3.2 The Mellin transform

$u : \mathbb{R}_> \to \mathbb{C}$. $\quad (\mathcal{M}u)(s) = \int_0^\infty u(x)\, x^s \frac{dx}{x}$

$\mathbb{R}_> := (0, \infty)$

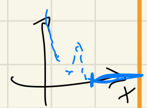
Mellin transform of u, $s \in \mathbb{C}$, where defined.

- works nicely together with polyhomogeneous functions
- useful for push-forward theorem and b-calculus
- $\mathcal{M}$ is for $(\mathbb{R}_>, \cdot)$ what $\mathcal{F}$ (Fourier transform) is for $(\mathbb{R}, +)$

Assumptions on u:

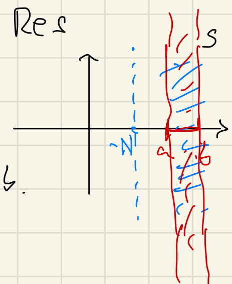
$(*_N)$ $\quad u \in L^1_{loc}(\mathbb{R}_>)$, supp u bounded

$(N \in \mathbb{R})$ $\quad u(x) = O(x^N)$



Note: $(*_N) \Rightarrow (\mathcal{M}u)(s)$ is defined and holomorphic in $\sigma > -N$, $\quad \sigma = \mathrm{Re}\, s$

and $\mathcal{M}u$ is bounded on each vertical strip $\{s : a \le \sigma \le b\}$, $-N < a \le b$.



Proof: Say supp $u \subset [0, C]$, $|u(x)| \le C' x^N$

$$\Rightarrow |\mathcal{M}u(s)| = \left| \int_0^\infty u(x)\, x^s \frac{dx}{x} \right| \le C' \cdot \int_0^C x^N x^\sigma \frac{dx}{x}$$

$$= C' \frac{x^{N+\sigma}}{N+\sigma} \Big|_0^C \underset{\substack{N+\sigma>0 \\ \text{i.e. } \sigma > -N}}{=} C' \cdot \frac{C^{N+\sigma}}{N+\sigma}$$

Holomorphic: $\frac{d}{ds} x^s = x^s \cdot \log x$, and $\int_0^C x^{N+\sigma} \log x \frac{dx}{x}$ converges if $N+\sigma > 0$.

Rem: Also $(\mathcal{M}u)'$ is bounded on strips.

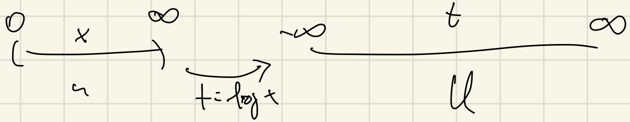
Relation to Fourier transform: $\mathcal{U}: \mathbb{R}_t \to \mathbb{C}$

$$(\mathcal{F}\mathcal{U})(\tau) = \int_{-\infty}^{\infty} e^{-it\tau}\,\mathcal{U}(t)\,dt$$

$x = e^t$

$$= \int_0^\infty x^{-i\tau}\,\mathcal{U}(\log x)\,\frac{dx}{x}$$

$$= (\mathcal{M}u)(-i\tau), \quad u(x) = \mathcal{U}(\log x)$$



Lemma (elementary properties of $\mathcal{M}$) (assume $(*_N)$)

$\circ$	$\mathcal{M}\circ$
u	$\mathcal{M}u$
$x\partial_x u$	$-s\cdot \mathcal{M}u$
$(\log x)\cdot u$	$(\mathcal{M}u)'$
$x^z u$	$(\mathcal{M}u)(\cdot + z)$
ex: $\chi = \chi_{[0,1]}$	$\frac{1}{s}$, $\operatorname{Re} s > 0$.
$x^z \chi$	$\frac{1}{s+z}$ $\operatorname{Re} s > -\operatorname{Re} z$
$x^z \log^k x \cdot \chi$	$\frac{c_k}{(s+z)^{k+1}}$, $c_k = (-1)^k k!$

$$\int_0^\infty x u' \, x^s \frac{dx}{x} = -\int_0^\infty u \cdot s \cdot x^s \frac{dx}{x}$$

if $N > 0$ and $(*_N)$ for $u, x\partial_x u$

$$\int_0^1 x^s \frac{dx}{x} = \frac{1}{s}$$

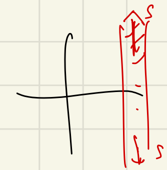
if $\operatorname{Re} s > 0$.

Lemma: (Conormal bounds and decay of $\mathcal{M}u$)

$(*_N)^J$: $(x\partial_x)^j u = O(x^N)$ $\forall j \le J$

(+ supp u bounded)

Then $|\mathcal{M}u(s)| \le \frac{C}{\langle s \rangle^J}$, $\langle s \rangle := 1 + |s|$



with uniform C on any strip in $\operatorname{Re} s > -N$.

Proof: $\underbrace{\mathcal{M}\left((x\partial_x)^j u\right)}_{\text{Ass'n: bounded on strips}} = (-s)^j \mathcal{M}u$ $\forall j \le J$

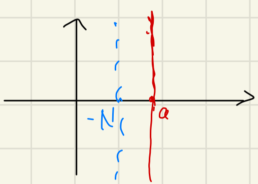
$\Rightarrow$ $\langle s \rangle^J \cdot \mathcal{M}u$ bounded on strips. qed

Thm (Mellin inversion formula): u satisfies $(*_N)^2$ then: for any $a > -N$

$$u(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} (\mathcal{M}u)(s)\, x^{-s}\, ds$$

Proof: $a = 0$, $N > 0$. Fourier inversion

$$\mathcal{U}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau} (\mathcal{F}\mathcal{U})(\tau)\, d\tau$$



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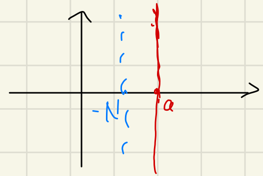

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$$U(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\tau} (\mathcal{F}U)(\tau)\, d\tau$$

$$u(x) = U(\log x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{i\tau} (\mathcal{M}u)(-i\tau)\, d\tau, \quad s = -i\tau$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-s} (\mathcal{M}u)(s)\, ds$$

General a, N: $u(x) = O(x^N) \Rightarrow v(x) = x^a u(x) = O(x^{N+a})$

$a > -N$ $\qquad N + a > 0$

$\rightsquigarrow$ use first part for v.

Cor: If $\operatorname{supp} u$ is bounded then

$u \in \mathcal{A}^N \Rightarrow \mathcal{M}u = O(\langle s \rangle^{-\infty})$ uniformly in strips $\subset \{\operatorname{Re} s > -N\}$

$u \in \mathcal{A}^{N-\varepsilon}\ \forall \varepsilon > 0 \Leftarrow$

Proof: "$\Rightarrow$": Recall $\mathcal{A}^N = \mathcal{A}^N(\mathbb{R}_+)$

$= \{u : (x\partial_x)^j u = O(x^N),\ x \in (0,1]\ \forall j\}$

✓ (see above)

"$\Leftarrow$": Let $a = -N + \varepsilon$

$$\Rightarrow |u(x)| = \left| \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-s} (\mathcal{M}u)(s)\, ds \right|$$

$$\le C \cdot x^{-a} = C x^{N-\varepsilon}$$

$$|(x\partial_x)^j u| = | \cdots \cdots (-s)^j \mathcal{M}u(s)\, ds |$$

$$\le C_j x^{-a}$$ ✓

qed

Thm (Polyhomogeneity + merom. continuation of $\mathcal{M}u$)

Assume $(*_{N_0})$ for u.

Let E be an index set. Write $E_{\mathbb{C}} = \{z : (z,0)\in E\}$, $k_z = \max\{k : (z,k)\in E\}$

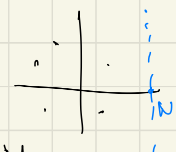
The following are equivalent:

(i) $u \in \mathcal{A}^E(\mathbb{R}_+)$.

(ii) a) $\mathcal{M}u$ has a meromorphic continuation to $\mathbb{C}$ with poles (at most) at $-z$, $z\in E_{\mathbb{C}}$, of order at most $1+k_z$.

b) $\mathcal{M}u = O(\langle s\rangle^{-\infty})$ $(|\mathrm{Im}\, s|\to\infty)$ uniformly in strips $a\le \mathrm{Re}\, s\le b$, $-\infty<a\le b<\infty$.

Proof: $B_{E,N}(x\partial_x) := \prod_{z\in E_{\mathbb{C}},<N} (x\partial_x - z)^{k_z}$



We know: (i) $\Leftrightarrow$ (i') $B_{E,N}(x\partial_x)u\in\mathcal{A}^N$ $\forall N$.

(i') $\overset{\mathcal{M}}{\Longrightarrow}$ $\forall N$ $\underbrace{B_{E,N}(-s)}_{\substack{\text{polynomial in } s,\\ \text{zeros at } s=-z, \text{ order } k_z+1}}\cdot \mathcal{M}u(s) = O(\langle s\rangle^{-\infty}) + \text{holom.}$ in $\mathrm{Re}\, s > -N$.

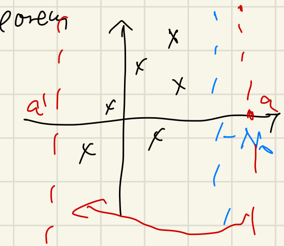
$\Rightarrow$ (ii)

(ii) $\Rightarrow$ $B_{E,N}(x\partial_x)u \in \mathcal{A}^{N-\varepsilon}$ $\forall N$

$\Rightarrow$ (i)

qed.

Rem: Could also use residue theorem for (ii) $\Rightarrow$ (i)



Rem: coefficients of $\frac{1}{(s+z)^{k+1}}$ in principal part of $\mathcal{M}u(s)$ around $s=-z$ $= c_k \cdot$ coeff of $x^z \log^k x$ in u

Ex: 1) $u(x)=e^{-x}$ $\Rightarrow$ $\mathcal{M}u=\Gamma$, poles at $-\mathbb{N}_0$

2) $u(x)=\frac{1}{e^x-1}=\sum_1^\infty e^{-nx}$

$\rightarrow \mathcal{M}u = \sum \mathcal{M}(e^{-nx}) = \sum_{n=1}^\infty n^{-s}\mathcal{M}(e^{-x})$

$= \zeta(s)\cdot\Gamma(s)$ $\Rightarrow$ merom. cont. of ζ, pole at $s=1$.

Regularized integral

Q: E index set. Is there a linear extension of the map

$$u \longmapsto \int_0^\infty u(x)\,\frac{dx}{x}$$

from $u \in \mathcal{A}_c^{E_{>0}}$ to all $u \in \mathcal{A}_c^{E}$?

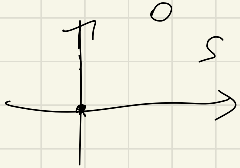
compact support

$$E_{>0} := \{(z,k)\in E : \operatorname{Re} z > 0\}$$

Yes. a $|E \setminus E_{>0}|$-dimensional set of them.

One is especially important:

Def: $\displaystyle\int_0^\infty u(x)\,\frac{dx}{x} :=$ s^0 term of Laurent expansion of $\mathcal{M}u$ around $s=0$.



Lemma: Let E be an index set, $u \in \mathcal{A}_0^{E}(\mathbb{R}_+)$.

a) For $x>0$ let $\displaystyle v(x) = \int_x^\infty u(t)\,\frac{dt}{t}$

Then $v \in \mathcal{A}_0^{E'}(\mathbb{R}_+)$ where

$$E' = E \cup \{(0,0)\} \cup \{(0,k+1) : (0,k)\in E\}$$

b) $\displaystyle\int_0^\infty u(x)\,\frac{dx}{x} =$ the coefficient of the x^0-term in the asymptotics of v as $x \to 0$.

Proof: a) integrate $x^z \log^k x$

b) $x\partial_x v = -u \implies s\cdot \mathcal{M}v = \mathcal{M}u$

$\implies$ s^0-term of $\mathcal{M}u$ at $s=0$
$=$ s^{-1}-term of $\mathcal{M}v$ at $s=0$
$=$ x^0-coeff. of asymp. of v as $x \to 0$.

q.e.d.