2021-01-06 # Conormal distributions

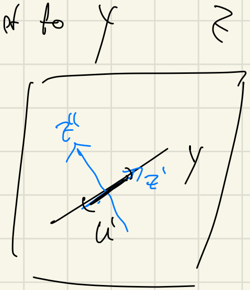
Def: Z manifold, $Y \subset Z$ submanifold.

$u \in \mathcal{D}'(Z)$ is conormal with respect to Y

$:\Leftrightarrow$ for some $m \in \mathbb{R}$:

(i) $\operatorname{sing\,supp} u \subset Y$

(ii) in any coord. system $z = (z', z'')$ where $Y = \{z'' = 0\}$:



locally

(*) $$u(z', z'') = \int_{\mathbb{R}^l} e^{i z'' \cdot \xi} a(z', \xi)\, d\xi$$

where $l = \operatorname{codim} Y$

for some $a \in S^m(U', \mathbb{R}^l)$.

If $\dim Z = 2 \cdot \dim Y$ then m is called the order of u.

Remark: We will always assume that a is a classical symbol, that is

$$a(x, \xi) \sim a_m(x, \xi) + a_{m-1}(x, \xi) + \ldots$$

$$a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x, \xi) \quad \forall t > 0 \quad \forall x, \xi$$

Non-trivial facts:

- if (*) holds in some coord. system near $p \in Y$ then it holds in any ($Y = \{z'' = 0\}$).
- if (*) holds with $a(z', z'', \xi)$, then $\exists \tilde{a}(z', \xi)$ so it holds for $\tilde{a}$. (reduction)
- Define the principal symbol of u as $a_m(z', \xi)$. This depends on the choice of coord's, but is defined invariantly if considered as a function on

$$N^*Y = \{(p, \alpha) : p \in Y, \alpha \in T_p^* Z, \alpha|_{T_p Y} = 0\}$$

$$= \text{(locally) } \operatorname{span}\{dz_j, j = 1 \ldots l\} \text{ if } z'' = \{z_1 \cdots z_l\}$$

$$= \left\{\sum_{j=1}^{l} \xi_j\, dz_j : \xi \in \mathbb{R}^l\right\}$$

$$= \text{dual bundle of } NY = TZ / TY$$

So we have $\sigma(u) = \sigma_m(u) \in S^{[m]}(N^*Y)$

$S^{[m]}(N^*Y) = \{$smooth functions on $N^*Y \setminus 0$, positive homogeneous of degree m in fibre variable$\}$

Examples:

- $H(x)=\begin{cases}1 & x>0\\ 0 & x\le 0\end{cases}$ on $Z=\mathbb{R}$, $Y=\{0\}$

  $\hat{H}(\xi) = \dfrac{1}{\xi - i0}$ $\quad (m=-1)$

- $\delta^{(\alpha)}$ on $\mathbb{R}^n$: $Z=\mathbb{R}^n$, $Y=\{0\}$.

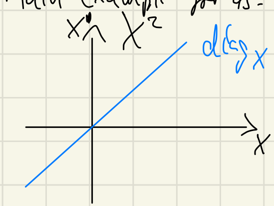
- $u = \text{p.v.}\,\dfrac{1}{x}$; $\hat{u} = \operatorname{sign}\xi$ $\quad (m=0)$

- $x_+^c = \begin{cases}x^c & x>0\\ 0 & x\le 0\end{cases}$ $\quad c>-1$ $\quad Z=\mathbb{R}$, $Y=\{0\}$

- r^c on $\mathbb{R}^n$, $c>-n$ $\quad Y=\{0\}$

Main example for us: X manifold, $Z = X^2 := X\times X$

$Y = \mathrm{diag}_X := \{(p,p) : p\in X\}$



(say $X=\mathbb{R}^n$)

$z' = x, \quad z'' = x-x'$

$k\in\mathcal{D}'(X^2)$ conormal w.r.t. diag_X of order m $\Leftrightarrow$

$$k(x,x') = \int_{\mathbb{R}^n} e^{i(x-x')\xi}\, a(x,\xi)\,d\xi, \quad a\in S^m(\mathbb{R}^n,\mathbb{R}^n)$$

If $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\,\xi^\alpha$ then

$$k(x,x') = \sum_{|\alpha|\le m} a_\alpha(x)\, D^\alpha \delta(x-x')$$

$=$ Schwartz kernel of $P = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$.

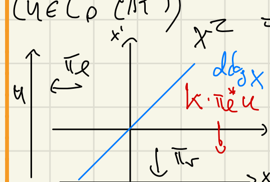
Def. Let X be a manifold. A ΨDO of order $m\in\mathbb{R}$ is an operator $P: C^\infty(X, |\Omega|^{1/2})$ ⟲ given by a Schwartz kernel $k\in\mathcal{D}'(X^2, |\Omega|^{1/2})$ which is conormal w.r.t. diag_X of order m.

$$\Psi^m(X) := \{\Psi\text{DOs of order } m \text{ on } X\}$$

Note: $\mathrm{Diff}^m(X) \subset \Psi^m(X)$ if $m\in\mathbb{N}_0$.

$X=\mathbb{R}^n$: $\quad (Pu)(x) = \iint e^{i(x-x')\xi}\, a(x,\xi)\, d\xi\; u(x')\,dx'$

$(u\in C_0^\infty(\mathbb{R}^n))$ $\quad = \int e^{ix\xi}\, a(x,\xi)\,\hat{u}(\xi)\,d\xi$



$Pu = (\pi_r)_*(k\cdot \pi_l^* u)$

Push-forward of conormal distributions

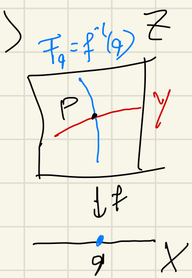
Let $f: Z \to X$ smooth map, $\mu \in \mathcal{D}'(Z, |\Omega|)$

Then $f_*\mu$ is defined if f is proper on $\operatorname{supp}\mu$: $\in \mathcal{D}'(X, |\Omega|)$. For $\varphi \in C_0^\infty(X)$ let

$$\langle f_*\mu, \varphi \rangle = \langle \mu, f^*\varphi \rangle$$

Note: • No further condition on f.

• Recall: If f is fibration then $\mu \in C^\infty \Rightarrow f_*\mu \in C^\infty$.



Thm (Push-forward of conormal distributions)
Let $f: Z \to X$ be a fibration,
$Y \subset Z$ submanifold so that

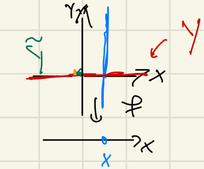
$$T_p F_q \cap T_p Y = 0 \quad \forall p \in Y,\ q = f(p).$$

Let $\mu \in \mathcal{D}'(Z)$ be conormal w.r.t. Y.

a) If $f_{|Y}: Y \to X$ is diffeo then $f_*\mu$ is smooth.

b) Otherwise, $Y' = f(Y)$ is a submanifold of X and $f_*\mu$ is conormal w.r.t. Y'.

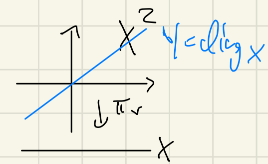
Ex: $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$



a) $f_*(\delta(y)\,dx\,dy) = \int \delta(y)\,dy\,dx$
$= dx$ smooth. (Singularity is integrated away.)

b) $f_*(\delta(x)\delta(y)\,dx\,dy)$ $\tilde{Y}$
$= \delta(x)\,dx$ conormal w.r.t. $f(\tilde{Y})$.

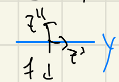
ex: $f = \pi_r$:



⇝ case a).

Thm implies $Pu \in C^\infty$, $P \in \Psi^m$, $u \in C^\infty$.

Pf a): Choose coord.



$$\mu = \left[\int e^{iz''\xi} a(z',\xi)\,d\xi\right] \rho(z'')\,dz \qquad \rho \in C_0^\infty.$$

$\Rightarrow f_*\mu =$ (integrate in z'') $= \left[\int \hat{\rho}(-\xi)\, a(z',\xi)\,d\xi\right] dz'$
smooth in z' as $\hat{\rho} \in \mathcal{S}(\mathbb{R}^{\ell})$.

b) Similar: only integrate some of the z'' variables.

Pull-back:

$f: X \to Z$

- u smooth $\Rightarrow f^* u$ smooth (no cond. on f)
- $f^* u$ not always defined if u is a distribution.
- if f fibration then $u \in \mathcal{D}' \Rightarrow f^* u \in \mathcal{D}'$ defined

For conormal distr. a weaker condition suffices:

Thm: Let $f: X \to Z$ smooth, $Y \subset Z$.

Assume f is transversal to Z

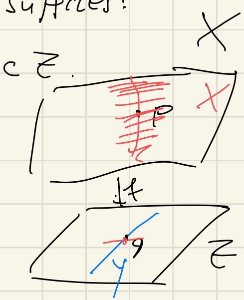
$\forall p \in X, \ q = f(p),$

(*) $T_q Z = T_q Y + df_p(T_p X)$

(df fills up directions of TZ not only in TY)

Then $Y' = f^{-1}(Y) \subset X$ is a submanifold, and

u conormal on Z wrt $Y \Rightarrow f^* u$ defined $\in \mathcal{D}'(X)$ and conormal wrt Y'.



Note: (*) satisfied if f is fibration.

Reference: Hörmander: Linear PDOs, I + III