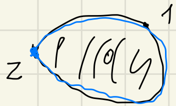
Manifolds with corners, modelled on

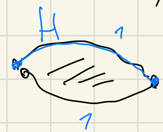
$$\mathbb{R}^n_k = \mathbb{R}^k_+ \times \mathbb{R}^{n-k}, \quad \mathbb{R}_+ = [0,\infty)$$

$\mathbb{R}^3_3 = \mathbb{R}^3_+$

~~Not:~~



weak mwc



mwc

boundary defining fcn. for H: $\rho : X \to \mathbb{R}_+$

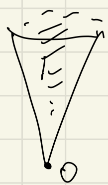
$\rho^{-1}(0) = H$

$d\rho|_p \neq 0 \quad \forall p \in H.$

b-maps, b-vector fields

PDE:

M.



$\Delta u = f, \quad u|_{\partial M \setminus 0} = 0$

Polar coord's.

$\tilde{u}(r,\omega)$

$x = r\cdot\omega$

$r = |x|$

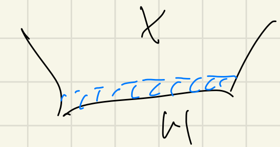
$\omega = \frac{x}{|x|}$

$\tilde{u}$ defined on $\{r>0\} \times \Omega, \quad \Omega \subset S^2$.

Guiding principle: $\boxed{\text{local product structure is important}}$

l. pr. str.: · mwc

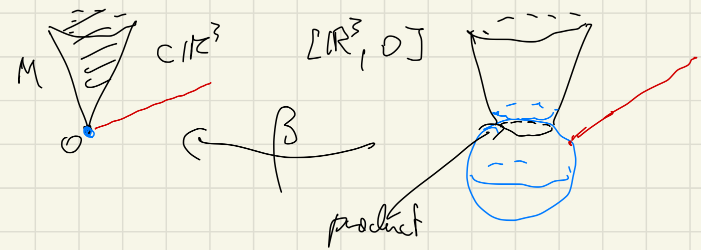
· b-maps



II.8 Blow-up

Blow-up serves to create local product structure if it's not there.

ex.:

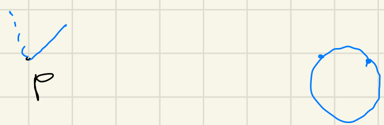


Blow-up is a geometric way of talking about introducing polar coordinates.

II.5.1 Blow-up of a point

Idea: X manifold (w. corners), $p \in X$

$[X,p] = (X \setminus p) \cup$ (one point for each direction of approach to p)



Start with local models.

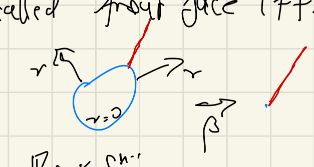
Def: The blow-up of 0 in $\mathbb{R}^n$ is the space

$$[\mathbb{R}^n, 0] := \mathbb{R}_+ \times S^{n-1}$$

together with the blow-down map

$$\beta : [\mathbb{R}^n, 0] \to \mathbb{R}^n$$
$$(r, \omega) \mapsto r\omega$$

$\beta^{-1}(0)$ is called front face (ff) of the blow-up.



Note: $\beta : ([\mathbb{R}^n, 0] \setminus ff) \to \mathbb{R}^n \setminus \{0\}$ is diffeomorphism.

Projective coordinates

$[\mathbb{R}^n, 0]$ is a manifold with boundary, parametrized by r, ω.

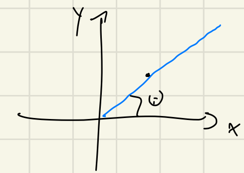
For calculations need coordinates.

Ex: $n=2$: $\omega = (\cos\theta, \sin\theta) \rightsquigarrow$ coord. r, θ

$n = 3$: ...

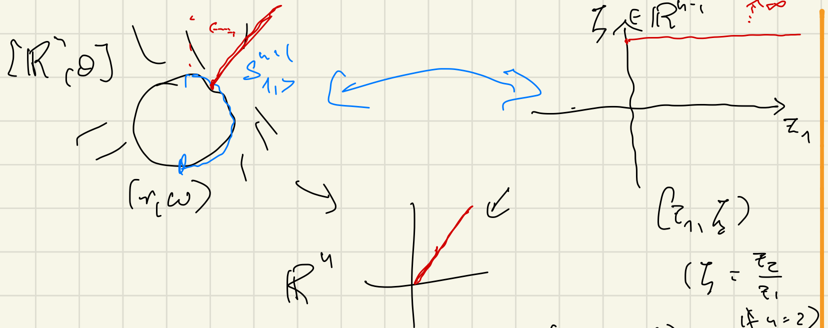
Better for calculations (any n; rational): projective coord's.

Idea, $\mathbb{R}^2$



- fix the blue ray by slope $(= \frac{y}{x})$ instead of angle θ.
- then, on a fixed ray, fix a point on it by x instead of r.

$\rightsquigarrow$ coordinates $x, \frac{y}{x}$.



$z = r \cdot \omega = (z_1, z_1 \zeta)$

Let $S^{n-1}_{1,>} := \{\omega \in S^{n-1} : \omega_1 > 0\}$

Lemma: the map $(r,\omega) \mapsto (z_1, \zeta)$ is a diffeo.

$$\{r > 0\} \times S^{n-1}_{1,>} \longrightarrow \{z_1 > 0\} \times \mathbb{R}^{n-1}$$

and extends to a diffeo

$$\{r \geq 0\} \times S^{n-1}_{1,>} \longrightarrow \{z_1 \geq 0\} \times \mathbb{R}^{n-1}.$$

Hence (z_1, ζ) are a local coord. system for $[\mathbb{R}^n, 0]$.

Proof: $S^{n-1}_{1,>} = \{(\omega_1, \omega') : |\omega'| < 1,\ \omega_1 = \sqrt{1 - |\omega'|^2}\}$

is graph over $\{|\omega'| < 1\} \subset \mathbb{R}^{n-1}$.

$\leadsto \omega'$ coords on $S^{n-1}_{>,1}$

In these coords the map is

$$(r, \omega') \mapsto z = \left(r\sqrt{1 - |\omega'|^2},\ r\omega'\right) \mapsto \left(r\sqrt{1-|\omega'|^2},\ \frac{\omega'}{\sqrt{1-|\omega'|^2}}\right)$$

which is smooth and extends smoothly to $r = 0$.

Its inverse is

$$(z_1, \zeta) \mapsto (z_1, z_1\zeta) \xrightarrow[\omega = \frac{z}{|z|}]{r = |z|} \left(z_1 \cdot \sqrt{1 + |\zeta|^2},\ \frac{\zeta}{\sqrt{1+|\zeta|^2}}\right)$$

which is smooth and extends smoothly to $z_1 = 0$. <u>qed</u>

<u>Prop</u>: $[\mathbb{R}^n, 0]$ is covered by $2n$ coord. systems.

$$U_i^{\pm} = \{(r, \omega) : r \geq 0,\ \pm\omega_i > 0\},\quad i = 1 \dots n$$

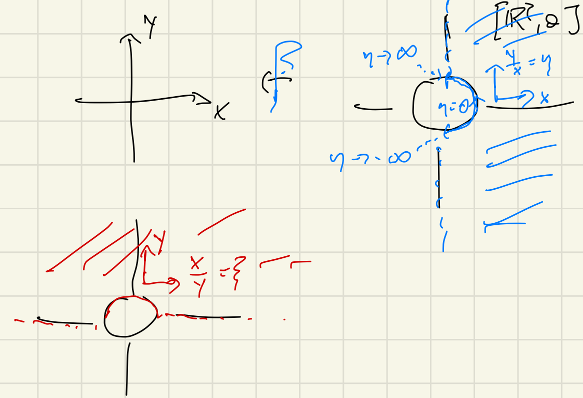
where coordinates on $U_i^{\pm}$ are

$$\left(z_i,\ \frac{z_1}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right)$$

[really, the pullbacks of these under β, extended to ff]

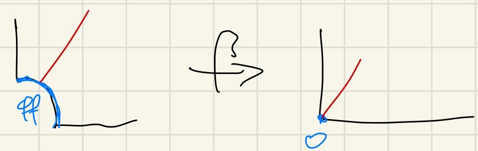
z_i is called the dominant variable in $U_i^{\pm}$

Ex: $n=2$:

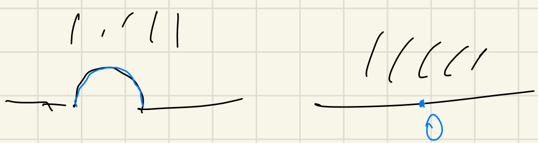


Def: $[\mathbb{R}^n_k, 0] := \mathbb{R}_+ \times S^{n-1}_k$, $S^{n-1}_k := \mathbb{R}^n_k \cap S^{n-1}$

ex: $[\mathbb{R}^2_2, 0]$:



$[\mathbb{R}^2_1, 0]$



Note: $[\mathbb{R}^n_k, 0]$ is a mwc.

ex: $[\mathbb{R}^2, 0]$. $f(x,y) = \sqrt{x^2+y^2}$ on $\mathbb{R}^2$
not smooth at $(0,0)$.

$(\beta^* f)(r,\omega) = r$ is smooth.

How about $f(x,y) = \sqrt{x^2+2y^2}$.

$\beta^* f$ is also smooth. Check in proj. coord's:

· $(x, \eta = \frac{y}{x})$: $(\beta^* f)(x,\eta) = \sqrt{x^2+2x^2\eta^2} = x\cdot\sqrt{1+2\eta^2}$
smooth in $x>0$.

$y = x\eta$

· $(\xi = \frac{x}{y}, y)$: $\sqrt{(\xi y)^2+2y^2} = y\cdot\sqrt{\xi^2+2}$
smooth.

Note: the dominant variable defines the front face.

Prop (Invariance under coord. change)

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open neighborhoods of 0.

$F: \Omega_1 \to \Omega_2$ a diffeo, $F(0)=0$.

Let $[\Omega_i, 0] = \{(r,\omega) \in \mathbb{R}_+ \times S^{n-1} : r\omega \in \Omega_i\}$

Then there is a unique diffeo $\tilde{F}: [\Omega_1, 0] \to [\Omega_2, 0]$ so that $(*)$ commutes.

$$\begin{tikzcd} \text{[}\Omega_1,0\text{]} \arrow[r, "\tilde{F}"] \arrow[d, "\beta"] & \text{[}\Omega_2,0\text{]} \arrow[d, "\beta"] \\ \Omega_1 \arrow[r, "F"] & \Omega_2 \end{tikzcd} \qquad (*)$$

Proof:

Uniqueness: $\tilde{F}$ uniquely defined on $[\Omega_1, 0] \setminus ff \to [\Omega_2, 0] \setminus ff$

since β is diffeo $[\Omega_i, 0] \setminus ff \to \Omega_i \setminus 0$.

By denseness, an extension is unique.

Existence: (explicit proof; other proof see Melrose: Real blow-up)

First: if true for F, G then true for $G \circ F$.

$\rightsquigarrow$ reduce to two cases:

1st case: F linear. By def'n, for $r > 0$,

$$\tilde{F}(r,\omega) = (r', \omega') \quad \text{where } F(r\omega) = r'\omega'$$

So

$$r' = |F(r\omega)| = r \cdot |F(\omega)| \qquad (F \text{ linear})$$

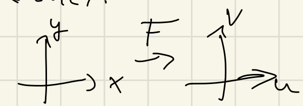
$$\omega' = \frac{F(r\omega)}{|F(r\omega)|} = \frac{F(\omega)}{|F(\omega)|}$$

this clearly extends smoothly to $r = 0$.

2nd case: F smooth, $dF|_0 = Id$.

We check the claim in projective coordinates, for $n=2$ (general case works the same).

Write $F(x,y) = (u,v)$



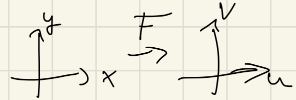
$$dF|_0 = Id \Rightarrow \begin{array}{l} u(x,y) = x + q(x,y) \\ v(x,y) = y + \tilde{q}(x,y) \end{array}$$

where $q, \tilde{q}$ are quadratic, i.e. (by Taylor)

$$q = x^2 a + xy\, b + y^2 c, \quad a, b, c \text{ smooth}.$$

(+ similar for $\tilde{q}$)

Write $F(x,y) = (u,v)$



$dF_{10} = Id \Rightarrow$ $u(x,y) = x + q(x,y)$

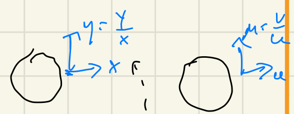
$v(x,y) = y + \tilde{q}(x,y)$

where $q, \tilde{q}$ are quadratic, i.e. (by Taylor)

$$q = x^2 a + xy\, b + y^2 c, \quad a, b, c \text{ smooth.}$$

(+ similar for $\tilde{q}$)

Now write $\tilde{F}$ in proj. coordinates



$\tilde{F}(x,y) = (u, \mu)$ where

$(y = x \cdot y)$

$$u = x + q(x, xy)$$

$$\mu = \frac{v}{u} = \frac{xy + \tilde{q}(x,xy)}{x + q(x,xy)} \quad (\text{in } x > 0)$$

Now $q(x,xy) = x^2 \cdot s(x,y)$
$\tilde{q}(x,xy) = x^2 \cdot \tilde{s}(x,y)$ $\quad s, \tilde{s}$ smooth.

$\Rightarrow$ $u = x + x^2 \cdot s$

$\mu = \dfrac{y + x \cdot \tilde{s}}{1 + x \cdot s}$ $\leftarrow$ this extends smoothly to $x = 0$.

Similar calculation in other proj. coord. system

Rem. The last eq'n also shows that

$\mu = y$ at $x = 0$, i.e.

$\tilde{F}_{|ff} = id_{ff}$. $\quad (dF_{10} = id)$

More generally, $\tilde{F}_{|ff} : ff \to ff$ is determined by dF_{10}, and is a projective map.