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3.4 Conormal distributions and classical ΨDO calculus

3.4.0 Review of basics on distributions

$U \subset \mathbb{R}^n$ open.

- $\mathcal{D}'(U) := \{$ continuous linear functionals $C_0^\infty(U) \to \mathbb{C} \}$

- $$L^1_{loc}(U) \hookrightarrow \mathcal{D}'(U)$$
$$f \longmapsto \left(f : C_0^\infty(U) \to \mathbb{C},\ \varphi \mapsto \int f(x)\varphi(x)\,dx \right)$$
(φ: test functions.)

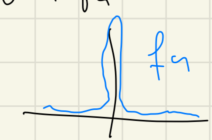
- image is dense w.r.t. weak topology on $\mathcal{D}'(U)$:
$f_n, f \in \mathcal{D}'(U)$: $f_n \longrightarrow f$
$:\Leftrightarrow \langle f_n, \varphi \rangle \longrightarrow \langle f, \varphi \rangle \ \forall \varphi$

- $\delta \in \mathcal{D}'(\mathbb{R}^n)$: $\langle \delta, \varphi \rangle = \varphi(0)$.

- Derivative $\partial_{x_j} : \mathcal{D}'(U) \to \mathcal{D}'(U)$
$$\langle \partial_{x_j} f, \varphi \rangle := -\langle f, \partial_{x_j} \varphi \rangle$$
ex: $\mathbb{R}$: $\langle \delta', \varphi \rangle = -\varphi'(0)$

- Multiply with smooth functions:
$$\langle af, \varphi \rangle := \langle f, a\varphi \rangle$$

- Cannot (in general) multiply distributions.
Ex: $\delta \cdot \delta$ does not make sense.
$\mathbb{R}$: $\delta = \lim f_n$ $\quad f_n(x) = n \cdot f(nx)$
$f \in C_0^\infty(\mathbb{R})$, $\int f = 1$.



then $\lim f_n^2$ does not exist in $\mathcal{D}'(\mathbb{R})$

- Informal way of writing things:

$$\int \delta(x)\varphi(x)\,dx = \varphi(0)$$

$$\left(= \lim \int f_n(x)\varphi(x)\,dx \right)$$

$c>0$: $\int \delta(cx)\varphi(x)\,dx \overset{y=cx}{:=} \int \delta(y)\varphi\left(\frac{y}{c}\right)\frac{dy}{c}$

$$= \frac{1}{c}\cdot\varphi(0)$$

$$\Rightarrow \boxed{\delta(cx) = \frac{1}{c}\delta(x)} \qquad \delta(-x) = \delta(x)$$

$$(\partial^\alpha\delta)(cx) = c^{-1-|\alpha|}\,\delta(x)$$

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}, \quad \alpha\in\mathbb{N}_0^n.$$

$$|\alpha| = \alpha_1+\dots+\alpha_n.$$

- $\langle\delta(x-1),\varphi\rangle = \varphi(1)$.

- $\varphi(x) = \int \delta(x-x')\varphi(x')\,dx'$

- Restriction: $U'\subset U$ open

$$f\in\mathcal{D}'(U) \Rightarrow f|_{U'}\in\mathcal{D}'(U').$$

- $\operatorname{supp} f = \left[\bigcup\{U'\subset U : f|_{U'}=0\}\right]^c$

- $\operatorname{singsupp} f = \dots\dots\dots f|_{U'}\in C^\infty(U')$

$$\operatorname{supp}\delta = \operatorname{sing\,supp}\delta = \{0\}$$

$$\operatorname{supp}\frac{1}{|x|^{1/2}} = \mathbb{R} \qquad (\mathbb{R})$$

$$\operatorname{singsupp}\frac{1}{|x|^{1/2}} = \{0\}.$$

- $\mathcal{E}'(U) := \{ f \in \mathcal{D}'(U) : \operatorname{supp} f \text{ compact} \}$

- Fourier transform: $(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx$

$\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : |D^\alpha f(x)| \le C_{\alpha,m} \langle x \rangle^{-m} \ \forall \alpha, m \}$

$\langle x \rangle = 1 + |x|$

$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ homeomorphism

- $\mathcal{S}'(\mathbb{R}^n)$ "tempered" distributions $\subset \mathcal{D}'(\mathbb{R}^n)$

e.g.: $|f(x)| \le C \langle x \rangle^N$, some C, N

$\Rightarrow f \in \mathcal{S}'(\mathbb{R}^n)$

since $\int f(x) \varphi(x)\,dx$ makes sense for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

- $$\begin{array}{ccc} \mathcal{S} & \subset & \mathcal{S}' \\ \mathcal{F} \downarrow & & \downarrow \mathcal{F} \\ \mathcal{S} & \subset & \mathcal{S}' \end{array}$$

$\langle \mathcal{F} f, \varphi \rangle := \langle f, \mathcal{F} \varphi \rangle$

$f \in \mathcal{S}', \varphi \in \mathcal{S}$.

$(\mathcal{F}^{-1} f)(x) = \int e^{ix\xi} \hat{f}(\xi)\, đ\xi$ $(f \in \mathcal{S}(\mathbb{R}^n))$. $đ\xi := (2\pi)^{-n} d\xi$

- $\hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx$

true for $f \in \mathcal{S}'(\mathbb{R}^n)$ in the weak sense:

$\langle \hat{f}, \varphi \rangle = \int \langle e^{-ix\cdot}, \varphi \rangle f(x)\,dx$

Ex: $\hat{\delta} = 1$ $\qquad \int e^{-ix\xi}\delta(x)\,dx = 1 \ \forall \xi.$

$$(*) \quad \int e^{ix\xi}\, d\xi = \delta(x)$$

$$D^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \qquad D_{x_j} = \frac{1}{i}\partial_{x_j}$$

$$(*) \Rightarrow \int e^{ix\xi}\, \xi^\alpha\, d\xi = (D^\alpha \delta)(x)$$

$$\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

On manifolds $\qquad X$ manifold.

$$\mathcal{D}'(X) := \left[C_0^\infty(X, |\Omega|)\right]'$$

Then

$$L^1_{loc}(X) \hookrightarrow \mathcal{D}'(X)$$

$$f \mapsto \left(\varphi \mapsto \int_X f\cdot\varphi\right)$$

Distribution densities

$$\mathcal{D}'(X, |\Omega|) := \left[C_0^\infty(X)\right]'$$

$$L^1_{loc}(X, |\Omega|) \hookrightarrow \mathcal{D}'(X, |\Omega|)$$

3.4.1 Conormal distributions

Motivation: (in $\mathbb{R}^n$)

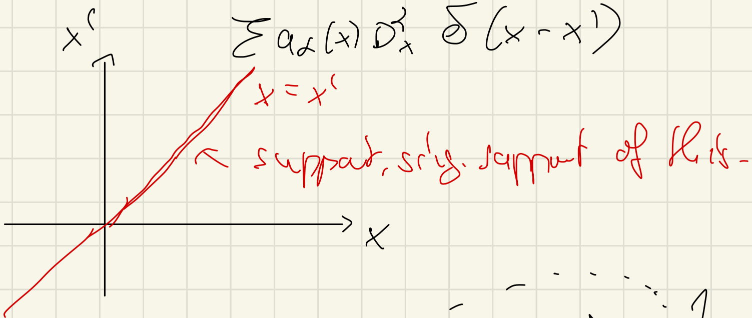
$$a_\alpha(x) D_x^\alpha \varphi(x) = \int a_\alpha(x) D_x^\alpha \delta(x-x')\varphi(x')\,dx'$$

This shows: The PDO

$$P = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$$

has Schwartz kernel

$$\sum a_\alpha(x) D_x^\alpha \delta(x-x')$$



Δ^{-1} in $\mathbb{R}^3$: $\frac{1}{|x-x'|}$ (F.T. of $\frac{1}{|\xi|^2}$ is $\frac{1}{|x|}$)

$\Delta u = f \iff |\xi|^2 \hat{u} = \hat{f} \iff \hat{u} = \frac{1}{|\xi|^2}\hat{f}$

$$(P\varphi)(x) = \int e^{i\xi(x-x')} \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \varphi(x')\,dx'\,d\xi$$

$$\delta(x-x') = \int e^{i\xi(x-x')}\,d\xi$$

Symbols:

Def: Let $m \in \mathbb{R}$, $U \subset \mathbb{R}^n$ open.

$$S^m(U,\mathbb{R}^N) := \Big\{ a \in C^\infty(\underset{x}{U} \times \underset{\xi}{\mathbb{R}^N}) : \forall \alpha,\beta\ \forall \text{ compact } K \subset U\ \exists C_{\alpha,\beta,K}: \ \left| D_\xi^\beta D_x^\alpha a(x,\xi)\right| \le C_{\alpha,\beta,K} \langle \xi \rangle^{m-|\beta|},\ x \in K \Big\}$$

Proposition:

- $m \leq m' \Rightarrow S^m \subset S^{m'}$
- $S^m \cdot S^{m'} \subset S^{m+m'}$
- $a \in S^m$, $|a| \geq c \cdot \langle \xi \rangle^m$, $c > 0$
  $\Rightarrow \frac{1}{a} \in S^{-m}$
- $D_x : S^m \to S^m$
- $D_\xi : S^m \to S^{m-1}$

Ex.: $a(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in S^m$, $m \in \mathbb{N}_0$.

- $\frac{1}{1+|\xi|^2} \in S^{-2}(\mathbb{R}^n, \mathbb{R}^n)$
- $e^{i\xi}$ is not a symbol on $\mathbb{R}$.

Notation: $S^\infty = \bigcup_m S^m$

$S^{-\infty} = \bigcap_m S^m$

Def: Let $m_1 \geq m_2 \geq \dots \to -\infty$,
$a_j \in S^{m_j}$, $a \in S^{m_1}$.

$a \sim \sum a_j \;:\Leftrightarrow\; \forall N$

$a = \sum_{m_j > N} a_j + r_N$, $r_N \in S^N$.

Def: A classical symbol of order m is an $a \in S^m(U, \mathbb{R}^N)$ for which there are $a_m, a_{m-1}, \dots$, $a_{m-j} \in S^{m-j}$

$$a \sim \sum_{j=0}^\infty a_{m-j}$$

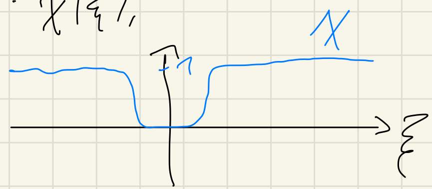
and: $a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x,\xi)$ $\forall t \geq 1$, $|\xi| \geq 1$.

Often we write

$$a \sim \sum a_{m-j}$$

with $a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x, \xi)$ $\quad \forall t > 0, \forall \xi$.

if $a \sim \sum a_{m-j} \cdot \chi(\xi)$,

$\chi =$ 

ex: $$\frac{1}{1+|\xi|^2} = \frac{1}{|\xi|^2} \frac{1}{1+\frac{1}{|\xi|^2}}$$

$$= \frac{1}{|\xi|^2} - \frac{1}{|\xi|^4} + \frac{1}{|\xi|^6} - \dots$$

$(|\xi| > 1)$

Fourier-Transform of symbols:

$U = \{pt\}$. $\quad S^m(pt, \mathbb{R}^N) =: S^m(\mathbb{R}^N_\xi)$

Note: • $S^m(\mathbb{R}^N) \subset S'(\mathbb{R}^N)$

• $S^{-\infty}(\mathbb{R}^N) = S(\mathbb{R}^N)$.

Lemma: $a \in S^m(\mathbb{R}^N)$

$\Rightarrow$ singsupp $\check{a} \subset \{0\}$

$\left(\check{a}(w) = \int e^{iw\xi} a(\xi)\, d\xi\right)$

non-ex: $a(\xi) = e^{i\xi} \Rightarrow \check{a} = \delta(w+1)$ has sing supp $= \{-1\}$

Proof: $(N=1)$: $\quad \xi^l D_\xi^k a = O(\langle \xi \rangle^{m-k+l})$

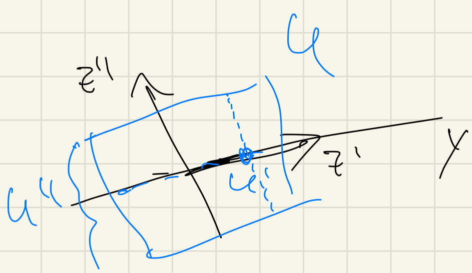
integrable if $m-k+l < -1$

$\Rightarrow (\xi^l D_\xi^k a)^\vee = \pm D_w^l\, w^k \check{a}$ $\quad$ continuous.

$\Rightarrow \check{a} \in C^l(\mathbb{R} \setminus 0)$: given l, choose k so $m-k+l < -1$.
qed.

Def: Let Z be a manifold,
$Y \subset Z$ a submanifold.
$u \in \mathcal{D}'(Z)$ is conormal with respect to Y
if, for some m:
(i) $\operatorname{sing\,supp} u \subset Y$
(ii) in any coord. system in which $Y = \{z'' = 0\}$ locally: (on $\mathcal{U} = \mathcal{U}' \times \mathcal{U}''$)

$$u(z', z'') = \int_{\mathbb{R}^l} e^{i\xi \cdot z''} a(z', \xi)\, d\xi$$



for some $a \in S^m(\mathcal{U}', \mathbb{R}^l)$

$l = \operatorname{codim} Y$

ex: $\delta^{(m)}$ conormal on $\mathbb{R}$ w.r.t. $\{0\}$