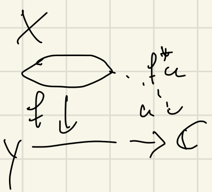
2020-12-02

3.3 Push-Forward Theorem and Pull-Back Theorem

Def: Let $f: X \to Y$ be a smooth map between manifolds with corners.

a) Pull-back under f is

$$f^*: C^\infty(Y) \to C^\infty(X)$$
$$u \mapsto f^*u := u \circ f$$



b) Push-forward by f is

$$f_*: \mathcal{M}_0(X) \to \mathcal{M}_0(Y)$$
$$\mu \mapsto f_*\mu$$

where $\mathcal{M}_0(X)$ = finite signed Borel measures on X, defined by

$$\int_Y \varphi \cdot f_*\mu := \int_X (f^*\varphi)\mu$$

$\forall \varphi \in C_c^\infty(Y)$.

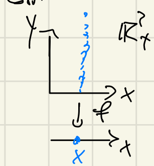
Recall: Measure μ on X is characterized by

either $A \rightsquigarrow \mu(A)$, A measurable $\subset X$

or $\varphi \mapsto \int_X \varphi\mu$, φ function on X

Question: How do f^*, f_* affect regularity of functions resp. measures

Simple ex: $f: \mathbb{R}^2_+ \to \mathbb{R}_+$, $f(x,y) = x$



$\mu = v(x,y)\,dx\,dy$.

$\varphi = \varphi(x)$: $(f^*\varphi)(x,y) = \varphi(x)$

$$\int \varphi\,(f_*\mu) = \int f^*\varphi \cdot \mu = \iint \varphi(x)\, v(x,y)\,dx\,dy$$
$$= \int \varphi(x) \cdot \left(\int v(x,y)\,dy\right) dx$$

$\Rightarrow$ $f_*\mu = w(x)\,dx$, $w(x) = \int_0^\infty v(x,y)\,dy$

= integral of v over $f^{-1}(x)$, the fibre of f over x.

So: Push-forward = integrating over the fibres of f

Recall: Applying an integral operator A to a function:

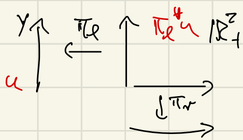
A fcn. on $\mathbb{R}_+ \times \mathbb{R}_+$, this defines an operator

also denoted A, $\int_y$ Schwartz kernel of the operator A

(*) $$(Au)(x) = \int_{\mathbb{R}_+} \underbrace{A(x,y)}\, u(y)\,dy$$

A: fcns on $\mathbb{R}_+$ $\to$ fcns on $\mathbb{R}_+$.

Rewrite (*):



$\pi_\ell(x,y) = y$

$\pi_r(x,y) = x$

(*) $\Leftrightarrow$ $\pi_{r*}\left(A \cdot \pi_\ell^* u\right)$ [use Lebesgue meas. to identify fcns + measures)

Observe: $A \in C_0^\infty \Rightarrow A: C_0^\infty \to C_0^\infty$

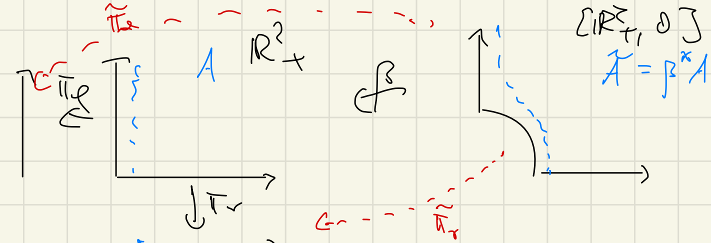
General: ∘ A polyhomogeneous. Then A: phg $\to$ phg

our interest $\Longrightarrow$ ∘ A smooth or phg after blow-up, e.g.

$A(x,y) = \tilde{A}(x, \frac{y}{x})$, $\tilde{A}$ smooth

or $\approx \sqrt{x^2+y^2}$

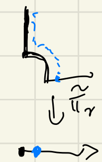
(ex: sol'n operator of $P(x\partial_x)u = f$, P polynomial & like this.) $\to \frac{y}{x}$ behavior



(*): $$\pi_{r*}\left(A \cdot \pi_\ell^* u\right) = \tilde{\pi}_{r*}\left(\tilde{A}\, \tilde{\pi}_\ell^* u\right)$$

(Change of coordinates)

Observe:



$\rightsquigarrow$ Need: · densities (smooth measures)

· mapping properties of PF, PB

· geometry: b-fibrations

· later: extend to distributions

3.3.1 Densities

Def.: A density on a manifold (w. corners) X is a smooth signed Borel measure:
for any local coordinates z on $U \subset X$
there is $v \in C^\infty(U)$ so that $\forall A \subset\subset U$:

$$\int_A \mu = \int_A v(z)\,dz$$

In short: $\mu = v(z)\,dz$ locally.
or $v(z)\,|dz|$

Rem.: · If X oriented: densities = n-forms ($n = \dim X$)

· Transformation: $v(z)\,dz = w(z')\,dz'$
iff $v = w \cdot \left|\det \frac{\partial z'}{\partial z}\right|$

Notation: $C^\infty(X, |\Omega|) = \{\text{densities on } X\}$.
or $|\Omega(X)|$
($|\Omega|$ = density bundle)

Q: f smooth map, μ smooth density.
Is $f_*\mu$ smooth density?

No

Ex.: $f: \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = x^3$ $\qquad$ $Y = x^3$, $x = Y^{1/3}$

$$f_*(dx) = \frac{1}{3\,Y^{2/3}}\,dy$$

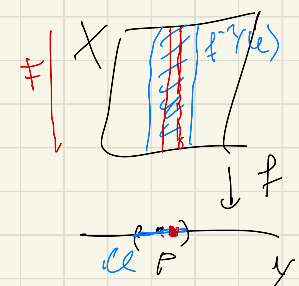
not smooth at $y = 0$.

Problem is: $f'(0) = 0$.

Def.: a) $f: X \to Y$ is submersion
$:\Leftrightarrow$ $df|_p: T_pX \to T_{f(p)}Y$ surjective $\forall p \in X$.

b) $f: X \to Y$ is fibration (or fibre bundle)
there is a mwc F:
for all $p \in Y$ $\exists$ nbhd. U of p
and a diffeo

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "f"'] & U \times F \arrow[d, dashed, "\text{proj. to } U"] \\ & U \end{tikzcd}$$



[recall: local product structure]

<u>Ehresmann theorem</u>:

If f is a proper submersion, Y connected then f is a fibration.

- proper: f^{-1}(compact) is compact
- f fibration $\Rightarrow$ f submersion
- Main point: by implicit fcn. theorem, a submersion is locally (in X) a projection.

<u>Thm</u> ("Differentiation under the $\int$")

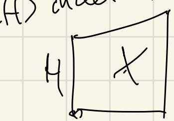
f fibration, $\mu \in C^\infty(X, |\Omega|)$

f proper on supp μ $\Longrightarrow$ $f_* \mu \in C^\infty(Y, |\Omega|)$

Pf: Part. of unity $\rightsquigarrow$ local calculation, wlog f projection $\rightsquigarrow$ like example above.

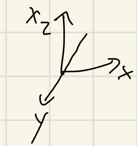
<u>Next</u>: · More general f
· ... μ

$\mathcal{E}(H)$ index set



<u>Def</u>: Let $\mathcal{E}$ smooth index family for X.

a) $\mu \in \mathcal{A}^{\mathcal{E}}(X, |\Omega|) :\Leftrightarrow \mu$ smooth densities on X°, so that in local coord. (x, y)



$\mu = v(x,y) \cdot dx_1 \cdots dx_k \, dy_1 \cdots dy_{n-k}$

$v \in \mathcal{A}^{\mathcal{E}}$

b) $\mu \in \mathcal{A}^{\mathcal{E}}(X, |\Omega_b|)$
(b-density phg with index family $\mathcal{E}$)

$:\Leftrightarrow$ ....

$\mu = v(x,y) \dfrac{dx_1}{x_1} \cdots \dfrac{dx_k}{x_k} \, dy_1 \cdots dy_{n-k}$

$v \in \mathcal{A}^{\mathcal{E}}$.

3.3.2 PFT: A special case

$f(x,y) = xy$. $\quad f : \underset{x,y}{\mathbb{R}^2_+} \to \underset{t}{\mathbb{R}_+}$

Let $\mu = u(x,y)\frac{dx}{x}\frac{dy}{y}$. Calculate $f_*\mu$:

$$\int_{\mathbb{R}_+} \varphi(t)\, f_*\mu = \int_{\mathbb{R}^2_+} (f^*\varphi)\mu = \iint \varphi(xy)u(x,y)\frac{dx}{x}\frac{dy}{y}$$

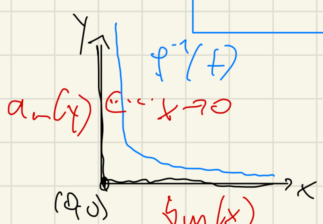
($y = \frac{t}{x}$ in y-int.) $= \iint \varphi(t)\, u(x, \frac{t}{x}) \frac{dx}{x}\frac{dt}{t}$

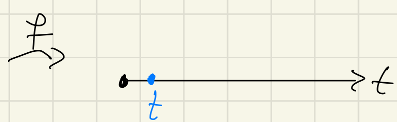
x fixed: $\frac{dy}{y} = \frac{dt}{t}$ (x cancels)

$$= \int_{\mathbb{R}_+} \varphi(t) \cdot \underbrace{\left(\int_{\mathbb{R}_+} u(x,\tfrac{t}{x})\frac{dx}{x}\right) \cdot \frac{dt}{t}}_{f_*\mu}$$

Therefore: $\mu = u\frac{dx}{x}\frac{dy}{y} \Rightarrow f_*\mu = v(t)\frac{dt}{t}$

$$\boxed{v(t) = \int_0^\infty u(x,\tfrac{t}{x})\frac{dx}{x}}$$





$df_{|(0,0)} = 0$

Note: f submersion on $\mathbb{R}^2_+ \setminus \{(0,0)\}$

Thm: $u \in \mathcal{A}_0^{E,F}(\mathbb{R}^2_+) \Rightarrow v \in \mathcal{A}_0^{E\bar{\cup}F}(\mathbb{R}_+)$

where

$E \bar{\cup} F := E \cup F \cup \{(z, k+l+1) : (z,k) \in E, (z,l) \in F\}$

(extended union)

+ explicit coefficient formulas.

In case $E = F = \mathbb{N}_0 \times \{0\}$ (i.e. u smooth)!

$E\bar{\cup}F = \mathbb{N}_0 \times \{0,1\}$ and

Rem:
- asympt of v is det. by asympt. of u at $\partial\mathbb{R}^2_+$
- log-coeff is det. by germ of u at $(0,0)$
- simple formulas due to $\frac{dx}{x}, \frac{dt}{t}, \ldots$

if $u \sim \sum_{m=0}^\infty a_m(y)x^m \quad (x \to 0)$

$u \sim \sum_0^\infty b_m(x)y^m \quad (y \to 0)$

$u \sim \sum c_{ml} x^m y^l$ (Taylor around $(0,0)$)

then $v(t) \sim \sum_{m=0}^\infty \left[\int_0^\infty \frac{1}{x^m} b_m(x)\frac{dx}{x} + \int_0^\infty \frac{1}{y^m} a_m(y)\frac{dy}{y}\right] t^m$

$$- \sum_{m=0}^\infty c_{mm}\, t^m \log t \quad (t \to 0)$$

(with derivatives)