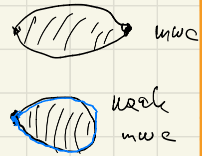
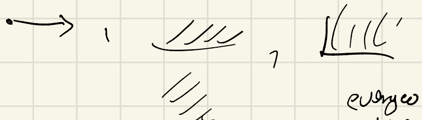
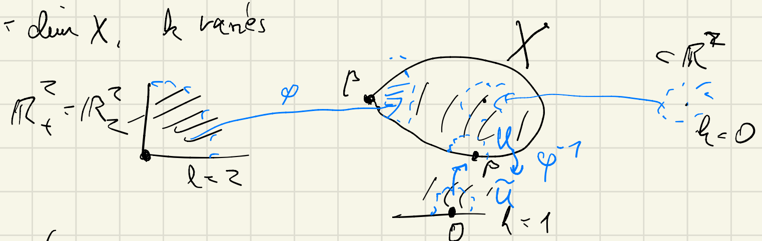
Manifold with corners (mwc)
t-manifold (weak mwc)




Model space $\mathbb{R}^n_k = \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$  $0 \le k \le n$
$\mathbb{R}_+ = [0, \infty)$.

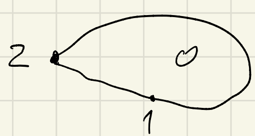


Weak mwc $X$ = space which everywhere locally looks like an $\mathbb{R}^n_k$.
$n = \dim X$, $k$ varies

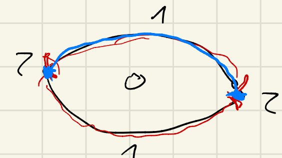



$\varphi$ chart
$\varphi^{-1}$ coordinate system  $\varphi^{-1}: U \to \tilde{U} \subset \mathbb{R}_+ \times \mathbb{R}$
$\varphi^{-1}(p) = (x(p), y(p))$  $x \ge 0$, $y \in$ open set of $\mathbb{R}$

Centered coord. system (at $p \in X$) has $p \mapsto 0$
Then $\operatorname{codim} p = k$




Face of codim $k$ := the closure of a connected component of $\{p \in X : \operatorname{codim} = k\}$

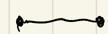
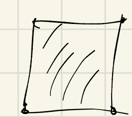



bls = boundary hypersurface = face of codim 1.

$M_k(X)$ := set of faces of codim $k$.
$\partial X := \bigcup_{H \in M_1(X)} H$,   $X^\circ = X \setminus \partial X$.

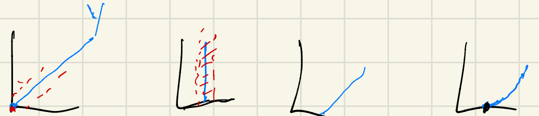
Note: If $X$ connected then $X$ is "face" of codim. 0.

Rem: $X, Y$ weak mwc $\Rightarrow$ $X \times Y$ is a weak mwc.
(since $\mathbb{R}^n_k \times \mathbb{R}^m_\ell = \mathbb{R}^{n+m}_{k+\ell}$)

ex: $X = Y =$  $\Rightarrow X \times Y =$ 

# II.2 Submanifolds

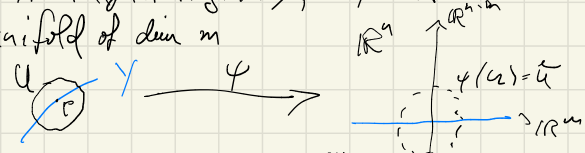
ex: $X = \mathbb{R}^2_+$.



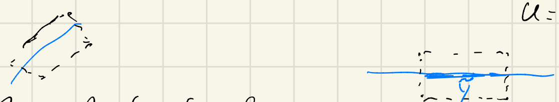

Y is: submanifold, not p-subm. | p-submanifold | not a submanifold

Recall: If $X$ manifold (e.g. $\mathbb{R}^n$), $Y \subset X$
$Y$ submanifold of dim $m$

$\Longleftrightarrow$




$\forall p \in Y$ $\exists$ coord. system $\varphi : U \to \tilde{U} \overset{\text{open}}{\subset} \mathbb{R}^n$, $U$ open nbhd. of $p$
so that $\varphi(Y \cap U) = (\mathbb{R}^m \times \{0\}) \cap \tilde{U}$



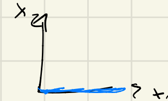

$\tilde{U} = \tilde{Y} \times U'$
$U' \subset \mathbb{R}^{n-m}$ open.

local product structure.

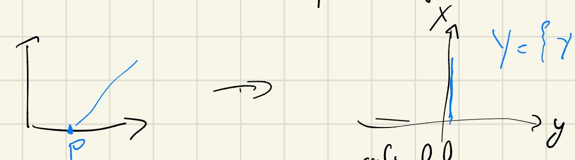
Def: Let $X$ be a weak mwc, $Y \subset X$. $Y$ is p-submanifold $:\Longleftrightarrow$
for each $p \in Y$ have coordinates centered at $p$ so that
in these coord's, $Y$ is locally a coordinate subspace:

$$\varphi = (x_1, \dots, x_k, y_1, \dots, y_{n-k}) : \quad \varphi(Y \cap U) = \left\{ \begin{array}{ll} x_i = 0, & \text{some of the } i = 1, \dots, k \\ y_j = 0, & \dots \quad j = 1, \dots, n-k \end{array} \right\}$$

ex:




$Y = \{x_2 = 0\}$
p-submfd.




$Y = \{y = 0\}$
p-submfd.

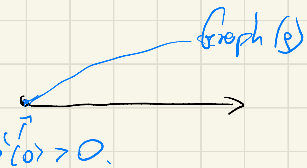
Rem: • $Y$ p-subm $\Rightarrow$ $Y$ is a weak mwc
• If $Y$ p-subm. then:
 · $Y$ is interior p-subm if $Y \not\subset \partial X$
 · $Y$ is boundary . . . . . $\subset$ .


$Y \setminus \{p\}$ is p-submfd., $Y$ is not!

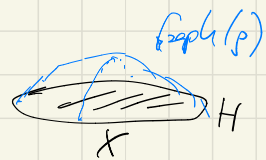
**Def**: Let $X$ be a weak mwc, $H \in \mathcal{M}_1(X)$ (a bhs).
A boundary defining function for $H$ is a smooth function $\rho : X \to \mathbb{R}_+$ so that: (bdf)

- $\rho^{-1}(0) = H$
- $d\rho_{|p} \neq 0 \quad \forall p \in H$.

ex: $X = \mathbb{R}_+$, $H = \{0\}$

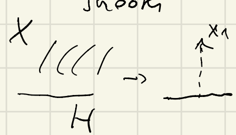



ex: $X$ compact man. with bd, $H = \partial X$




**Note**: If $\rho, \tilde{\rho}$ bdf's for $H$ then $\exists !\, a : X \to (0,\infty)$ smooth
$$\tilde{\rho} = a \cdot \rho$$

Proof: First, note that: $\rho$ bdf $\Rightarrow$ can choose, for each $p \in H$, coord's centered at $p$ so that $\rho = x_1$.




[Proof: Choose any coord's centered at $p$: $x, y$ so that $H = \{x_1 = 0\}$ locally.
Write $\rho$ in coordinates:
$$\rho(x,y) = 0 \text{ if } x_1 = 0.$$

Taylor: $\rho(x_1, \dots) = \underbrace{\rho(0, \dots)}_{\text{here: } = 0} + x_1 \cdot b(x,y)$, $b$ smooth.

$$\Rightarrow \boxed{\rho(x,y) = x_1 \cdot b(x,y).}$$

Then $d\rho = (dx_1) \cdot b + x_1 \cdot db$
$\underset{\text{at } H\ (x_1 = 0)}{=} b \cdot dx_1 \Rightarrow b \neq 0$ at $H$.
($\Rightarrow b > 0$ near $H$)

Replace $x_1, x_2, \dots$ by new coord. system $\tilde{x}_1, x_2, \dots$ where $\boxed{\tilde{x}_1 = x_1 \cdot b = \rho}$

Jacobian:
$$\begin{pmatrix} \frac{\partial \tilde{x}_1}{\partial x_1} & \frac{\partial \tilde{x}_1}{\partial x_2} & \cdots \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \cdots \\ & & \ddots \end{pmatrix} \underset{\text{at } H}{=} \begin{pmatrix} b & * & \cdots & * \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 \\ & 0 & & \ddots \\ & & & 1 \end{pmatrix}$$

$b \neq 0 \Rightarrow$ invertible. ]

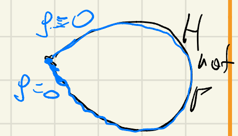
Set $a = \frac{\tilde{\rho}}{\rho} > 0$ on $X \setminus H$, $a$ smooth there. ($\rho > 0$)

Near a point of $H$: $\rho = x_1 \cdot b$, $\tilde{\rho} = x_1 \cdot \tilde{b}$, $b, \tilde{b} > 0 \Rightarrow a = \frac{\tilde{\rho}}{\rho} = \frac{\tilde{b}}{b}$
extends smoothly to $H$.

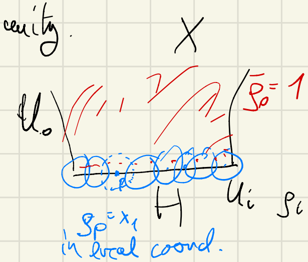
qed.

**Lemma:** Let $X$ be a weak mwc, $H \in \mathcal{M}_1(X)$.
Then TFAE (the following are equivalent):

(i) $H$ is a p-submanifold
(ii) $H$ has a bdf.




**Proof:** (ii) ⇒ (i): can take $\rho$ as $x_1$, so $H = \{x_1 = 0\}$ locally.
(i) ⇒ (ii): Use partition of unity.




P. of u.: $\chi_i \in C^\infty(X)$,
supp $\chi_i \subset U_i$
$\chi_i \geq 0$.
$\sum_i \chi_i \equiv 1$.
(locally finite)

Set $\rho := \sum_i \chi_i \rho_i$.

**Summarize:** Let $X$ be a weak mwc. Then:
$X$ mwc $\Leftrightarrow$ every $H \in \mathcal{M}_1(X)$ has a bdf.

**Lemma:** $X$ mwc, $Y \subset X$ p-submfd $\Rightarrow$ $Y$ is a mwc.

## II.3 b-maps

b = behaves well with respect to the boundary

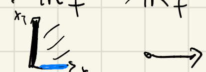
b-maps are smooth maps between mwc's that respect the bd. structure in a certain way.
First, in the local model:

**Def:** Let $\Omega \subset \mathbb{R}^n_k = \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$, $\Omega' \subset \mathbb{R}^{n'}_{k'} \subset \mathbb{R}^{k'}_+ \times \mathbb{R}^{n'-k'}$, $F: \Omega \to \Omega'$ smooth.
$F$ is a b-map if, for $F = (F_1, \dots, F_{k'}, \dots)$ where
for each $j = 1, \dots, k'$,

(i) either $F_j \equiv 0$
(ii) or $F_j(x_1, \dots, x_k, y) = a_j(x,y) \cdot \prod_{i=1}^{k} x_i^{e_{ij}}$

where $e_{ij} \in \mathbb{N}_0$ and $a > 0$ smooth.

ex: $F: \mathbb{R}^2_+ \to \mathbb{R}_+$, $(x_1, x_2) \mapsto x_1^r x_2^s$ is b-map
for $r, s \in \mathbb{N}_0$.



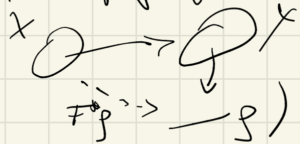
But: $(x_1, x_2) \mapsto x_1 + x_2$ is not a b-map.

Note: $x_i$ are bdf's on $\mathbb{R}^n_k$. $x_j'$ = bdf for $\mathbb{R}^{n'}_{k'}$
$F_j(x,y) = x_j'(F(x,y)) = (F^* x_j')(x,y)$

(Recall: $F: X \to Y$, $\rho \in C^\infty(Y)$

then $F^*\rho \in C^\infty(X)$ pull-back of $\rho$ by $F$

$F^*\rho := \rho \circ F$.)




Global version of that definition:

Def: Let $X, Y$ be weak mwc, $F: X \to Y$ smooth.

Choose bdf's $\rho_G$ for $G \in M_1(X)$, $\rho'_H$ for $H \in M_1(Y)$.

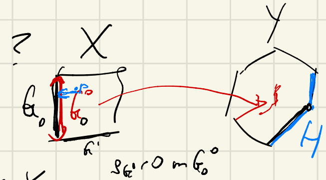
$F$ is a b-map if for each $H$:

(i) either $F^*\rho_H = 0$

(ii) or $F^*\rho_H = a_H \cdot \prod_{G \in M_1(X)} \rho_G^{e(G,H)}$

where $a_H > 0$ smooth, $e(G,H) \in \mathbb{N}_0$ $\forall G, H$.

What does this mean geometrically?




(i): $F^*\rho_H = 0 \iff \rho_H(F(p)) = 0 \ \forall p \in X$

$\iff F(p) \in H \ \forall p \in X$

$\iff F(X) \subset H$.

(ii) Fix $G_0, H$. Two cases:

a) $e(G_0, H) = 0$. Then $F^*\rho_H > 0$ on $G_0^\circ$

$\iff \rho_H(F(p)) > 0 \ \forall p \in G_0^\circ$

$\iff F(p) \notin H \ \forall p \in G_0^\circ$

$\iff F(G_0^\circ) \cap H = \emptyset$

ie image pts of $G_0^\circ$ stay away from $H$.

b) $e(G_0, H) > 0$.