Examples for b-calculus

· **Bessel equation** (ODE)

$$P = x^2\partial_x^2 + x\partial_x + (x^2 - \nu^2) \quad \text{on } \mathbb{R}_+$$

$$= (x\partial_x)^2 + (x^2 - \nu^2) \qquad \nu \in \mathbb{C}$$

b-op. near $x = 0$, b-elliptic

$$\hat{I}_P(z) = z^2 - \nu^2 \Rightarrow \operatorname{spec}_b(P) = \{\pm\nu\}$$

$Pu = 0$: u Bessel functions.

$\rightsquigarrow$ expect solutions (for $\nu > 0$):

$$J_\nu \sim x^\nu \,(1 + c_1 x^{\nu+1} + \dots)$$

$$J_{-\nu} \sim x^{-\nu} \,(\dots \qquad)$$

(obtain by power series Ansatz $u = x^\nu \sum_{j=0}^{\infty} c_j x^j$)

But: If $\nu = n \in \mathbb{N}$ then $J_{-n} = \pm J_n$

In this case, have solutions

$$J_n \sim x^n$$

$$Y_n \sim x^{-n}, \text{ having } x^{n+j} \log x$$

$(j \in \mathbb{N}_0)$

log arises since $n - (-n) \in \mathbb{Z}$.

(if $\nu \in \frac{1}{2} + \mathbb{Z}$ then $\nu - (-\nu) \in \mathbb{Z}$ but still no log's)

$\nu = 0$:

$$J_n \sim 1$$

$$Y_n \sim \log x$$

Cone

$C = (\overset{r}{0,\infty}) \times Y$

Y closed manifold

$g = dr^2 + r^2 h$, h Riem. metric on Y.

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_h$$

near $r=0$: $\Delta = r^{-2}\left[(r\partial_r)^2 + (n-2)r\partial_r + \Delta_h\right]$

near $r=\infty$: $x = \frac{1}{r}$: $\partial_r = -x^2\partial_x$

$$\Delta = (x^2\partial_x)^2 - (n-1)x^3\partial_x + x^2\Delta_h$$
$$= x^2\left[(x\partial_x)^2 - (n-2)x\partial_x + \Delta_h\right]$$

Take $X = [0,\infty] \times Y$ compact.

$\Delta = a \cdot P$, $a(r) = r^{-2}$

P is elliptic b-operator.

Note: can use sep. of variables,

but: b-calculus useful for perturbations of this.

We get from b-calculus:

- solutions of $\Delta u = 0$ (or $\Delta u = f$, $\operatorname{supp} f \subset (0,\infty)\times Y$),

  if u is pol. bdd ($u \in x^\alpha H_b^{-\infty}$)

  then u is phg as $r \to 0$ or $r \to \infty$.

Ex: $Y = (S^{n-1}, h_{std})$: $C = \mathbb{R}^n \setminus 0$.

Liouville thm: polyn. bdd harmonic fcns are polynomials.

- Fredholm between weighted b-Sobolev spaces.

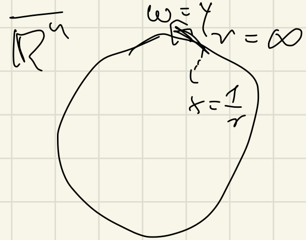
$z \in \mathbb{R}^n$

(for $\mathbb{R}^n$ (at ∞): these are not the standard $H^s(\mathbb{R}^n)$ Sobolev spaces, since ∂_{z_j} are not b-vector fields

In fact:

∂_{z_i} are spans of

$x^2\partial_x, \; x\partial_{y_j}$



Rem: $\Delta + c$: near 0 still reducible to $\mathcal{V}_b$
near ∞ not b

Note: $\Delta u = f \iff Pu = \frac{1}{a}\cdot f$

$\Delta = a \cdot P$

Recall guiding principles:

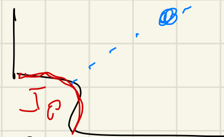
- local product structure
  used for:
  - (approx.) sep. of variables
    ⇝ iterative solution using model problems
  - regularity notions:
    phg, conormal singularities

  embodied in: mwc, p-submfd, b-maps, ...

  blow-up yields l.p.s.
- b-stuff is good: $\mathcal{V}_b = \langle x\partial_x, \partial_y \rangle$
  - as a tool for phg etc.
  - behave well under blow-up
- idea of model problems

model problems

eg b-calculus:



- σ_p model pr. at diagonal (≙ freezing coeff's)

- I_p: model pr. at bd. (≙ ff)

M.P. are simpler to solve than original problem because they have some symmetry:

σ_p: translation invariance

I_p: dilation invariance (in x-variable)

⇝ reduce to calculus „one step simpler"

I_p reduce b-calc. to $\Psi^*(\partial X)$

- separate geometric and analytic aspects

General setting for singular analysis:

- X mfd with corners
- $\mathcal{V} \subset \mathcal{V}_b$ Lie algebra of vector fields.

$\mathcal{V}$ defines $\mathrm{Diff}^*_{\mathcal{V}}$.

Goal: Find $\Psi^*_{\mathcal{V}} \supset \mathrm{Diff}^*_{\mathcal{V}}$ containing parametrices of elliptic elements.

[also: heat calculus ...]

$\mathcal{V}$ = „boundary fibration structure"
(see Kyoto 1990)

Examples: (X = mwb here)

• b: $\mathcal{V} = \mathcal{V}_b = \langle x\partial_x, \partial_y \rangle$ $(\rightsquigarrow {}^bTX)$

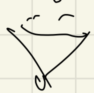
dual: $\langle \frac{dx}{x}, dy \rangle$ $({}^bT^*X)$

b-metric: pos. def. qu. form on ${}^bTX$

e.g. $g = \left(\frac{dx}{x}\right)^2 + dy^2$ $\left(+ \frac{dx}{x} dy\right)$ ...

$\rightsquigarrow \Delta_g \in \mathrm{Diff}_b$.

Geometry: • infinite cyl. ends



• cones near tip (up to leading factor)

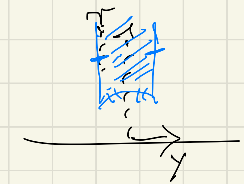
$dx^2 + x^2 dy^2 = x^2 \cdot (\text{b-metric})$.

• cusp: $\mathcal{V}_{cu} = \langle x^2\partial_x, \partial_y \rangle$

ex: 

$dx^2 + x^4 dy^2 = x^4\left[\left(\frac{dx}{x^2}\right)^2 + dy^2\right]$

ex: hyperbolic cusp: $\mathbb{H} / SL(2,\mathbb{Z})$



$g = \frac{dr^2 + dy^2}{r^2}$ $\quad r = \frac{1}{x}$

$= \left(\frac{dx}{x}\right)^2 + (x\,dy)^2$ $\quad y \in S^1$

• sc (scattering): $\mathcal{V}_{sc} = \langle x^2\partial_x, x\partial_y \rangle$

(cone at ∞, $\mathbb{R}^n$ at ∞)

eg $\Delta + c \in \mathrm{Diff}_{sc}$.

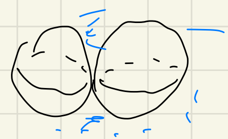
- φ-calculus: $\mathcal{V}_\varphi = \langle x^2\partial_x, x\partial_y, \partial_z \rangle$

ex: 1) $\mathbb{R}^4_{x,y} \times F_z$ F compact

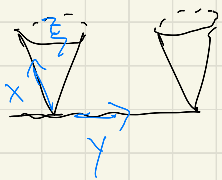
2) loc. symm. spaces $\times S^1$

3) complement of

(F has boundary)



- e (edge) $\mathcal{V}_e = \langle x\partial_x, x\partial_y, \partial_z \rangle$

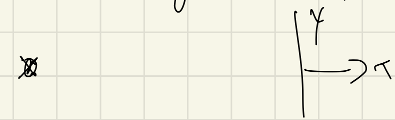


$$dx^2 + x^2 dz^2 + dy^2$$
$$= x^2\left[\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{x}\right)^2 + dz^2\right]$$

- 0 (zero): $\langle x\partial_x, x\partial_y \rangle$

- hyperbolic space at ∞ (conformally compact)

- Boundary value problems:

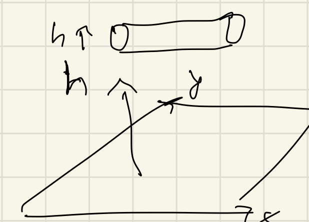


$$dx^2 + dy^2$$
$$= x^2\left[\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{x}\right)^2\right]$$

- parameter dependent problems, e.g.:

$P_h = h^2\partial_x^2 + \partial_y^2$, $h > 0$, $h \to 0$.

$P_h u = f$, $P_h u = \lambda u$.



$\mathcal{V}_{ad} = \langle h\partial_x, \partial_y \rangle$

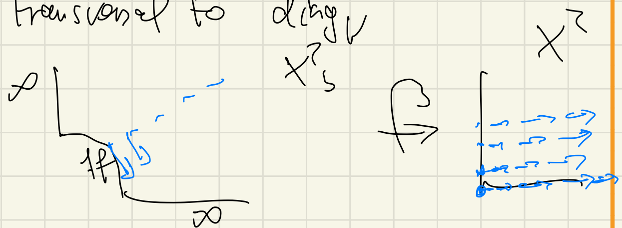
(adiabatic calculus)

[survey DG: u Grasimoles, ...]

General procedure for $(X, \mathcal{V})$

1) Construct $X^2_{\mathcal{V}} \xrightarrow{\beta} X^2$

so that $\mathcal{V}$ lifts from right factor to be transversal to $\mathrm{diag}_{\mathcal{V}}$



2) $\Psi_{\mathcal{V}}^{*} = \{$ distr. on $X^2_{\mathcal{V}}$

conormal wrt $\mathrm{diag}_{\mathcal{V}}$

smoothly up to ff $::$

those face meeting $\mathrm{diag}_{\mathcal{V}}$

∞ order vanishing at other faces $\}$

"small $\mathcal{V}$-calculus"

$> \mathrm{Diff}^{*}_{\mathcal{V}}$

· need $X^3_{\mathcal{V}} \longrightarrow X^3$ (t, b-fibrations to $X^2_{\mathcal{V}}$) to show

$\Psi^{*}_{\mathcal{V}}$ is closed under composition.

3) obstruction to compactness of

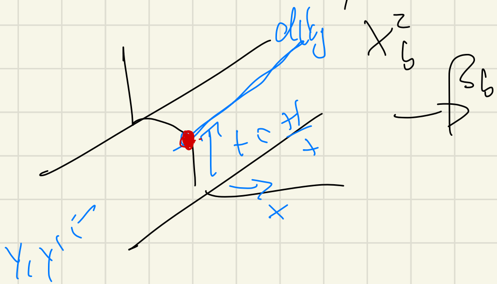
$R \in \Psi_{\mathcal{V}}^{-\infty}$

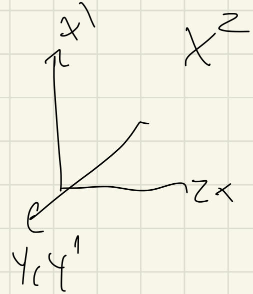
$\rightsquigarrow$ normal operators (e.g. indicial op.)

[model op.]

· may need larger Ψ to invert normal op's and get parametrix with compact errors.

$V_{sc} = \langle x^2\partial_x,\ x\partial_y\rangle$



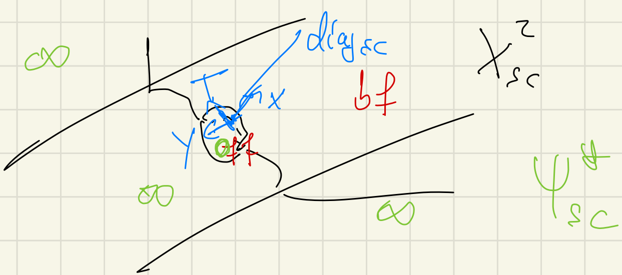


$\beta_b^* \; x^2\partial_x = x t \partial_t$ (t small)

$\approx x\partial_t$ near diag, $t = 1$.

$\beta_b^* \; x\partial_y = x\partial_y$

⇝ blow-up: $t = 1,\ y = y',\ x = 0$.



$T = \frac{t-1}{x}$, $Y = \frac{y-y'}{x}$, x, y'

⇝ Def. of Ψ^*_{sc}.

composition: ....

- Obstruction to compactness of $R \in \Psi^{-\infty}_{sc}$ is $(K_R')|_{ff}$.

⇝ interpret $(K_R')|_{ff}$ as operator.

$P(x^2\partial_x, x\partial_y; x, y) \rightsquigarrow$

$N_p = P(\partial_T, \partial_Y; 0, y_0)$

family of ops param. by $y_0 \in \partial X$, constant coeff. in T, Y.

$\hat{N}_p = P(\tau, \eta; 0, y_0)$.

Def: $P \in \Psi^{\infty}_{sc}$ is <u>fully elliptic</u>
if it is sc-elliptic
(σ_P (at dicy) is inv.)
and $\hat{N}_P(i\tau, \eta; 0, y_0) \neq 0$ $\forall \tau, \eta, y_0$.

Thm: P fully elliptic $\Rightarrow$
- $\exists$ parametrix, error $\in \rho_{ff}^{\infty} \Psi^{-\infty}_{sc}$
  [it in small calc.]
- Fredholm
- regularity

ex: Δ on $\mathbb{R}^n$ or C $\quad \rho = \frac{1}{x}$

$$(x^2\partial_x)^2 + \underbrace{c \cdot x^3\partial_x}_{x \cdot x^2\partial_x} + x^2\Delta_y$$

$$N(\Delta) = \partial_T^2 + 0 + \Delta_y$$

$$\hat{N}(\Delta) = -\tau^2 - |\eta|^2.$$

not fully elliptic ($=0$ at $(\tau,\eta)=0$).

but $\hat{N}(\Delta - 1) = -\tau^2 - |\eta|^2 - 1 \neq 0$ $\forall \tau, \eta$

$\Rightarrow \Delta - 1$ <u>is</u> fully elliptic. $z \in \mathbb{R}^n$

Explicitly: kernel of inverse is $K(z - z')$

$$K(z) = \int e^{iz\xi} \frac{1}{|\xi|^2 + 1} \, d\xi$$

$K(z) = O(\langle z \rangle^{-\infty})$ ~ in Ψ^{-2}_{sc}.