2021-02-03

4.7.3 Inverting the indicial operator (cont'd.)

$P \in \Psi_b^m(X)$ elliptic

$\rightsquigarrow$ $I_P \in \Psi_{b,I}^m(\tilde{X})$, $\tilde{X} = \partial X \times \mathbb{R}_+$

$K_{I_P}(s, y, y') \;\hat{=}\; (\beta^* K_P)|_{ff}$

$\hat{I}_P(z) \in \Psi^m(\partial X)$, $z \in \mathbb{C}$

a) kernel of $\hat{I}_P(z)$ is $(\mathcal{M} K_{I_P,ff})(-z)$

b) $\hat{P}(z) = \hat{I}_P(z) = (x^{-z} P x^z)_\partial$

Meaning of b): $\hat{P}(z) v = (x^{-z} P x^z \tilde{v})|_{x=0}$

$v \in C^\infty(\partial X)$, $\tilde{v} \in C^\infty(X)$, $\tilde{v}|_{x=0} = v$.

This says

$$\boxed{P(x^z \tilde{v}) = x^z \cdot \hat{P}(z) v + O(x^{z+1})}$$

Prop: (formal solutions)

$P \in \Psi_b^m$ elliptic, $f \in \mathcal{A}^E(X)$.

Then $\exists u \in \mathcal{A}^F$ so that

$$Pu = f \mod \underbrace{\dot{C}^\infty(X)}_{O(x^\infty)}$$

where $F_{\mathbb{C}} = E_{\mathbb{C}}$.

Proof (sketch):

Remove asymptotic terms of f iteratively, then apply Borel lemma.

1) $f = x^z \cdot w(y) + \text{h.o.t.}$

Case I: $z \notin \operatorname{spec}_b(P)$, i.e. $\hat{P}(z)$ invertible.

Take $u = x^z v + u'$, $v = \hat{P}(z)^{-1} w$, $u' = \text{h.o.t.}$

From $P(x^z v) = x^z w + \text{h.o.t.}$, so need only solve

$$Pu' = f' = o(x^z)$$

Case I: $z_0 \in \mathrm{spec}_b P$. Say simple pole.

$$\hat{P}(z)^{-1} = \frac{1}{z-z_0} C(z), \quad C \text{ holom.}$$

Apply to ω $\Rightarrow$

$$(z-z_0)\omega = \hat{P}(z) \underbrace{C(z)\omega}_{=: v(z)}$$

then $P(x^z v(z)) = x^z (z-z_0)\omega + \text{h.o.t}$

$$\frac{d}{dz}\Big|_{z=z_0} : \quad P(x^z \log x \cdot v(z_0) + x^z \cdot v'(z_0)) = x^{z_0} \cdot \omega + \text{h.o.t}$$

$\Rightarrow$ $f = x^z \log^k x \cdot \omega + \text{h.o.t}$ (similar)

qed

The Mellin transform on $\mathbb{R}_+$.

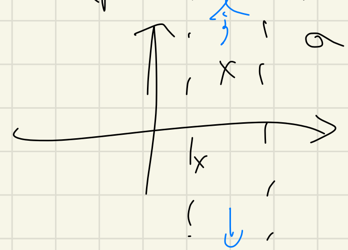
Need: $\mathcal{M}u$ for u having asymptotics as $x \to 0$ and as $x \to \infty$.

Recall: If $\mathrm{supp}\, u \subset [0, C]$, $u \in \mathcal{A}^E(\mathbb{R}_+)$

$\Rightarrow$ • $\mathcal{M}u(\sigma) = \int_0^\infty u(x) x^\sigma \frac{dx}{x}$ is defined for $\mathrm{Re}\,\sigma > -\inf E$

• $\mathcal{M}u$ has meromorphic cont. to $\mathbb{C}$, with $(-E)$-poles: $(z,k) \in E$ (with maximal k) then pole at $-z$ of order $k+1$.

• rapid decrease in vertical strips



u on $\mathbb{R}_+$

u_∞: change variable $x \to \frac{1}{x}$

Let $\overline{\mathbb{R}}_+ = [0,\infty]$, boundary def for $\frac{1}{x}$ at ∞.

⇝ $\mathcal{A}^{(\mathcal{E}_0,\mathcal{E}_\infty)}(\overline{\mathbb{R}}_+)$ is defined

supp u near ∞: $(\mathcal{M}u)(\sigma) = \int_0^\infty u(x)\, x^\sigma \frac{dx}{x}$ is

defined if $\sigma < \inf \mathcal{E}_\infty$

$\mathcal{E}_\infty \ni z$ ⇝ x^{-z} in u

Prop: Let $\mathcal{E}_0, \mathcal{E}_\infty$ be index sets satisfying

$$\inf \mathcal{E}_0 + \inf \mathcal{E}_\infty > 0.$$

Then $\mathcal{M}u$ is defined for $u \in \mathcal{A}^{(\mathcal{E}_0,\mathcal{E}_\infty)}$

if $-\inf \mathcal{E}_0 < \mathrm{Re}\,\sigma < \inf \mathcal{E}_\infty$.

Also, $\mathcal{M}$ defines an isomorphism

$$\mathcal{A}^{(\mathcal{E}_0,\mathcal{E}_\infty)}(\overline{\mathbb{R}}_+) \longrightarrow \left\{ \begin{array}{l} \text{merom. fcts on } \mathbb{C} \\ \text{having } ((-\mathcal{E}_0)\cup\mathcal{E}_\infty) \text{ poles} \\ \text{decaying rapidly in} \\ \text{vertical strips} \end{array} \right\}$$

with inverse

$$\left[(\mathcal{M}^{-1})_\alpha \hat{u}\right](x) = \frac{1}{2\pi i} \int_{\alpha + i\mathbb{R}} \hat{u}(\sigma)\, x^{-\sigma}\, d\sigma$$

for any $\alpha \in (-\inf \mathcal{E}_0, \inf \mathcal{E}_\infty)$.

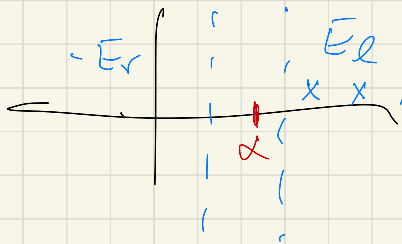
Note: Given $S \subset \mathbb{C} \times \mathbb{N}_0$, vertically finite, there is an $\mathcal{M}$ map (and inverse) for any interval in $\mathbb{R} \setminus \mathrm{Re}\,S$. Each such interval determines a decompos. $S = (-\mathcal{E}_0) \cup \mathcal{E}_\infty$.

The Prop. extends directly to smoothing op's:

$$\tilde{P} \in \Psi^{-\infty,\mathcal{E}}_{b,I} \xrightarrow{M(-z)} \hat{P}(z) \text{ meromorph.}$$

$(M^{-1})_\alpha$

with $[-E_r] \cup E_l$ poles



(Need: $\inf E_l + \inf E_r > 0$)

Def:

$\mathrm{Spec}_b(P) = \mathrm{Spec}_b(I_P) = \{ (z,k) : \hat{I}_P^{-1}$ has a pole at z of order $\geq k+1 \}$

Thm: Let $\tilde{P} \in \Psi^m_{b,I}(\tilde{X})$ be elliptic and $\tilde{R} \in \Psi^{-\infty}_{b,I}(\tilde{X})$.

Then for each $\alpha \notin \mathrm{Re}\,\mathrm{spec}_b(\tilde{P})$ there is $\tilde{Q}_\alpha \in \Psi^{-\infty,\mathcal{E}(\alpha)}_{b,I}(\tilde{X})$ solving

$$\tilde{P}\tilde{Q}_\alpha = \tilde{R}$$

where $\mathcal{E}(\alpha) = (E_l(\alpha), E_r(\alpha))$

$E_l(\alpha)$ = part of $\mathrm{Spec}_b(\tilde{P})$ to right of α

$-E_r(\alpha)$ = ... left.

Proof: Need: $\hat{P}\hat{Q} = \hat{R}$

$\rightsquigarrow$ take $\hat{Q} = \hat{P}^{-1}\hat{R} \in \Psi^{-\infty}$ + meromorph.

apply $(M^{-1})_\alpha$

qed

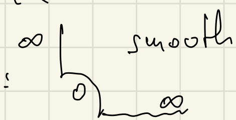
Note: One needs to complete $\mathcal{E}_{l/r}(\alpha)$ to smooth index sets.

4.24 The parametrix construction in the full b-calculus

Choose $\alpha \notin \operatorname{spec}_b P$.

$P \in \Psi_b^m(X)$ elliptic. Outline:

1) $PQ_1 = I + R_1$, $R_1 \in \Psi_b^{-\infty}$ $\quad R_1$:



2) $PQ_2 = I + R_2$, $\quad R_2$:

3) $PQ_3 = I + R_3$ $\quad R_3$:

Neumann series

4) $\quad R_4$

(For right parametrix; transpose trick for left param.)

1) ⇝ 2) Use I_P^{-1}:

Have: $PQ_1 = I + R_1$

If $Q_2 = Q_1 + Q_1'$ then $PQ_2 = I + R_2$

where $R_2 = R_1 + R_1'$, $R_1' = PQ_1'$.

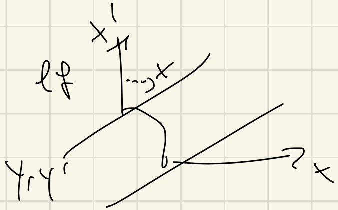
Choose Q_1' so $I_P I_{Q_1'} = -I_{R_1}$

Then we get:

- $I_{R_2} = 0$
- $Q_1' \in \Psi_b^{-\infty, \mathcal{E}(\alpha)}$
- $R_1' = PQ_1' \in \Psi_b^m \Psi_b^{-\infty, \mathcal{E}(\alpha)} = \Psi_b^{-\infty, \mathcal{E}(\alpha)}$

$\Rightarrow R_2 \in \rho_{ff} \Psi_b^{-\infty, \mathcal{E}(\alpha)}$

2) ⇝ 3): Remove asymptotics at lf.



$$lf = \underset{Y}{\partial X} \times \underset{x',y'}{X}$$

has nbhd. $\underset{x,y}{\mathbb{C}l} \times \underset{p'=(x',y')}{X}$

Need to solve

$$PQ_2' = -R_2 \pmod{x^\infty} \text{ near } lf.$$

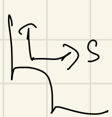
$$P\, Q_2'(x,y;x',y') = -R_2(x,y;x',y')$$

P acts in x,y variables. (x',y' parameters)

Find Q_2' by formal solution prop.

→ get Q_2'. $\quad Q_3 := Q_2 + Q_2'$.

Default: near ff!



should use s, not x.

$s = \frac{x}{x'}$, e.g.:

$$P = \sum a_k(x)(x\partial_x)^k = \sum a_k(x's)(s\partial_s)^k \quad \checkmark$$

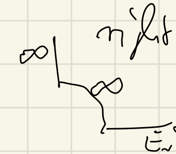
b-op. in s (and y), parameter x'.

Thm (full Parametrix in b-calculus)

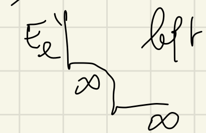
Let $P \in \Psi_b^m(X)$ be elliptic, $\alpha \in \mathbb{R}$, $\alpha \notin \operatorname{Re}\operatorname{spec}_b(P)$.

Then there are $Q_\alpha \in \Psi_b^{-m,\, \mathcal{E}'(\alpha)}$

and $R_{\alpha,r} \in \rho_{ff}^\infty \Psi_b^{-\infty,(\emptyset,\, \mathcal{E}_r'(\alpha))}$



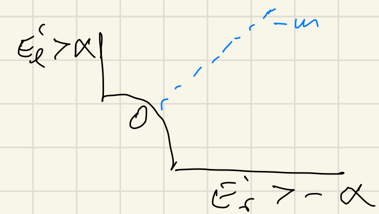
$R_{\alpha,l} \in \rho_{ff}^\infty \Psi_b^{-\infty,(\mathcal{E}_l'(\alpha),\, \emptyset)}$

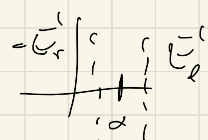


so that

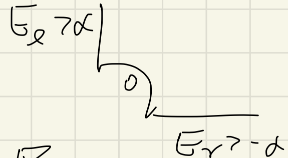
$$PQ_\alpha = I + R_{\alpha,r}$$
$$Q_\alpha P = I + R_{\alpha,l}$$

Note: Q_α:





4.2.5 Consequences

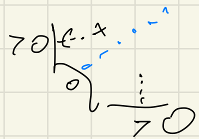


Lemma: $A \in \Psi_b^{m,\mathcal{E}}$, $\alpha \in \mathbb{R}$.

If $E_r > -\alpha$, $E_l > \alpha$ then

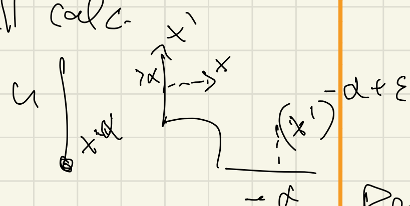
$$A : x^\alpha H_b^{s+m} \to x^\alpha H_b^s \quad \forall s.$$

Proof idea: If $\alpha = 0$:
$m = 0$
$s = 0$



main pt. is $x^\alpha \in L^2$ if $\alpha > 0$.
+ L^2 bddness of small calc.

- general: any α:



Au

$$\alpha + (-\alpha + \varepsilon) > 0$$

$\rightsquigarrow$ in L^2.

Thm: $P \in \Psi_b^m$ elliptic, $\alpha \notin \mathrm{Re}\,\mathrm{spec}_b P$

$$\Rightarrow \quad P : x^\alpha H_b^{s+m} \to x^\alpha H_b^s$$

is Fredholm $\forall s$.

Proof: the parametrix Q_α,

remainders $R_{\alpha,l}$:

$R_{\alpha,l}$ maps $x^\alpha H_b^s \to x^{\alpha+\varepsilon} H_b^\infty$

compact embedding $\dashrightarrow x^\alpha H_b^s$

+ similar for $R_{\alpha,r}$ ✓.

Rem: index depends on α but on s.
larger α $\rightsquigarrow$ kernel gets smaller
cokernel gets larger

Thm (b-elliptic regularity)

$P \in \Psi_b^m$ elliptic. Let $\alpha \in \mathbb{R}$.

Suppose $u \in x^\alpha H_b^{-\infty}$.

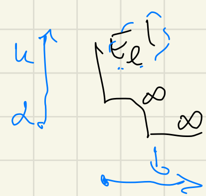
If $Pu = f$, $f \in \mathcal{A}^F$

then $u \in \mathcal{A}^G$, $G = F \,\bar{\cup}\, E_\ell'(\alpha)$.

Proof: First, note $u \in x^\alpha H_b^{-\infty}$

$\Rightarrow Pu \in x^\alpha H_b^{-\infty}$

$\Rightarrow F > \alpha$.



Apply left parametrix Q_α:

$$\underbrace{Q_\alpha f}_{\in \mathcal{A}^{F \bar{\cup} E_\ell'(\alpha)}} = Q_\alpha P u = u + \underbrace{R_{\alpha,\ell} u}_{\in \mathcal{A}^{E_\ell'(\alpha)}}$$

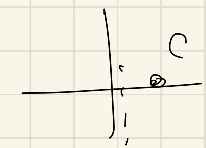
qed

Cor: $u \in x^\alpha H_b^{-\infty}$, $Pu = 0 \Rightarrow u \in \mathcal{A}^{E_\ell'(\alpha)}$.

ex: $P = xD_x - c$ $\quad$ $\operatorname{spec}_b P = \{c\}$

$\hat{P} = z - c$

$\ker P = \{x^c\}$.



$$\boxed{\text{Cor: (Fredholm) inverses are in the calculus.}}$$

ex: in scattering theory: want $(P - z)^{-1}$ has poly Schwartz kernel.