Relation of conormal functions and conormal distributions

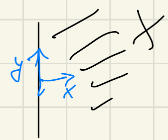
- X mfd with boundary (or corners)

$u \in \mathcal{A}^s(X)$ (conormal function of order $s \in \mathbb{R}$)

$\Leftrightarrow \quad V_b^k u = \mathcal{O}(x^s)$, x bound. def. fcn. for all $k \in \mathbb{N}_0$

$V_b = \{b\text{-vector fields on } X\}$

$= \operatorname{span}\{x\partial_x, \partial_y\}$



- $a \in S^m(\underset{x}{U}, \underset{\xi}{\mathbb{R}^n}) \overset{(*)}{\Leftrightarrow} a \in \mathcal{A}^{-m}(U \times \overline{\mathbb{R}^n})$

where $\overline{\mathbb{R}^n}$ = radial compactification of $\mathbb{R}^n$

$= \mathbb{R}^n \cup$ sphere at ∞.

as a manifold, outside unit ball:

$\mathbb{R}^n \setminus \overline{B}_1 \cong [1,\infty) \times S^{n-1} \subset [1,\infty] \times S^{n-1} \cong \overline{\mathbb{R}^n} \setminus \overline{B}_1$

$\overline{\mathbb{R}^n}$ is man. w. bd., take $t = \frac{1}{|\xi|}$ as b.d.f.

($\xi \in \mathbb{R}^n$) so $f \in C^\infty(\overline{\mathbb{R}^n}) :\Leftrightarrow$ smooth in interior $\mathbb{R}^n$ and smooth as fcn. of t, $\omega \in S^{n-1}$ near $t = 0$.

$(*)$ holds (say $U = pt$) since

$u \in S^m(\mathbb{R}^n) \Leftrightarrow \xi_i \partial_{\xi_j} u = \mathcal{O}(|\xi|^m) \quad \forall i,j$

(in $|\xi| > 1$) (+ same for higher derivs. and u)

By earlier calculation: $\quad \xi = \rho \cdot \omega, \ \rho = |\xi|$.

$\xi_i \partial_{\xi_j} = f_{ij}(\omega)\rho\partial_\rho + V_{ij}$, V_{ij} smooth vector field on S^{n-1}. f smooth fcn.

Also, $t = \frac{1}{\rho} \Rightarrow \rho\partial_\rho = -t\partial_t$.

So $\xi_i \partial_{\xi_j}$ b-vector field.

- Classical symbols $\underset{\text{classical}}{S^m_{cl}(U, \mathbb{R}^n)} = \underset{\text{polyhomogeneous}}{\mathcal{A}^{(-m+\mathbb{N}_0)\times\{0\}}(U \times \overline{\mathbb{R}^n})}$

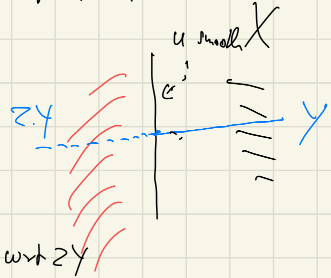
- u conormal distr. on X w.r.t. Y

$\Leftrightarrow$ u has stable regularity when applying vector fields on X tangent to Y, iteratively.

(see Hörmander 18.2: Besov spaces)

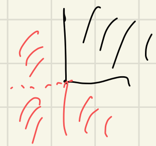
Conormal distributions on a manifold with corners X with respect to a p-submanifold Y:

Let $2X$ = double of X across boundary



u conormal distr. on X wrt Y

$:\Leftrightarrow \exists\, \tilde{u}$ con. distr. on $2X$ wrt $2Y$ s.t. $u = \tilde{u}|_X$.

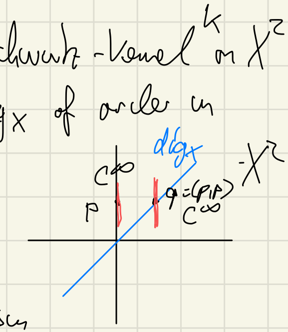


i.e. in local representation the symbol is smooth up to the boundary.

3.4.2 Classical ΨDO calculus

Recall: $P \in \Psi^m(X) :\Leftrightarrow P$ has Schwartz-kernel K on X^2 conormal wrt diag_X of order m

(X manifold, compact)



Recall: $\sigma(K) \in S^{[m]}(N^* \mathrm{diag}_X)$

Lemma: there is a natural isomorphism
$$N^* \mathrm{diag}_X \cong T^* X$$

So we define $\sigma(P) = \sigma(K)$, considered as element of $S^{[m]}(T^*X)$.

Proof: Let $p \in X$, $q = (p,p) \in \mathrm{diag}_X$.

- $T_pX \hookrightarrow T_qX^2$, $v \longmapsto (v,0)$
- The image is $\pitchfork$ $T_q \mathrm{diag}_X$, since the intersection is 0. $T_q \mathrm{diag}_X = \{(v,v) : v \in T_pX\}$
- Therefore $T_pX \cong T_qX^2 / T_q \mathrm{diag}_X = N_q \mathrm{diag}$. natural.
- Take dual spaces.

qed

Rem.: If $P \in \mathrm{Diff}^m(X)$ then this notion of (principal) symbol is the usual one.

So we have:

- spaces of operators/Schwartz kernels $\Psi^m = \Psi^m(X)$, $m \in \mathbb{R}$.
  $\Psi^m \subset \Psi^{m+1}$
- spaces of symbols: S^m (fcts on T^*X, having classical expansion as $\xi \to \infty$)
  $S^m \subset S^{m+1}$
- symbol maps: $\Psi^m \xrightarrow{\sigma_m} S^{[m]} := S^m / S^{m-1}$
  (classical symbols: $S^{[m]}$ = fcts homog. of degree m in ξ)

These satisfy: $\Psi^* = \bigcup \Psi^m$, $S^* = \bigcup S^m$

(Alg) Ψ^*, S^* are graded algebras and σ is an algebra homomorphism.

$(\Psi^*, \circ)$: $P \in \Psi^m, Q \in \Psi^\ell \Rightarrow P \circ Q \in \Psi^{m+\ell}$

$(S^*, \cdot)$: $p \in S^m, q \in S^\ell \Rightarrow pq \in S^{m+\ell}$

σ linear and $\sigma(P \circ Q) = \sigma(P) \cdot \sigma(Q)$
$\sigma(I) = 1$

(Exact): There is a short exact sequence for each m:

$$0 \to \Psi^{m-1} \to \Psi^m \xrightarrow{\sigma_m} S^{[m]} \to 0$$

That is:
- σ_m surjective
- If $P \in \Psi^m$ then $\sigma_m(P) = 0 \Leftrightarrow P \in \Psi^{m-1}$.

Def.: $P \in \Psi^m$ elliptic $:\Leftrightarrow \sigma(P)$ is invertible. (Then $\sigma(P)^{-1} \in S^{[-m]}$).

- A "parametrix of order k" for $P \in \Psi^m$ is a $Q \in \Psi^{-m}$ satisfying
  $PQ = I + R$, $QP = I + R'$, $R, R' \in \Psi^k$ $(k < 0)$.

note: $\Psi^{-\infty} := \bigcap \Psi^m \subset \dots \Psi^{-2} \subset \Psi^{-1} \subset \Psi^0 \subset \Psi^1 \subset \Psi^2$

note: $\exists$ parametrix to order $-1 \Leftrightarrow P$ elliptic.

Thm: (Parametrix construction for elliptic ΨDOs): (Alg), (Exact) imply: If $P \in \Psi^m$ is elliptic then it has a parametrix to any order.

Proof: P elliptic $\Rightarrow \sigma(P)$ invertible, $\sigma(P)^{-1} \in S^{[-m]}$

$\overset{\sigma_{-m}\ \mathrm{surj.}}{\Rightarrow}$ $\exists Q_0$: $\sigma_{-m}(Q_0) = \sigma_m(P)^{-1}$.

$\overset{(\mathrm{Alg})}{\Rightarrow}$ $\sigma_0(PQ_0 - I) = \sigma_m(P)\sigma_{-m}(Q_0) - \sigma_0(I) = 0$

$\overset{(\mathrm{Exact})\ m=0}{\Rightarrow}$ $PQ_0 - I \in \Psi^{-1} \Rightarrow PQ_0 = I - R_0$, $R_0 \in \Psi^{-1}$

$PQ_0 = I - R_0, \quad R_0 \in \Psi^{-1}$

$\Rightarrow PQ_0(I + R_0 + \dots + R_0^{k-1}) \overset{(A_3)}{=} (I - R_0)(I + \dots + R_0^{k-1})$
$= I - R_0^k$

and $R_0^k \in \Psi^{-k}$ (by (A_3)).

So $Q_k := Q_0(I + R_0 + \dots + R_0^{k-1})$ is right parametrix of order k.

By the same procedure, get left param. of order k:

$Q_k' P = I + R_k'. \qquad PQ_k = I + R_k$

$Q_k + R_k' Q_k = Q_k' P Q_k = Q_k'(I + R_k) = Q_k' + Q_k' R_k$

$\Rightarrow Q_k - Q_k' \in \Psi^{-k-m}$

$\Rightarrow Q_k P = Q_k' P + (Q_k - Q_k')P$
$= I + R_k'', \; R_k'' \in \Psi^{-k}.$

qed

Refinement:

(AC) Asymptotic completeness: If $P_i \in \Psi^{m-i}, i \in \mathbb{N}_0$ then there is $P \in \Psi^m$ with $P \sim \sum_{i=0}^{\infty} P_i$, meaning by def'n: $P - \sum_{i=0}^{N} P_i \in \Psi^{m-N-1} \;\; \forall N.$

Then: P elliptic $\Rightarrow \exists$ parametrix Q to order $-\infty$ $(R, R' \in \Psi^{-\infty})$.

In order to apply this, need:

(Diff) $\mathrm{Diff}^m \subset \Psi^m \quad \forall m \in \mathbb{N}_0$.

(Ell) The "usual" elliptic operators (Dirac, Laplace) are elliptic.

(Map) Mapping properties, e.g. $P \in \Psi^m \Rightarrow$
$P: H^{s+m}(X) \to H^s(X) \quad \forall s \in \mathbb{R}$
($H^s(X)$: Sobolev space of order s).

(Neg) the remainders, i.e. elements in $\Psi^{-\infty}$, are "negligible", i.e.

- compact in H^s
- smoothing: $H^s \to C^\infty = \bigcap_t H^t$

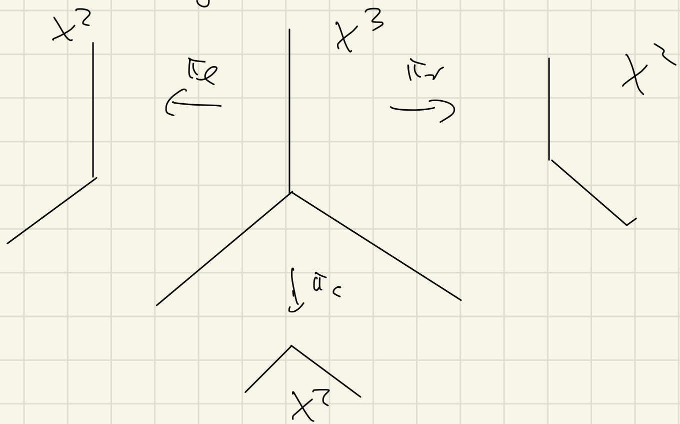
Remarks on (Alg):

$$P \in \Psi^m, \; Q \in \Psi^l \overset{!}{\Longrightarrow} P \circ Q \in \Psi^{m+l}$$

Two approaches:

		S. kernel
(1) Kernel formula:	P	K
	Q	L
	$P \circ Q$	M

$$M(z,z'') = \int K(z,z')\, L(z',z'')\, dz'$$



$$M = (\pi_c)_* \left(\pi_l^* K \cdot \pi_r^* L \right)$$

Problem: Product of distributions.

Solution: use Wave front sets.

references:

- Hörmander vol III
- ΨDO: Shubin
- DG: Basics of b-calculus
- Melrose: online book
- ΨDO lec. notes (in German, DG web page)

(2) Hands-on: local rep's of P, Q:

$$K(x,x') = \int e^{i(x-x')\xi}\, a(x,\xi)\, d\xi \quad \ldots \; ! \text{ use } Q^t$$

$$L(x',x'') = \int e^{i(x'-x'')\eta}\, b(x'',\eta)\, d\eta$$

$$(Pu)(x) = \int e^{ix\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi$$

$$(Qv)^\wedge(\xi) = \int e^{-ix''\xi}\, b(x'',\xi)\, v(x'')\, dx''$$

$$\Rightarrow (PQv) \underset{u=Qv}{=} \iint e^{ix\xi} e^{-ix''\xi}\, a(x,\xi)\, b(x'',\xi)\, v(x'')\, d\xi\, dx''$$

$$= \int M(x,x'')\, v(x'')\, dx''$$

$$M(x,x'') = \int e^{i(x-x'')\xi}\, c(x,x'',\xi)\, d\xi$$

$$c(x,x'',\xi) = a(x,\xi)\, b(x'',\xi)$$

we can rewrite as $\int e^{i(x-x'')\xi}\, \tilde{c}(x,\xi)\, d\xi$.

$$\boxed{\begin{aligned} \tilde{c}(x,\xi) &= c(x,x,\xi) + \text{l.o.t.} \\ &= a(x,\xi)\cdot b(x,\xi) + \text{l.o.t.} \\ &\Rightarrow \sigma \text{ is algebra homom.} \end{aligned}}$$

q.e.d.