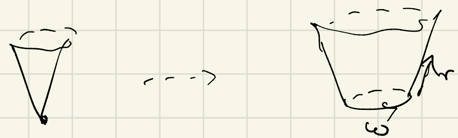
2020-11-12

Blow-ups in our standard example:

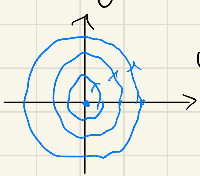
1) 

X

2) Integral kernels of solution operators of $\Delta u = f$.
$K(r,\omega; r',\omega')$, behaviour as $r \to 0$, $r' \to 0$.
behaviour like $\frac{r}{r'}$ ⇝ blow-up $r = r' = 0$ in double space $X \times X$.

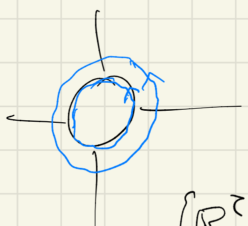
Correction: $[X,Y]$ for X mwc, Y closed p-submanifold

• Resolving group actions: ex: S^1 acting on $\mathbb{R}^2$.



not free action

$\mathbb{R}^2$



free action

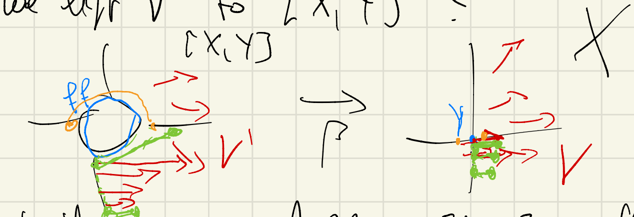
$[\mathbb{R}^2, 0]$

(General theorem by Albin-Melrose 2010)

• Resolving vector fields: later

Lifting vector fields under a blow-up

Question: X mwc, $Y \subset X$ closed p-submfd. V smooth vector field on X.
Can we lift V to $[X,Y]$?

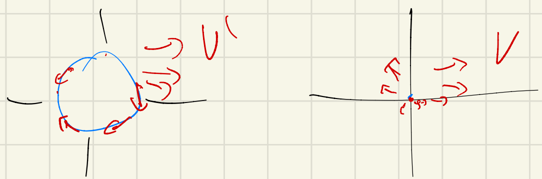


I.e.: Is there a vector field on $[X,Y]$ so that
$\forall p \in [X,Y]$: $d\beta_p(V'_p) = V_{\beta(p)}$.

Notes: • Yes on $[X,Y] \setminus ff$ since $\beta: [X,Y]\setminus ff \to X \setminus Y$ is Diffeo.
• If lift exists, it is unique.

Prop: $V \in \mathcal{V}(X)$ lifts to $[X, Y]$ if and only if V is tangent to Y, and then the lift is tangent to ff.

Ex: $Y = pt = \{p\}$ $\quad V$ tangent to pt $\Leftrightarrow$ $V_p = 0$.



Pf: If V lifts then its flow lifts, so it must be tangential.

Assume V is tangential to Y.

Special case $X = \mathbb{R}^n$, $Y = \{0\}$.

- either check in proj. coord.
- or use scaling / homogeneity ←

$V = \sum a_i(z)\partial_{z_i}$. $\quad V(0) = 0 \Leftrightarrow a_i(0) = 0 \ \forall i$

$\underset{\text{Taylor}}{\Leftrightarrow} a_i(z) = \sum_j z_j \cdot a_{ij}(z)$

(a_{ij} smooth).

⇒ it is enough to prove the prop. for $V = z_j \partial_{z_i}$.

$[\mathbb{R}^n, 0] = \mathbb{R}_{+,r} \times S^{n-1}_{\omega}$. In $r > 0$ write the lift as

$$V' = a(r,\omega)\,\partial_r + W(r), \quad W(r) \in \mathcal{V}(S^{n-1})$$

for each r, smooth in r.

- Under the scaling $r \mapsto tr$, $t > 0$, V' becomes

$$V'_t = a(tr,\omega)\frac{1}{t}\partial_r + W(tr) \qquad \boxed{\partial_r = \frac{\partial}{\partial r}}$$

V is scale invariant since $r \mapsto tr$ corresponds to $z \mapsto tz$.

$$tz_j \frac{\partial}{\partial(tz_i)} = z_j \frac{\partial}{\partial z_i},$$

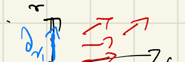
so $V_t = V$, so $V'_t = V'$ $\forall t > 0$.

⇒ $\cdot$ $a(tr,\omega)\frac{1}{t} = a(r,\omega) \overset{t = \frac{1}{r}}{\Rightarrow} a(r,\omega) = r \cdot \overbrace{a(1,\omega)}^{=:\tilde{a}(\omega)}$

$\cdot$ $W(tr) = W(r) \Rightarrow W$ is independent of r.

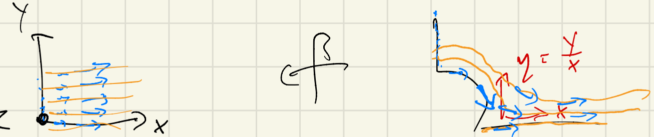
⇒ $V' = \tilde{a}(\omega) \cdot r\partial_r + W$. $\quad$ (in $r > 0$).

This extends smoothly to $r \geq 0$ (i.e. to the front face), and is tangential to $ff = \{r = 0\}$.



q.e.d.

ex: $X = \mathbb{R}^2_+$, $Y = \{0\}$



lift of $x\partial_x$:

$$\beta^*\left(x\frac{\partial}{\partial x}\right) = x\left(\frac{\partial x}{\partial x}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\right)$$

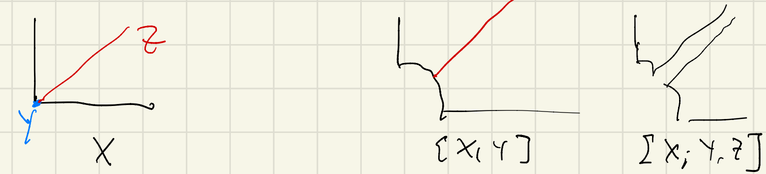
$$= x\cdot\left(1\cdot\partial_x + \underbrace{\left(-\frac{Y}{x^2}\right)}_{-\frac{y}{x}}\partial_y\right)$$

$$= x\partial_x - y\partial_y$$

Addition: X mwc, Y p-subm, $V \in \mathcal{V}(X)$.
If $V \in \mathcal{V}_b(X)$ then the lift $V' \in \mathcal{V}_b(X)$.

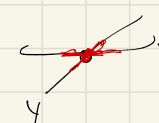
Commuting blow-ups

Notation: X mwc, $Y \subset X$ closed p-submfd, $Z \subset X$ closed subset.
If $\beta^* Z \subset [X, Y]$ is a p-submfd, then
$$[X; Y, Z] := [[X, Y], \beta^* Z]$$

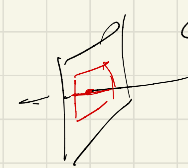


Def: X mwc, $Y, Z \subset X$ p-submfds.
Y, Z intersect cleanly if $\forall p \in Y \cap Z)$ $\exists$ coord's centered at p in which Y, Z are coordinate subspaces.

ex: $\subset \mathbb{R}^3$



not $\pitchfork$



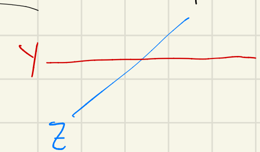
$\subset \mathbb{R}^3$ $\pitchfork$

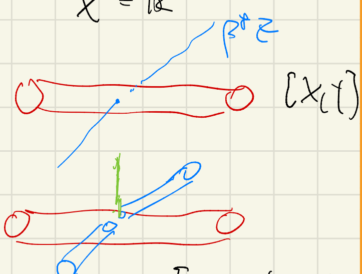
Def: Y, Z intersect transversally, $Y \pitchfork Z$ if they intersect cleanly and $T_p Y + T_p Z = T_p X$ $\forall p \in Y \cap Z$.

Thm: Let X be a mwc, $Y, Z \subset X$ cleanly intersecting p-submanifolds (closed).
Then
$$[X; Y, Z] \overset{(*)}{=} [X; Z, Y]$$
if and only if $Y \pitchfork Z$ or $Y \subset Z$ or $Z \subset Y$.

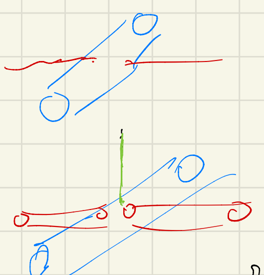
Note: If Y, Z intersect cleanly then $\beta^* Z$, $\beta : [X; Y] \to X$ is a p-submfd.

Pictures: • Not $\pitchfork$, not $\subset, \supset$: $X = \mathbb{R}^3$





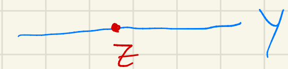
$[X; Y, Z]$

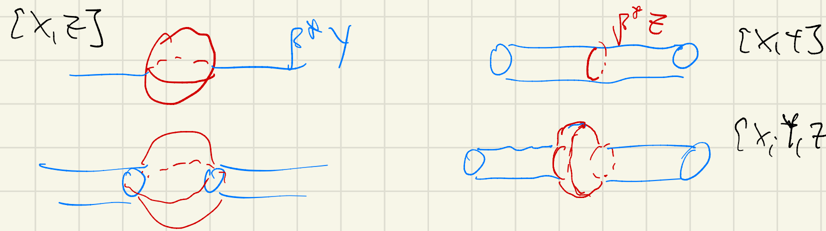


$[X; Z, Y]$

By def., $(*)$ means the following: the identity
$$X \setminus (Y \cup Z) \xrightarrow{id} X \setminus (Y \cup Z)$$
(which lifts to $[X; Y, Z] \setminus (\text{front faces})$
$[X; Z, Y] \setminus (\text{front faces})$)
extends smoothly to the boundary (as a diffeo).

• Case of $Y \supset Z$: $X = \mathbb{R}^3$





Why? This will be used in the composition theorem for b-pseudodifferential operators.

II.6 The b-tangent bundle

$\mathcal{V}(X) = \{ \text{smooth vector fields} \}$

$\mathcal{V}_b(X) = \{ V \in \mathcal{V}(X) : \text{tangent to boundary} \}$

in coordinates: x, y on $U \subset X$

$V \in \mathcal{V}(X)$: $\quad V = \sum a_i \partial_{x_i} + \sum b_j \partial_{y_j}$, a_i, b_j smooth.

so $\{\partial_{x_i}, i = 1..k, \partial_{y_j}, j = 1 ... n-k\}$ are basis

(*) • basis of $\mathcal{V}(U)$, over $C^\infty(U)$

• at each $p \in U$, $\partial_{x_i}, \partial_{y_j}$ are basis of T_pX.

General fact: (Serre-Swan theorem)

$\mathcal{V}(X)$ locally free sheaf of $C^\infty(X)$-modules (i.e. (*) holds)

$\Rightarrow$ then there is a unique vector bundle E so that

$$\mathcal{V}(X) = C^\infty(X; E) = \{\text{sections of } E\}$$

$$= \{ V : X \to E,\ p \mapsto V_p \in E_p \}$$

Here: $E = TX$.

Apply this to $\mathcal{V}_b(X)$ instead:

$V \in \mathcal{V}_b(X)$ then in coordinates

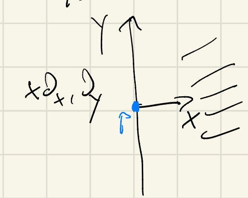
$$V = \sum a_i x_i \partial_{x_i} + \sum b_j \partial_{y_j}, \quad a_i, b_j \text{ smooth.}$$

$\rightsquigarrow$ $x_i \partial_{x_i}, \partial_{y_j}$ are local basis for $\mathcal{V}_b(X)$.

Def: ${}^bTX$ is the vector bundle over X (of rank n) whose space of sections is $\mathcal{V}_b(X)$.

So in coord.: basis of ${}^bTX$ is $x_1\partial_{x_1}, \dots, x_k\partial_{x_k}$, $\partial_{y_1}, \dots, \partial_{y_{n-k}}$

Important: $x_i \partial_{x_i}$ are non-zero elements of ${}^bT_pX$ even for $p \in \partial X$.



Formally: $\quad {}^bT_pX = \mathcal{V}_b(X) \big/ I_p \cdot \mathcal{V}_b(X)$

where $I_p = \{ f \in C^\infty(X, \mathbb{R}) : f(p) = 0 \}$.

Exercise: check that $x\partial_x = \tilde{x}\partial_{\tilde{x}}$, where $(\tilde{x}, \tilde{y})$ are new coords, with $\tilde{x}$ a bound. def. fcn.