Polyhomogeneous functions

on $\mathbb{R}_+$: $u(x) \sim \sum_{(z,k)\in E} a_{z,k}\, x^z \log^k x$

$\mathcal{A}^E(\mathbb{R}_+)$: $\{$all such $u\}$ — phg E-smooth

$\mathcal{A}^s(\mathbb{R}_+)$: conormal functions

Characterization by differential operators

Lemma. a) $(x\partial_x - z)\, x^z \log^k x = k \cdot x^z \log^{k-1} x$

$ x^z \qquad\qquad\qquad\quad 0$

b) $\ker (x\partial_x - z)^{k+1} = \operatorname{span}\{x^z, x^z \log x, \dots, x^z \log^k x\}$

Remark a): Useful rule:

$$\boxed{x\partial_x (x^z u) = x^z (x\partial_x + z) u}$$

For an index set E let (for $s \in \mathbb{R}$)

$$B_{E,s} = \prod_{(z,k)\in E_{\le s}} (x\partial_x - z)$$

Note:
- $x\partial_x - z$ appears k_z times, $k_z = \max\{k : (z,k) \in E_{\le s}\}$
- $\ker B_{E,s} = \operatorname{span}\{x^z \log^k x : (z,k) \in E_{\le s}\}$

Prop: $\mathcal{A}^E(\mathbb{R}_+) = \{u \in C^\infty(\mathbb{R}_+^\circ) : B_{E,s} u \in \mathcal{A}^s(\mathbb{R}_+) \ \forall s \in \mathbb{R}\}$

Note: The coeff's $a_{z,k}$ don't appear on RHS.

Proof: "$\subset$": $u \in \mathcal{A}^E \Rightarrow u = \sum_{(z,k)\in E_{\le s}} a_{z,k}\, x^z \log^k + r_s$, $r_s \in \mathcal{A}^s$

$\Rightarrow B_{E,s} u = B_{E,s} r_s \in \mathcal{A}^s$.

"$\supset$": Let $B_{E,s} u = v \in \mathcal{A}^s$.

$B_{E,s}$ linear $\Rightarrow$ $u = (\text{el't of } \ker B_{E,s}) + u_0$

u_0 some sol'n of $B_{E,s} u_0 = v$.

$\rightsquigarrow$ need to show that there is $u_0 \in \mathcal{A}^s$ solving this. (then $r_s = u_0$)

sketch: just do $(x\partial_x - z) u_0 = v$, then iterate

- conjugate by x^z $\rightsquigarrow$ $x\partial_x (x^{-z} u_0) = x^{-z} v$
- $\rightsquigarrow$ replace $x^{-z} v$ by v, $x^{-z} u_0$ by u_0

$\rightsquigarrow$ w.l.o.g. may take $z = 0$, $s > 0$.

$v \in \mathcal{A}^s$, $s > 0$. $\qquad x\partial_x u_0 = v$

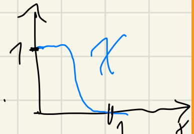
let $u_0(x) = \int_0^x v(t) \frac{dt}{t}$ $\rightsquigarrow$ $u_0 \in \mathcal{A}^s$. q.e.d.

Borel lemma:

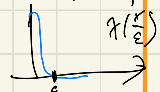
Given an index set $\mathcal{E}$ and any $a_{z,k} \in \mathbb{C}$

$\exists u \in \mathcal{A}^{\mathcal{E}}(\mathbb{R}_+)$: $\quad u(x) \sim \sum_{(z,k)\in \mathcal{E}} a_{z,k}\, x^z \log^k x \quad (x \to 0)$

Idea of proof: Choose $\chi \in C_0^\infty(\mathbb{R}_+)$, $\chi \equiv 1$ near $x=0$.



Let $u(x) = \sum_{\mathcal{E}} a_{z,k}\, x^z \log^k x \cdot \chi\left(\frac{x}{\varepsilon_z}\right)$



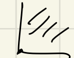
where $\varepsilon_z \to 0$ for $\mathrm{Re}\, z \to \infty$ sufficiently fast.

Rem.: $\mathcal{E} = \mathbb{N}_0 \times \{0\}$: $\forall (a_n)$: $\exists u \in C^\infty(\mathbb{R}_+)$
$u^{(n)}(0) = \frac{a_n}{n!}$

· u is not unique, e.g. can add $e^{-\frac{1}{x}}$.
$e^{-\frac{1}{x}} \sim 0$

3.1.2 Phg functions on manifolds with corners

3 things to think about:

- Half space: $\mathbb{R}^n_1 = \mathbb{R}_+ \times \mathbb{R}^{n-1}$
- Quarter plane: $\mathbb{R}^2_+$ 
- behavior under coord. changes

Some general definitions: Let X be a mwc.

Def.: b-differential operators of order $m \in \mathbb{N}_0$:

$$\mathrm{Diff}_b^m(X) = \left\{ a + \sum_{\ell=1}^m V_{\ell 1} \cdots V_{\ell \ell} \;:\; \text{all } V_{\ell i} \in \mathcal{V}_b(X),\ a \in C^\infty(X) \right\}$$

$$\mathrm{Diff}_b^*(X) = \bigcup_m \mathrm{Diff}_b^m(X).$$

Locally, $P \in \mathrm{Diff}_b^m(X)$ looks like

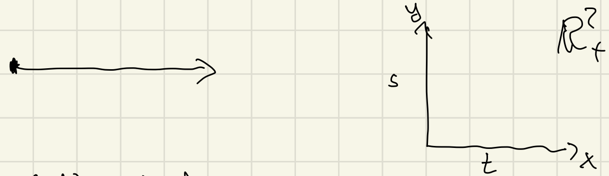
$$P = \sum_{|\alpha|+|\beta|\le m} a_{\alpha\beta}(x,y)\, (x_1\partial_{x_1})^{\alpha_1} \cdots (x_k \partial_{x_k})^{\alpha_k}\, \partial_{y_1}^{\beta_1} \cdots \partial_{y_{n-k}}^{\beta_{n-k}}$$

Def.: Index family for X: $\mathcal{E} = (\mathcal{E}_H : H \in \mathcal{M}_1(X))$
$\mathcal{E}_H$ smooth index set

· weight family: $s = (s_H \in \mathbb{R} : H \in \mathcal{M}_1(X))$

$$\mathcal{A}^s(X) = \{ u \in C^\infty(X^\circ) : u = \mathcal{O}(\rho^s) \} \quad \text{conormal functions}$$

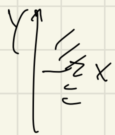
where $\rho^s = \prod_{H \in \mathcal{M}_1(X)} \rho_H^{s_H}$, ρ_H is bound. def. fcn. for H.



$\mathcal{A}^{(s,t)}(\mathbb{R}^2_+)$: $u(x,y) = \mathcal{O}(x^s \cdot y^t)$

How to define $\mathcal{A}^{\mathcal{E}}(X)$?

• Half space:



$$u(x,y) \sim \sum_{(z,k)\in\mathcal{E}} a_{z,k}(y)\, x^z \log^k x \quad \text{as } x\to 0.$$

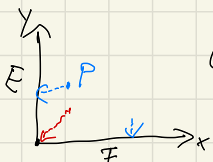
all $a_{z,k} \in C^\infty(\mathbb{R}^{n-1})$ (locally uniformly)

Rem: If $\mathcal{E}$ smooth index set then equivalently could require

$$u(x,y) \sim \sum a_{z,k}(x,y)\, x^z \log^k x$$

$a_{z,k}$ smooth. (Use Taylor: $a_{z,k}(x,y) = a_{z,k}(0,y) + x\cdot \partial a_{z,k}(0,y) + \dots$)

• $\mathbb{R}^2_+$:



E, F index sets.

Def of $\mathcal{A}^{E,F}(\mathbb{R}^2_+)$ should involve: Behavior of $u(p)$ as

• $p \to \{x=0\}^\circ$

• $p \to \{y=0\}^\circ$

• $p \to (0,0)$

Def: $u \in \mathcal{A}^{E,F}(\mathbb{R}^2_+) :\iff u \in C^\infty((\mathbb{R}^2_+)^\circ)$ and $\forall s,t$: $\exists N$ $\exists a_{z,k}, b_{\ell,w}$

$$u(x,y) = \sum_{(z,k)\in E_{\le s}} a_{z,k}(y)\, x^z \log^k x + r_s(x,y), \quad r_s \in \mathcal{A}^{(s,-N)}(\mathbb{R}^2_+)$$

$$u(x,y) = \sum_{(w,\ell)\in F_{\le t}} b_{w,\ell}(x)\, y^w \log^\ell y + r'_t(x,y), \quad r'_t \in \mathcal{A}^{(-N,t)}(\mathbb{R}^2_+)$$

$a_{z,k} \in \mathcal{A}^F(\mathbb{R}_+)$, $b_{w,\ell} \in \mathcal{A}^E(\mathbb{R}_+)$.

Ex: · $E = F = \mathbb{N}_0 \times \{0\} \Rightarrow \mathcal{A}^{E,F}(\mathbb{R}_+) = C^\infty(\mathbb{R}^2_+)$

· $\frac{1}{xy}$ is phg, $E = \{(-1,0)\} = F$

· $\frac{1}{x+y}$: fix $y>0$, let $x \to 0$

$$= \frac{1}{y}\cdot\frac{1}{\frac{x}{y}+1} = \frac{1}{y} - \frac{1}{y^2}x + \frac{1}{y^3}x^2 - + \dots$$

geometric series $(x<y)$

But: there is no index set F so that $\frac{1}{y^n} \in \mathcal{A}^F(\mathbb{R}_+)$ $\forall n$.

$\Rightarrow \frac{1}{x+y}$ is not phg.

• $x+y$ is phg (smooth).

Notes on $\mathbb{R}^2_+$ case:

- compatibility conditions:

$$a_{zk}(y) \sim \sum_{w\ell} c_{zkw\ell}\, y^w \log^\ell y \qquad (y\to 0)$$

$$b_{w\ell}(x) \sim \sum_{zk} c'_{zkw\ell}\, x^z \log^k x \qquad (x\to 0)$$

(*) Then $c_{zkw\ell} = c'_{zkw\ell}$ $\quad \forall z k w \ell$

- Borel Lemma: Given a's, b's satisfying (*) there is u with these coefficient functions.
- Differential operators: $B^x_{E,s} = \prod_{(z,k)\in E} (x\partial_x - z)$

$$B^y_{F,t} = \prod (y\partial_y - w)$$

$$u \in A^{E,F}(\mathbb{R}^2_+) \iff \exists N\ \forall s,t:$$

$$B^x_{E,s}\, u \in A^{(s,-N)}$$

$$B^y_{F,t}\, u \in A^{(-N,t)}$$

Analogous: $\mathbb{R}^n_k$.

Coordinate changes:

in $\mathbb{R}_+$: $\quad \tilde{x} = x \cdot c(x), \quad c > 0$ smooth.

$$\log \tilde{x} = \log x + \underbrace{\log c(x)}_{\text{smooth}}$$

$$\tilde{x}^z = x^z \cdot \underbrace{c(x)^z}_{\text{smooth}}$$

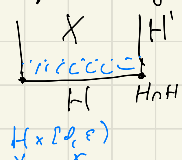
Taylor expand $c, \log c$ $\Longrightarrow$ $\tilde{x}^z \log^k \tilde{x}$ = sum of terms $x^{z+m} \log^\ell x$, $m \in \mathbb{N}_0,\ 0 \le \ell \le k$.

⇝ Lemma: If $\mathcal{E}$ is a smooth index family for $\mathbb{R}^n_k$ then $A^{\mathcal{E}}(\mathbb{R}^n_k)$ is invariant under coord. changes.

Def: X mwc, $\mathcal{E}$ (smooth) index family for X.

$A^{\mathcal{E}}(X) := \{u \in C^\infty(X^\circ) : u$ is phg. in each chart for corresponding index sets$\}$

Equivalently: $u \in A^{\mathcal{E}}(X) \iff \forall H : u(x,y) \sim \sum_{(z,k)\in \mathcal{E}_H} a^H_{zk}(y)\, x^z \log^k x$



and $a^H_{z,k} \in A^{\mathcal{E}(H)}(H)$

where $\mathcal{E}(H)$ is index family for H defined by $[\mathcal{E}(H)]_F := \mathcal{E}_{H'}$ if F is a component of $H \cap H'$.

Def: $V \in \mathcal{V}_b(X)$ radial w.r.t. $H \in \mathcal{M}_1(X)$

$:\Leftrightarrow \quad V|_H = \rho_H \partial_{\rho_H}$, ρ_H bdf for H.

Thm: Let $B^H_{\mathcal{E},s} := \prod_{\mathcal{E}_H \leq s} (V_H - z)$

for each $H \in \mathcal{M}_1(X)$, where V_H is radial for H.

Then $u \in \mathcal{A}^{\mathcal{E}}(X) \Longleftrightarrow u \in C^\infty(X^\circ)$

and $\exists N \ \forall s \ \forall H$

$$B^H_{\mathcal{E},s} u \in \mathcal{A}^{s_H}(X), \quad s_H(H') = \begin{cases} s & H' = H \\ N & H' \neq H \end{cases}$$