2021-01-21

4.1.5 b-ΨDOs on b-Sobolev spaces

X compact manifold with boundary.

Choose ν positive b-density.

$$L^2_b(X) = L^2(X,\nu) = \{ f : X \to \mathbb{C} : \int_X |f|^2 d\nu < \infty \}$$

Topology independent of choice of ν.

Half-densities: $u \in L^2(X, |\Omega|^{1/2}) :\Leftrightarrow \int |u|^2 < \infty$

$$\overset{\shortparallel}{L^2(X, |\Omega_b|^{1/2})}$$

$$u(x,y)\sqrt{dx\,dy} = \sqrt{x}\,u(x,y)\sqrt{\tfrac{dx}{x}dy}$$

$$L^2_b(X) \to L^2(X, |\Omega|^{1/2})$$
$$f \mapsto f\sqrt{\nu}$$

<u>Prop</u>: $P \in \Psi^0_b(X) \Rightarrow \exists C$

$$\|Pu\|_{L^2} \le C\|u\|_{L^2} \quad \forall u \in C^\infty_0(\mathring{X}).$$

Therefore P extends to a bdd. operator

$$P : L^2 \to L^2$$

<u>Lemma 1</u>: Let $P \in \Psi^0_b$. Then there is $C > 0$ and $Q \in \Psi^0_b$, $R \in \Psi^{-\infty}_b$ so that

$$P^*P + Q^*Q = C + R$$

Here P^* = adjoint of P.

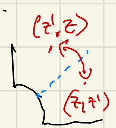
- choose ν, one adjoint wrt $L^2(X,\nu)$
- for half-densities: $L^2(X, |\Omega|^{1/2})$ is naturally a Hilbert space $\rightsquigarrow P^*$ defined.

Schwartz-kernel: $K_{P^*}(z,z') = \overline{K_P(z',z)}$.

This shows that $P \in \Psi^m_b \Rightarrow P^* \in \Psi^m_b$.

$$(Pu, v) = (u, P^*v)$$

<u>Fact</u>: $\sigma(P^*) = \overline{\sigma(P)}$



Lemma 1: Let $P \in \Psi_b^0$. Then there is $C > 0$ and $Q \in \Psi_b^0$, $R \in \Psi_b^{-\infty}$ so that

$$P^* P + Q^* Q = C + R$$

Proof: Let $p_0 = {}^b\sigma_0(P) \in S^{[0]}({}^bT^*X)$, p_0 bounded function, choose $C > \max |p_0|^2$.

Let $$q_0 = \sqrt{C - |p_0|^2} \in S^{[0]}({}^bT^*X).$$

Choose $Q_0 \in \Psi_b^0$ so that ${}^b\sigma_0(Q_0) = q_0$.

Then $R_1 = C - P^*P - Q_0^* Q_0$ is in Ψ_b^0 and has

$${}^b\sigma_0(R_1) = C - \bar{p}_0 p_0 - \bar{q}_0 q_0 = C - |p_0|^2 - q_0^2 = 0$$

so $R_1 \in \Psi_b^{-1}(X)$.

We look for $Q_1 \in \Psi_b^{-1}$ so that

$$R_2 := C - P^*P - (Q_0 + Q_1)^*(Q_0 + Q_1)$$

is in Ψ_b^{-2}.

We have $R_2 = \underset{-1}{R_1} - \underset{-1}{Q_0^* Q_1} - \underset{-1}{Q_1^* Q_0} - \underset{-2}{Q_1^* Q_1}$

orders!

$\Rightarrow R_2 \in \Psi_b^{-1}$, ${}^b\sigma_{-1}(R_2) = r_1 - q_0 q_1 - \bar{q}_1 q_0$

where $r_1 = {}^b\sigma_{-1}(R_1)$ (q_0 real)

$q_1 = {}^b\sigma_{-1}(Q_1)$

$\rightsquigarrow$ we need to choose q_1 so that

$$q_1 + \bar{q}_1 = \frac{r_1}{q_0} \qquad (\text{recall } q_0 > 0)$$

Choose q_1 to be real, $q_1 = \frac{r_1}{2 q_0}$.

Proceeding inductively, we get $Q_j \in \Psi_b^{-j}$ so

$$R_N = C - P^*P - (Q_0 + \dots + Q_{N-1})^*(Q_0 + \dots + Q_{N-1})$$

is in Ψ_b^{-N}, for every N.

Let $Q \sim \sum_{j=0}^{\infty} Q_j \in \Psi_b^0$, then

$R := C - P^*P - Q^*Q \in \Psi_b^{-\infty}$ qed

Lemma 1: Let $P \in \Psi_b^0$. Then there is $C > 0$ and $Q \in \Psi_b^0$, $R \in \Psi_b^{-\infty}$ so that

$$P^*P + Q^*Q = C + R$$

Apply this to $u \in C_0^\infty(X^\circ)$, take scalar product with u

$$\langle P^*Pu, u\rangle = \langle Pu, Pu\rangle = \|Pu\|^2 \implies$$

$$\|Pu\|^2 + \underbrace{\|Qu\|^2}_{\geq 0} = C\cdot\|u\|^2 + \langle Ru, u\rangle$$

this implies $\|Pu\| \leq C'\|u\|$ if we prove:

Lemma 2: $R \in \Psi_b^{-\infty} \implies |\langle Ru, u\rangle| \leq C\|u\|^2$

Rem: If X closed then this is obvious, R has smooth Schwartz kernel

$$\langle Ru, u\rangle = \int \underbrace{k_R(z, z')}_{\text{smooth} \Rightarrow \text{bdd}} u(z')\overline{u(z)}\, dz\, dz'$$

+ Cauchy-Schwarz, $\mathrm{vol}(X) < \infty$.

Proof: First recall Young's inequality: (a simple form of it)

$$\sup_x \int |k(x, x')| \frac{dx'}{x'} + \sup_{x'} \int |k(x, x')| \frac{dx}{x} < \infty \qquad (*)$$

Then $\left|\int k(x, x')\, u(x) u(x') \frac{dx}{x} \frac{dx'}{x'}\right| \leq C\|u\|^2$

(for a proof use $|u(x)u(x')| \leq \frac{|u(x)|^2 + |u(x')|^2}{2}$)

So we only need to check $(*)$ for the kernel k of R.

By symmetry, we only need to show

$$\exists C\ \forall x \quad \int_0^1 |k(x, x')| \frac{dx'}{x'} \leq C.$$

(here we may assume k is supported near $(\partial X)^2$, say in $x < 1$, $x' < 1$ and we suppress the y variables)

Now $k(x, x') = \kappa(x, \frac{x'}{x})$ where $\kappa(x, t)$ vanishes rapidly as $t \to 0$ or $t \to \infty$ unif in x, and is smooth, supported in $x < 1$,

so
$$\int_0^1 |k(x, x')| \frac{dx'}{x'} = \int_0^1 \left|\kappa\left(x, \frac{x'}{x}\right)\right| \frac{dx'}{x'} \leq \int_0^\infty |\kappa(x, t)| \frac{dt}{t} < C$$

qed (Lemma 2 and L^2-boundedness)

[note: this page changed from live lecture]

Def: For $m \in \mathbb{R}$ let

$$H_b^m(X) = \{u \in C^{-\infty}(X) : Qu \in L^2 \text{ for all } Q \in \Psi_b^m(X)\}$$

For $m \in \mathbb{N}_0$:

$$H_b^m(X) = \{u \in L^2 : \mathrm{Diff}_b^m u \subset L^2\}$$

$$\dot{C}^\infty \subset \dots \subset H_b^1 \subset H_b^0 = L^2 \subset H_b^{-1} \subset \dots \subset C^{-\infty}$$

Prop. $P \in \Psi_b^m \Rightarrow P : x^\gamma H_b^s \to x^\gamma H_b^{s-m}$ $(*)$

for all $s \in \mathbb{R}, \gamma \in \mathbb{R}$.

Proof: (1) Reduction to $\gamma = 0$:

$$x^\gamma H_b^s := \{x^\gamma u : u \in H_b^s\}$$

$$P(x^\gamma u) \overset{!}{=} x^\gamma f$$

$u \in H_b^s$ then $f \in H_b^{s-m}$

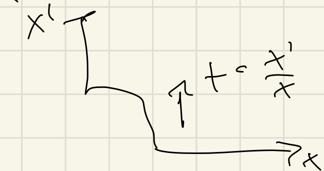
So with $P_\gamma := x^{-\gamma} P x^\gamma$

the claim $(*)$ is equiv. to $P_\gamma : H_b^s \to H_b^{s-m}$.

Lemma: $P \in \Psi_b^m$, $\gamma \in \mathbb{C}$

$\Rightarrow x^{-\gamma} P x^\gamma \in \Psi_b^m$.

Pf: $K_{x^{-\gamma} P x^\gamma}(x, x') = \left(\frac{x'}{x}\right)^\gamma \cdot K_P(x, x')$



(2) Pf for $\gamma = 0$ as exercise:

First: $s = m$, $s = 0$ (from def.)

Second: Choose elliptic $\Lambda_\ell \in \Psi_b^\ell$ for all ℓ.

(any s) Use this and its parametrix to reduce order.

4.1.6. Why the small b-calculus is not enough

(1) **Regularity**: If P b-elliptic, $Pu = 0$

we proved $u \in \mathcal{A}^s$ for some s.

Want: u phg.

(2) **Mapping in L^2**: $P \in \Psi_b^0 \Rightarrow$ bdd $L^2 \to L^2$.

Recall: $R \in \Psi^{-\infty} \Rightarrow R$ compact $: L^2 \to L^2$

(X closed) $\Rightarrow$ elliptic P is Fredholm.

Prop: Let $R \in \Psi_b^{-\infty}(X)$. The following are equivalent:

(i) $R: L^2 \to L^2$ is compact.

(ii) $R: L^2 \to L^2$ is Hilbert-Schmidt

(iii) $K_R|_{ff} = 0$

So $R \in \Psi_b^{-\infty} \not\Rightarrow R$ compact!!

Pf: (ii) $\Leftrightarrow$ (iii): Recall R is Hilbert-Schmidt

$\Leftrightarrow \quad K_R \in L^2(X^2)$

$$\|K_R\|_{L^2}^2 = \iint (K_R(x,x'))^2 \frac{dx}{x}\frac{dx'}{x'}$$

assume: supported near ∂X $\dashrightarrow$
$$= \iint \left|\kappa\left(x, \frac{x'}{x}\right)\right|^2 \frac{dx}{x}\frac{dx'}{x'}$$

$$= \iint |\kappa(x,t)|^2 \frac{dx}{x}\frac{dt}{t}$$

t-integral is ok.

x-integral is finite iff $\kappa(0,t) = 0$ for each t.

$\Updownarrow$

$K_R|_{ff} \equiv 0$.

(iii) $\Rightarrow$ (i) standard func ana

(i) $\Rightarrow$ (iii) exercise. qed

(3) A simple example

Consider $X = \mathbb{R}_+$, $P = x\partial_x - c$, $c \in \mathbb{R}$.

Solve $\quad Pu = f \quad$ (f given)

$$(x\partial_x - c)u = f \qquad \Big| \cdot x^{-c-1}$$

$$\iff \partial_x(x^{-c}u) = x^{-c-1} f$$

$$\iff u(x) = a \cdot x^c + x^c \int_0^x (x')^{-c-1} f(x')\,dx'$$

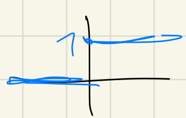
if $f = O(x^s)$, $s > c$ $(x \to 0)$

First consider $a = 0$:

$$u(x) = \int_{\mathbb{R}_+} K_>(x,x') f(x') \frac{dx'}{x'} =: Q_> f$$

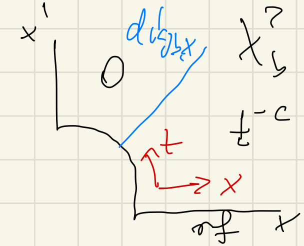
where

$$K_>(x,x') = \left(\frac{x'}{x}\right)^{-c} H(x - x') = \boxed{t^{-c} H(1-t)}$$

$H =$ 

$t = \frac{x'}{x}$

Note:



- $\beta^* K_>$ is conormal on X_b^2 w.r.t. $\mathrm{diag}_{b,X}$
- but: no ∞ order vanishing at rf.

  $\rightsquigarrow K \notin \Psi_b^*$

Note: If $s > c$ and $f \in \mathcal{A}^s(\mathbb{R}_+)$ then we get

a) $\exists$ sol'n $u_0 = Q_> f$, $u_0 \in \mathcal{A}^s$

b) general solution is $a x^c + u_0$.

($a \cdot x^c$ = sol'n of homog. eq'n $(x\partial_x - c)x^c = 0$)

[in fact: $Q_>$ is inverse of $P : \mathcal{A}^c \to \mathcal{A}^s$]

Q: What if $s < c$?

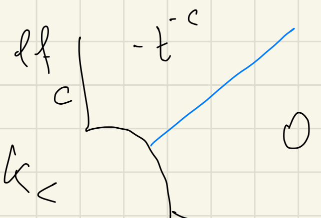
If $s < c$: a solution (say for $f \in \mathcal{A}_0^s$) is

$$u(x) = x^c \int_\infty^x (x')^{-c} f(x') \frac{dx'}{x'}$$

$$= \int_{\mathbb{R}_+} K_<(x,x') f(x') \frac{dx'}{x'}$$

where $K_<(x,x') = -\left(\frac{x'}{x}\right)^c H(x'-x)$

$$= -t^{-c} H(t-1)$$



non-trivial asymptotics at lf: index set $= \{c\}$

Lesson:

- there are different inverses (with different Schwartz kernels) depending on the decay rate s of f.
- what matters is the number c.

c: exponent of $x^c \in \ker(x\partial_x - c)$.