2} \hat{g} \right)_{L^2(\mathbb{R}^n)}$$

Definition 2.5 $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined as the completion of $\mathcal{S}(\mathbb{R}^n)$ wrt $(\cdot,\cdot)_s$.

Properties of Sob-spaces

- $H^s(\mathbb{R}^n)$ is a Hilbert space with inner product $(\cdot,\cdot)_s$
- For $s < s'$ we have a continuous inclusion $H^{s'}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$.
- For $f \in \mathcal{S}(\mathbb{R}^n)$ mult-n by f is continuous $H^s \to H^s$ for all s
- The scalar product $\int fg$ on $\mathcal{S}(\mathbb{R}^n)$ extends to pairing $H^s \times H^{-s} \to \mathbb{C}$.

For integer $s \in \mathbb{Z}_+$, the order of the Sobolev-space $H^s(\mathbb{R}^n)$ is a statement on "regularity" of functions $f \in H^s(\mathbb{R}^n)$ under differentiation, which follows from the following:

Proposition 2.6 Let $s \in \mathbb{Z}_+$. Then on $H^s(\mathbb{R}^n)$ we may define a norm equivalent to $\|f\|_s := \sqrt{(f,f)_s}$ by

$$\left(\sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |D^\alpha f|^2 \right)^{1/2}$$

{ on $\mathcal{S}(\mathbb{R}^n)$! Statement: H^s can equivalently be defined as completion wrt other norm

Proof: For fixed $s\in\mathbb{Z}_+$ there exist constants c_1, c_2 st. $(k \stackrel{!}{=} s)$

$$\forall \xi\in\mathbb{R}^n: \quad c_1(1+\|\xi\|^2)^k \le \sum_{|\alpha|\le k} |\xi^\alpha|^2 \le c_2(1+\|\xi\|^2)^k$$

Consequently: $(f\in \mathcal{S}(\mathbb{R}^n))$

$$\|f\|_s^2 = \int_{\mathbb{R}^n} (1+\|\xi\|^2)^k |\hat{f}(\xi)|^2 \, d\xi$$

$$\le \frac{1}{c_1} \sum_{|\alpha|\le k} \int |\xi^\alpha \hat{f}(\xi)|^2 \, d\xi$$

$$= \frac{1}{c_1} \sum_{|\alpha|\le k} \int |\widehat{D^\alpha f}(\xi)|^2 \, d\xi$$

(Plancherel) $$= \frac{1}{c_1} \sum_{|\alpha|\le k} \int |D^\alpha f(x)|^2 \, dx$$

Similarly $$\|f\|_s^2 \ge \frac{1}{c_2} \sum_{|\alpha|\le k} \int |D^\alpha f(x)|^2 \, dx$$ ▨

$\longrightarrow$ short account on weak derivatives inserted!

cts ✓

Proposition 2.7 For $s\in\mathbb{R}$, $\alpha\in\mathbb{Z}_+^n$, $D^\alpha: H^s(\mathbb{R}^n) \to H^{s-|\alpha|}(\mathbb{R}^n)$, ie application of $|\alpha|$ derivatives decreases Sobolev-regularity by $|\alpha|$. In particular for $s\in\mathbb{Z}_+$, $|\alpha|=s$:

$$D^\alpha: H^{|\alpha|}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \text{ ctsly.}$$

(ie $f\in H^{|\alpha|}(\mathbb{R}^n)$ remains L^2 under differentiation of at most order $|\alpha|$)

Proof: $f\in S(\mathbb{R}^n)$; then $$\|D^\alpha f\|_{s-|\alpha|}^2 = \int (1+\|\xi\|^2)^{s-|\alpha|} |\widehat{D^\alpha f}(\xi)|^2 \, d\xi$$

$$= \int \underbrace{(1+\|\xi\|^2)^{s-|\alpha|} \cdot |\xi^{2\alpha}|}_{\le c\cdot(1+\|\xi\|^2)^s} \cdot |\hat{f}(\xi)|^2 \, d\xi$$

$$\le c \cdot \|f\|_s^2 .$$ ▨

Spaces of continuously differentiable functions

$C^k(\mathbb{R}^n)$ — k-times continuously differentiable fns on $\mathbb{R}^n$.

$C_b^k(\mathbb{R}^n) = \{ f \in C^k(\mathbb{R}^n) \mid \|f\|_{\infty,k} < \infty \}$ Banach space

$$\text{with } \|f\|_{\infty,k} := \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\le k} |D^\alpha f(x)|$$

$$C_0^k(\mathbb{R}^n) = \text{closure of } \mathcal{S}(\mathbb{R}^n) \subset C_b^k(\mathbb{R}^n)$$

$$= \{ f \in C^k(\mathbb{R}^n) \mid \forall |\alpha| \le k: \lim_{\|x\|\to\infty} |D^\alpha f(x)| = 0 \}$$

(Exercise)

Theorem 2.9 (Sobolev-embedding theorem)

Let $k \in \mathbb{Z}_+$, $s > k + \frac{n}{2}$. Then the inclusion

$$H^s(\mathbb{R}^n) \hookrightarrow C_0^k(\mathbb{R}^n) \quad \text{is continuous}$$

$\boxed{k < s - \frac{n}{2}}$

Proof: It suffices to check $\|f\|_{\infty,k} \le c \cdot \|f\|_s$ for $f \in S(\mathbb{R}^n)$. For $|\alpha| \le k$:

$$|\widehat{D^\alpha f}(\xi)| = |\xi^\alpha \hat{f}(\xi)|$$

$$= \underbrace{(1+\|\xi\|^2)^{\frac{s-k}{2}} |\xi^\alpha|} \cdot |\hat{f}(\xi)| \cdot (1+\|\xi\|^2)^{-\frac{(s-k)}{2}}$$

$$\le C \cdot (1+\|\xi\|^2)^{s/2} |\hat{f}(\xi)| \cdot (1+\|\xi\|^2)^{-\frac{(s-k)}{2}}$$

$$\Rightarrow \int |\widehat{(D^\alpha f)}(\xi)| \, d\xi \underset{\substack{\uparrow \\ \text{Cauchy-Schwarz}}}{\le} C \cdot \|f\|_s \cdot \underbrace{\sqrt{\int_{\mathbb{R}^n} (1+\|\xi\|^2)^{k-s} \, d\xi}}_{<\infty, \text{ since } (k-s) < -\frac{n}{2}}$$

$$\Rightarrow \|D^\alpha f\|_\infty \le C \cdot \|f\|_s$$

$$\Rightarrow \|f\|_{\infty,k} \le C \cdot \|f\|_s \qquad \boxtimes$$

Theorem 2.10 (Rellich-Lemma)

Let $K \subset \mathbb{R}^n$ be compact, and define

$$H^s_K(\mathbb{R}^n) := \{ u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset K \}$$

Then a) $H^s_K(\mathbb{R}^n)$ is a closed subspace of $H^s(\mathbb{R}^n)$.

b) For $t < s$ the inclusion map $H^s_K \hookrightarrow H^t$ is compact, ie any bded sequence $(f_n) \subset H^s_K$ admits a ~~convergent~~ subsequence (f_{n_k}) which converges in H^t for any $t < s$.

Proof: a) For any $f \in C_0^\infty(\mathbb{R}^n)$ we define

$$f^\perp := \{ u \in H^s(\mathbb{R}^n) \mid (u, f)_s = 0 \}$$

which is an orthogonal complement in a Hilbertspace and therefore closed. Statement now follows from

$$H^s_K(\mathbb{R}^n) = \bigcap_{\substack{f \in C_0^\infty(\mathbb{R}^n) \\ \operatorname{supp} f \cap K = \emptyset}} f^\perp$$

b) Choose $g \in C_0^\infty(\mathbb{R}^n)$ st $g\big|_{\text{open nbd of } K} \equiv (2\pi)^{-n}$.

Then for any bded sequence $(f_n) \subset H^s_K(\mathbb{R}^n)$ we find:

- $f_n = (2\pi)^n \cdot f_n \cdot g$
- $\hat{f}_n = \hat{f}_n * \hat{g} = \hat{g} * \hat{f}_n$
- $\partial_j (\hat{g} * \hat{f}_n) = \partial_j \int \hat{g}(\xi - \xi') \hat{f}(\xi') \, d\xi'$
  $= (\partial_j \hat{g}) * \hat{f}_n$

$$\Rightarrow \quad |D_\xi^\alpha \hat{f}_n(\xi)| \leq \int_{\mathbb{R}^n} |(D^\alpha \hat{g})(\xi-\eta)| \cdot |\hat{f}_n(\eta)| \, d\eta$$

$$\underset{\substack{\uparrow \\ (CSU)}}{\leq} \|f_n\|_s \left(\int_{\mathbb{R}^n} (1+|\eta|^2)^{-s} |D^\alpha \hat{g}(\xi-\eta)|^2 d\eta \right)^{1/2}$$

Hence $|D_\xi^\alpha \hat{f}_n(\xi)|$ is bounded uniformly in $n \in \mathbb{N}$ and by mean value theorem, $(\hat{f}_n)_n$ is uniformly bded and equicts (gleichmäßig beschränkt und gleichgradig stetig) on compact subsets of $\mathbb{R}^n_\xi$

Arzela-Ascoli $\Rightarrow$ $\exists$ subsequence that converges uniformly on cpt subsets of $\mathbb{R}^n_\xi$ ("compact convergence")
We denote this subsequence again by $(\hat{f}_n)$.

<u>Claim:</u> (f_n) converges in H^t, $t < s$,
ie we prove that since $(f_n) \subset H^s(\mathbb{R}^n)$ is bded and ~~converges~~ $(\hat{f}_n)$ converges compactly, then (f_n) converges in any $H^t(\mathbb{R}^n)$ with $t < s$.

<u>Proof of the claim:</u> $\|f_j - f_k\|_t^2 = \int |\hat{f}_j(\xi) - \hat{f}_k(\xi)| (1+|\xi|^2)^t d\xi$

$$= \int_{|\xi| \leq r} + \int_{|\xi| \geq r}$$

$$\int_{|\xi| \geq r} \cdots \leq \|f_j - f_k\|_s^2 (1+r^2)^{t-s} \leq \varepsilon/2 \text{ for } r \geq r_0 \text{ large}$$

since $\|f_j - f_k\|_s^2$ is bded. This fixes r.

$$\int_{|\xi| \leq r} \cdots \leq C(r) \cdot \sup_{|\xi| \leq r} |\hat{f}_j(\xi) - \hat{f}_k(\xi)|^2 \leq \frac{\varepsilon}{2} \text{ for } j,k \geq j_0$$

since $(\hat{f}_n)$ is compactly convergent and hence by def uniformly conv. on $\{|\xi| \leq r\}$.

$\boxtimes$

Theorem 2.11 (Trace theorem)

Let $s > 1/2$. Then the map

$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1})$$
$$f \longmapsto f \restriction \mathbb{R}^{n-1}$$

extends to a bounded linear map $H^s(\mathbb{R}^n) \longrightarrow H^{s-1/2}(\mathbb{R}^{n-1})$

Proof: Let $f \in \mathcal{S}(\mathbb{R}^n)$, $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and define $g(x') := f(0, x')$. Then

$$g(x') \underset{x_1 = 0}{=} \int_{\mathbb{R}^n} e^{i\langle \xi', x'\rangle} \hat{f}(\xi_1, \xi')\, d\xi$$

$$= \int_{\mathbb{R}^{n-1}} e^{i\langle \xi', x'\rangle} \left(\int_{\mathbb{R}} \hat{f}(\xi_1, \xi')\, d\xi_1 \right) d\xi'$$

This shows:
$$\hat{g}(\xi') = \int_{\mathbb{R}} \hat{f}(\xi_1, \xi')\, d\xi_1$$

$$|\hat{g}(\xi')|^2 \underset{\substack{(CSU) \\ \times (1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2}}}{\leq} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi_1 \cdot \underbrace{\int_{\mathbb{R}} (1+|\xi|^2)^{-s}\, d\xi_1}_{\circledast}$$

$a = 1 + |\xi'|^2$

$$\circledast = \int_{\mathbb{R}} \left(1 + \xi_1^2 + |\xi'|^2\right)^{-s} d\xi_1 = \int_{\mathbb{R}} (a + \xi_1^2)^{-s}\, d\xi_1$$

$$= a^{-s} \int_{\mathbb{R}} \left(1 + \left(\xi_1/\sqrt{a}\right)^2\right)^{-s} d\xi_1 \underset{t = \xi_1/\sqrt{a}}{=} a^{-s+1/2} \underbrace{\int_{\mathbb{R}} (1+t^2)^{-s}\, dt}_{< \infty \text{ since } s > 1/2}$$

Altogether we obtain:

$$|\hat{g}(\xi')|^2 (1 + |\xi'|^2)^{s-1/2} \leq C \cdot \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi_1$$

Integrating over $\mathbb{R}^{n-1}$ yields
$$\|g\|_{H^{s-1/2}(\mathbb{R}^{n-1})} \leq C \cdot \|f\|_{H^s(\mathbb{R}^n)}$$

Since H^* is defined as completion of $\mathcal{S}$ under $\|\cdot\|_*$, we prove the thm. ☒