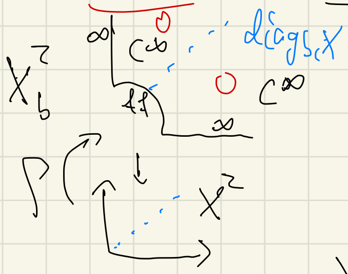
2021-01-27

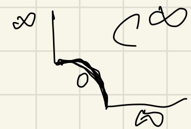
4.2 Full b-calculus

X = manifold with compact boundary ∂X

$\mathrm{Diff}_b^*(X) \subset \Psi_b^*(X)$



$R \in \Psi_b^{-\infty}(X)$



X compact:

R compact operator $\Leftrightarrow K_R'|_{ff} = 0$

K_P = int. kernel of P

$K_P' = \beta^* K_P$

Need: improve parametrix so error $= 0$ on ff

Plan

- identify a "second symbol" of P

$$I_P \;\hat{=}\; K_P'|_{ff}$$

(indicial operator)

- show that $P \mapsto I_P$ is algebra homom.
- Then: Given $PQ = I + R$

$P \in \Psi_b^m$ elliptic, $Q \in \Psi_b^{-m}$, $R \in \Psi_b^{-\infty}$

we want to add Q' to Q satisfying

$$I_{P(Q+Q')-I} = 0$$

$$\overset{\shortparallel}{I_R + I_{PQ'}} = I_R + I_P I_{Q'}$$

$\rightsquigarrow$ need: $I_P I_{Q'} = -I_R \rightsquigarrow$ invert I_P!

History: · 1967 Kondratyev: first paper on operators on spaces with conic singularities

· Melrose 1981 (Transformation of BVP)

· Rempel-Schulze 1986 „cone calculus"

Lauter-Seiler '99: b-calculus = cone calculus)

4.2.1 The indicial operator and indicial family

Def: For $P \in \mathrm{Diff}_b^m(X)$,

$$P = \sum_{k+|\alpha|\le m} a_{k\alpha}(x,y)(x\partial_x)^k D_y^\alpha$$

$$= \sum_{k=0}^m A_k(x)(x\partial_x)^k$$

where $A_k(x) \in \mathrm{Diff}^{m-k}(\partial X)$

define the indicial operator

$$I_P := \sum_{k=0}^m A_k(0)\,(x\partial_x)^k$$

and the indicial family

$$\hat{I}_P(z) = \sum_{k=0}^m A_k(0)\, z^k \quad , \quad z \in \mathbb{C}.$$

· $I_P \in \mathrm{Diff}^m_{b,I}(\tilde{X})$, $\tilde{X} = \underset{x}{\mathbb{R}_+} \times \partial X$

invariant under dilations $x \mapsto a x$, $a > 0$.

· $\hat{I}_P(z) \in \mathrm{Diff}^m(\partial X)$, Polynomial in z.

Rem: We fix a tubular neighborhood of ∂X:

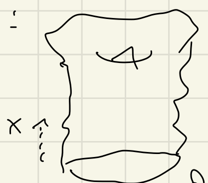
$$X \supset U \cong \underset{x}{[0,1)} \times \underset{y}{\partial X}$$

Note: I_P contains lower order derivatives.

Ex: $P = (x\partial_x)^2 + a(x)\, x\partial_x + b(x)$ $\qquad X = \mathbb{R}_+$

$$I_P = (x\partial_x)^2 + a(0)\, x\partial_x + b(0)$$

Remark: $I(P)$ arises as model operator at ∂X from zooming in:

$\hat{X} = \mathbb{R}_+ \times \partial X$  $\xrightarrow{M_\varepsilon}$ $(x,y) \mapsto (\varepsilon x, y)$  X, ∂X, x

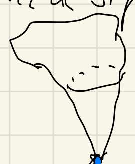
$$M_\varepsilon^* P = \sum_0^m A_k(\varepsilon x)\left(\varepsilon x \frac{\partial}{\partial(\varepsilon x)}\right)^k \qquad P$$

$$\downarrow \varepsilon \to 0$$

$$I(P)$$

If X arises by blow-up of a conical singularity: this corresponds to understand 'zooming in at p.'

(y = angular variables)



p

Remark: How does $\hat{I}_P(z)$ arise?

Try to solve $Pu = 0$

$$\left[\sum_{k=0}^m A_k(x)(x\partial_x)^k\right]\left(x^z v_0(y) + x^{z+1} v_1(y) + \dots\right) \overset{!}{=} 0$$

Leading x-power of this as $x \to 0$.
Coefficient of x^z is

$$\left(\sum_0^m A_k(0)\, z^k\right) v_0 = \hat{I}_P(z)\, v_0$$

$\Rightarrow$ any sol'n of this form must have $v_0 \in \ker \hat{I}_P(z)$.

$\Rightarrow$ possible z are the following:

Def: $\operatorname{spec}_b(P) := \{z : \ker \hat{I}_P(z) \neq 0\}$

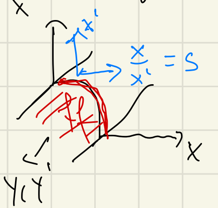
$\Updownarrow$ $\hat{I}_P(z)$ not invertible

(we'll see.

[Careful: Melrose uses a rotated version of this as def'n

Extending I_P to $P \in \Psi_b^*$:

- find relation $k_P \leftrightarrow k_{I_P}$, $P \in \mathrm{Diff}_b^*$
- define I_P for $P \in \Psi_b^*$ in same way



$$\beta^*(x\partial_x) = s\partial_s$$

$$\boxed{s = \frac{x}{x'}}$$

Lemma: Let $P \in \mathrm{Diff}_b^*(X)$ and let

$$k_P' = \beta^* k_P = k_P(s, x')$$

(valued in operators on Y / Schwartz kernels on Y^2)

Then

$$(I_P u)(x) = \int_0^\infty k_P\left(\frac{x}{x'}, 0\right) u(x') \frac{dx'}{x'}$$

$x' = 0$, i.e. restrict to ff

That is: I_P is given by:

(i) restricting k_P' to ff

(ii) multiplicative convolution in the x-variable (or s-)

Def: $(u * v)(x) := \int_0^\infty u\left(\frac{x}{x'}\right) v(x') \frac{dx'}{x'}$.

Proof: $k_P(x, x') = \sum A_k(x)(x\partial_x)^k \delta(x - x')$

$\Rightarrow$ if $x' > 0$: $k_P(s, x') = k_P(x's, x')$

$$= \sum A_k(x's)(s\partial_s)^k \delta(x'(s-1))$$

$\frac{1}{x'}$ disappears in density factor

$$= \sum A_k(x's)(s\partial_s)^k \delta(s-1)$$

$$\Rightarrow k_P(s, 0) = \sum A_k(0)(s\partial_s)^k \delta(s-1)$$

$$= k_{I_P}(s, 0) = k_{I_P}(s, x') \quad \forall x'$$

So:
- I_P ≙ restrict k_P to ff
- k_{I_P} indep. of x', write as $k_{I_P}(s)$.

Also:

$$(I_P u)(x) = \int_0^\infty k_{I_P}(x, x') u(x') \, dx'$$

$$= \int_0^\infty k_{I_P}\left(\frac{x}{x'}\right) u(x') \frac{dx'}{x'}$$

qed

Formula:

$$\boxed{I_P u = \kappa_{P,\mathrm{ff}} * u} \quad (*)$$

Def: For $P \in \Psi_b^m(X)$ define $I_P \in \Psi_{b,I}^m(\tilde{X})$ by $(*)$.

dilation invariant

$\tilde{X} = [\mathbb{R}_t \times \partial X$

Rem: (invariant perspective)

- Invariantly, $\tilde{X} = N^+ \partial X$ (inward pointing normal bundle) and naturally:

$$\begin{array}{ccc} X_b^2 & & \tilde{X}_b^2 \\ \cup & & \subset \\ & \mathrm{ff}(X_b^2) \cong \mathrm{ff}(\tilde{X}_b^2) & \end{array}$$

natural, use $\mathrm{ff}(X_b^2) = [N^+((\partial X)^2)]/\text{dilations}$.

$P \mapsto \kappa_P'|_{\mathrm{ff}} \mapsto I_P$ unique dil. inv. op. on $\tilde{X}$ having the same restriction to ff.

Short: $\boxed{I_P \cong \text{restrict } P \text{ to ff}}$

Extending $\hat{I}_P(z)$ to $P \in \Psi_b^*$

Recall: u fcn of $x \in (0,\infty)$

$$(\mathcal{M}_- u)(z) = (\mathcal{M} u)(-z) = \int_0^\infty u(x)\, x^{-z}\, \frac{dx}{x}$$

a) $\mathcal{M}_-(x\partial_x u) = z \cdot \mathcal{M}_- u$

b) $\mathcal{M}_-(u * v) = \mathcal{M}_- u \cdot \mathcal{M}_- v$

For $P \in \mathrm{Diff}_b^m$ we get: $I_P = \sum A_k(0)(x\partial_x)^k$

a) $\mathcal{M}_-(I_P u) = \hat{I}_P \cdot \mathcal{M}_- u$

b) together with $(*)$:

$\mathcal{M}_-(I_P u) = \mathcal{M}_-(\kappa_{P,\mathrm{ff}}) \cdot \mathcal{M}_- u$

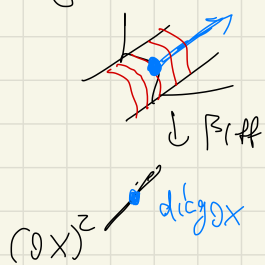
therefore, $\hat{I}_P(z) = \mathcal{M}_-(\kappa_{P,\mathrm{ff}})(z)$

for $P \in \mathrm{Diff}_b^*$.

Def: For $P \in \Psi_b^m(X)$ let

$$\hat{I}_P(z) = \mathcal{M}(\kappa_{P|ff})(-z) = \int_0^\infty \kappa_{P|ff}(s)\, s^{-z} \frac{ds}{s}$$

Note: $\hat{I}_P(z) \in \Psi^m(\partial X)$ by PFT for conormal distr.



Next goal: Show that $I, \hat{I}$ are algebra homom. (preserve products).

Non-obvious from def's.

⇝ We'll use another characterization of $\hat{I}_P$, not using the Schwartz kernel.

Def: $P \in \Psi_b^m(X)$ define $P_\partial \in \Psi^m(\partial X)$ by: For $v \in C^\infty(\partial X)$ let

$$P_\partial v = (P\tilde{v})_{|\partial X}, \quad \tilde{v} \in C^\infty(X)$$

some extension of v: $\tilde{v}_{|\partial X} = v$.

This is well-defined, i.e. indep. of choice of $\tilde{v}$:

Need to show: $v_{|\partial X} = 0 \Rightarrow (Pv)_{|\partial X} = 0$.

Proof: $v = x \cdot w$, $w \in C^\infty$

$$\Rightarrow Pv = P(xw) = x\, x^{-1} P x w = x \cdot \bar{P} w$$

where $\bar{P} = x^{-1} P x$.

We know $\bar{P} \in \Psi_b^m$, so $\bar{P} w \in C^\infty(X)$

$\Rightarrow$ $Pv = x \cdot \text{smooth}$ $\Rightarrow$ zero on ∂X.

Note: This would be wrong for $P \in \mathrm{Diff}^1(X)$ instead of $P \in \mathrm{Diff}_b^1(X)$.

$$\hat{I}_p[z] = \left(x^{-z} P x^{z} \right)_{a}$$