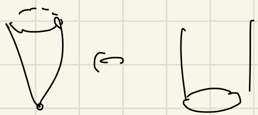
2021-01-14

b-calculus

$\mathrm{Diff}_b^m(X)$, X manifold with compact boundary

Geometric settings:

(1) M  X

∂X "cross section of cone"

We don't allow the cross section to have boundary.

not:  the full cone (with its interior)

$r^2 \Delta = P \in \mathrm{Diff}_b^2(X)$

only "boundary" of .

(2) Infinite cylindrical metrics: cylindrical end

X_0  R

$X_0 \supset [0,\infty) \times Y =: \mathrm{cyl}$

(Y compact mfd, $\partial Y = \emptyset$)

with metric $g|_{[0,\infty)\times Y} = dR^2 + h_Y$

(h_Y = Riem. Metric on Y).

$R = -\log x$, $x \in (0,1]$ $\Rightarrow$ $dR = -\frac{dx}{x}$

$\Rightarrow g|_{\mathrm{cyl}} = \left(\frac{dx}{x}\right)^2 + h_Y$

$\Rightarrow \Delta = (x\partial_x)^2 + \Delta_Y \in \mathrm{Diff}_b^2(X)$

where X = compactification of X_0

(add $x=0$: $\{0\} \times Y$).

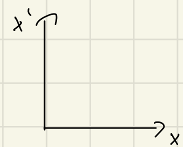
Perturbations of order $x = e^{-R}$ are permitted.

4.1 The small b-calculus

4.1.1 Motivation: Schwartz kernels of b-diff. operators

First, consider $X = \mathbb{R}_+$

X^2



K_P = Schwartz kernel of op. P.

$K_{Id}(x,x') = \delta(x-x')$ (neglect densities for now)

$\Rightarrow K_{x^m\partial_x^m} = x^m\partial_x^m\,\delta(x-x') = x^m\,\delta^{(m)}(x-x')$

[Note: b-diff ops on $\mathbb{R}_+$: $\sum a_k (x\partial_x)^k = \sum b_k x^k \partial_x^k$]

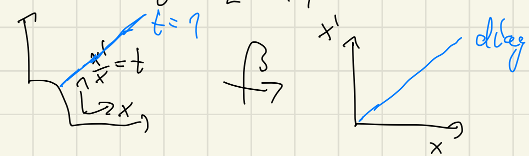
Remarks on this:

- $x^m\partial_x^m$ is elliptic in $x>0$, but not uniformly elliptic as $x\to 0$.
- reflected in x^m-factor in K
- recall $\delta^{(m)}(tx) = t^{-(m+1)}\delta(x)$

so
$$K_{x^m\partial_x^m} = x^{-1}x^{m+1}\delta^{(m)}(x-x') = x^{-1}\delta^{(m)}\left(\tfrac{1}{x}(x-x')\right) = x^{-1}\delta^{(m)}\left(1-\tfrac{x'}{x}\right)$$

- $\frac{x'}{x}$ appearing $\rightsquigarrow$ should consider this on the blown-up space

$$X_b^2 := [\mathbb{R}_+^2, 0]$$



$$\beta^* K_{x^m\partial_x^m} = x^{-1}\delta^{(m)}(1-t)$$

- Note: this is singular at $t=1$, i.e. the lifted diagonal.

- in terms of half-densities:

$$K_{Id} = \delta(x-x')\sqrt{dx\,dx'}$$

[check: If $\mu = u(x)\sqrt{dx}$ then

$$\int K_{Id}\,\mu(x') = \int \delta(x-x')u(x')\underbrace{\sqrt{dx'}\sqrt{dx'}}_{dx'}\sqrt{dx} = u(x)\sqrt{dx} = \mu$$

$t = \frac{x'}{x}$

· $k_{Id} = \delta(x-x')\sqrt{dx\,dx'}$ $(x>0)$

$= \sqrt{x}\sqrt{x'}\,\delta(x-x')\sqrt{\frac{dx}{x}\frac{dx'}{x'}}$

$= x\,\delta(x-x')\sqrt{\frac{dx}{x}\frac{dx'}{x'}}$ (since $a(x)\delta(x-x') = a(x')\delta(x-x')$ here: $a(x)=\sqrt{x}$)

$= \delta(1-\frac{x'}{x})\sqrt{\frac{dx}{x}\frac{dx'}{x'}}$

$$\boxed{\frac{dx}{x}\frac{dx'}{x'} = \frac{dx}{x}\frac{d(tx)}{tx} = \frac{dx}{x}\frac{t\,dx + x\,dt}{tx} = \frac{dx}{x}\frac{dt}{t}}$$

$\Rightarrow \beta^* k_{Id} = \delta(1-t)\sqrt{\frac{dx}{x}\frac{dt}{t}}$

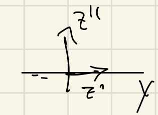
The same works for $k_{x^n\partial_x^n}$ (x^{-1} disappears).

Recall: · smooth half-density on manifold with corners: locally in coords (x,y):

$$a\sqrt{\frac{dx_1}{x_1}\cdots\frac{dx_k}{x_k}dy_1\cdots dy_{n-k}}$$

Def: Let $Y\subset Z$ be a p-submanifold, Z mwc. Smooth Dirac distribution on Z at Y of order m

$$:= \sum_{|\alpha|\le m} a_\alpha(z)\,\partial_{z''}^{\alpha}\,\delta(z'')$$



a_α smooth.

Note: · Can w.l.o.g. take $a_\alpha(z')$

· this is a special conormal distr. on Z w.r.t. Y.

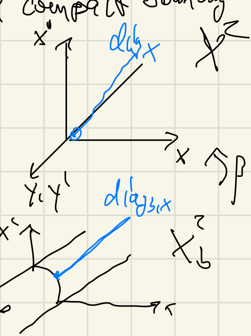
· If $E\to Z$ is vector bundle then have notion of smooth Dirac section of E (a_α sections of E).

Here: $E = |\Omega_Z|^{1/2}$.

Def: If X is a manifold with connected compact boundary let

$X_b^2 := [X^2, (\partial X)^2]$

$\mathrm{diag}_{b,X} := \beta^*\,\mathrm{diag}_X$



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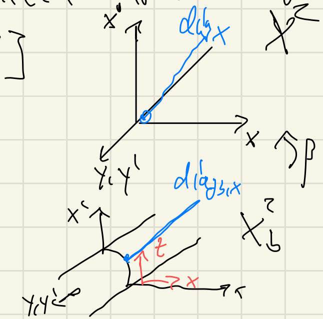


In coordinates:

$$\mathrm{diag}_{b,X} = \{t = 1,\ y = y'\} \qquad (x, y \text{ arbitrary})$$

Prop: the map $P \longmapsto \beta^* K_P$

defines an isomorphism

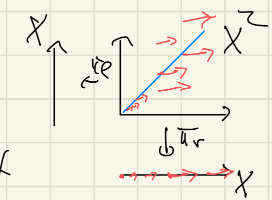
$\mathrm{Diff}_b^m(X, |\Omega_b|^{1/2}) \longrightarrow$ smooth Dirac sections of $|\Omega_b|^{1/2}$ on X_b^2 of order m at $\mathrm{diag}_{b,X}$

One proof: local calculation as above.

Another way to understand this:

(1) For $P = \mathrm{Id}$: $\beta^* K_{\mathrm{Id}}$ is smooth, non-vanishing Dirac section of $|\Omega_b|^{1/2}$

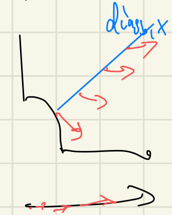
(2) Note that $K_P = (\pi_r^* P) K_{\mathrm{Id}}$



$\pi_r(p, p') = p$ (projection to left factor)

($\pi_r^* P$: P acts on left factor)

If P is a vector field $P = V$, then the lift $\pi_r^* V$ degenerate at $x = 0$. However, it does not after blow-up!



Lemma:

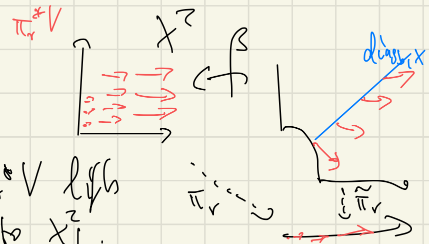
a) If $V \in \mathcal{V}_b(X)$ then $\pi_r^* V$ lifts smoothly from X^2 to X_b^2.

b) These lifts span an n-dim'l subbundle of $T_{\mathrm{diag}_{b,X}} X_b^2$ which is transversal to $T\mathrm{diag}_{b,X}$

Lemma:

a) If $V \in \mathcal{V}_b(X)$ then $\pi_r^* V$ lifts smoothly from X^2 to X^2_b.

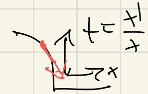
b) These lifts span an n-dim'l subbundle of $T_{\mathrm{diag}_{b,X}} X^2_b$ which is transversal to $T\,\mathrm{diag}_{b,X}$



__Proof:__ a) By lifting lemma for vector fields this follows since $\pi_r^* V$ is tangential to $(\partial X)^2$

center of blow-up.

b) Clear in the interior, and near bd. in coordinates:



$$\beta^*(x\partial_x) = x\partial_x - t\partial_t$$
$$= -\partial_t \quad \text{at } t=1,\ x=0$$

$$\beta^*(\partial_y) = \partial_y .$$

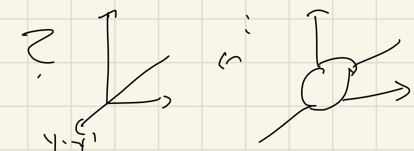
The lemma implies Proposition.

__Remarks:__

1.) This implies that $\tilde\pi_r = \pi_r \circ \beta$ induces an isomorphism

$$N\,\mathrm{diag}_{b,X} = T X^2_b \big/ T\,\mathrm{diag}_{b,X} \xrightarrow{d\tilde\pi_r} {}^b TX$$

2.) In $n>1$: Why do we blow up $\partial X \times \partial X$ $(x=x'=0,\ \text{all } y, y')$ and not $\mathrm{diag}_{\partial X} = (x=x'=0,\ y=y')$?



Answer: then a) would be false. In fact: $(\partial X)^2$ is the smallest closed submanifold

- containing $\mathrm{diag}_{\partial X}$ $(= \mathrm{diag}_X \cap (\partial X)^2)$
- so that $\pi_r^* \mathcal{V}_b$ is tangent to it.

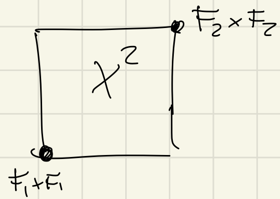
__I.o.w.:__ $(\partial X)^2$ is the flow-out of $\mathrm{diag}_{\partial X}$ under $\pi_r^* \mathcal{V}_b(X)$. (Starting from $y=y'$ can reach any (y,y') along some curve in ∂X)

∂X is connected $\longrightarrow$

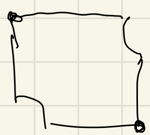
3.) If ∂X is disconnected, $\partial X = \bigcup_{i=1}^{N} F_i$

then we define $X_b^2 = [X^2; F_1 \times F_1, \dots, F_N \times F_N]$

Ex: $X = [0,1]$



$X_b^2 =$



(only blow-up "diagonal corners")

The principal symbol of $P \in \mathrm{Diff}_b^m(X)$

X m.w.b.

Def: In coordinates (x,y) we define for

$$P = \sum_{k+|\alpha|\le m} a_{k,\alpha}(x,y)\,(xD_x)^k D_y^\alpha$$

the b-symbol

$$^b\sigma(P) := \sum_{k+|\alpha|=m} a_{k,\alpha}(x,y)\,\lambda^k \eta^\alpha$$

$\lambda \in \mathbb{R},\ \eta \in \mathbb{R}^{n-1}$.

ex: $x^2\Delta$ on $\mathbb{R}^2$ in polar coords: $(x\partial_x)^2 + \partial_y^2$ $\quad$ y = angle var., x = radial

$\rightsquigarrow$ $^b\sigma(x^2\Delta) = -\lambda^2 - \eta^2$.

Where is $^b\sigma$ defined invariant.

Recall: X mfd:

symbol of κ_P on $N^*_{12}\mathrm{diag}_X$

symbol of P on T^*X

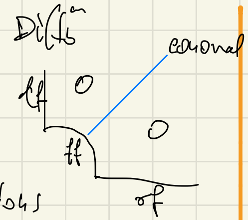
Use previous lemma/remark:

Lemma: The (principal) b-symbol defined above equals the (principal) symbol of the conormal distr. $\beta^*\kappa_P$ under the given isomorphism $N^*\mathrm{diag}_{b,X} \cong {}^bT^*X$

Therefore, $^b\sigma(P)$, $P \in \mathrm{Diff}_b^m(X)$, is defined invariantly as a function on $^bT^*X$.

Here, λ, η_j are coords on $^bT^*X$ w.r.t. the basis $\frac{dx}{x}, dy_j$.

Definition of small b-ΨDO calculus:



$\Psi_b^m(X) = \{ |\Omega_b|^{1/2}$-valued distributions on X_b^2 which are

a) conormal w.r.t. $\mathrm{diag}_b X$, smoothly up to ff

b) vanishing to ∞ order at lf, rf. $\}$