Prof. Dr. Daniel Grieser 29.10.2020

First problem set 'Singular Analysis'

We will talk about these on Wednesday, November 4. Please think about them and present your thoughts. If you want to write up and turn in solutions please do, I will read them!

1. (Extension and chain rule) Let $\Omega \subset \mathbb{R}^n_k$ be relatively open.
   (a) Prove the finite regularity extension theorem: Let $r \in \mathbb{N}_0$. If $u : \Omega \to \mathbb{C}$ is C^r (in the sense that $u_{|\Omega^\circ} \in C^r(\Omega^\circ)$ and derivatives up to order r extend continuously to the boundary) then there is an open extension $\Omega' \subset \mathbb{R}^n$ of Ω and a C^r extension of u to Ω'.
   Hint: First do $n = k = 1$. Reduce to the case $u(0) = \cdots = u^{(r-1)}(0) = 0$, then extend as even/odd function depending on the parity of r. Induct on r.
   (b) Conclude that the chain rule continues to hold up to the boundary: if $\Omega' \subset \mathbb{R}^m_l$ is relatively open and $F : \Omega \to \Omega'$, $u : \Omega' \to \mathbb{C}$ are C^1 then $u \circ F$ is C^1 and $d(u \circ F)_p = du_{F(p)} \circ dF_p$ for all $p \in \Omega$.
   Why does this not follow trivially by 'extension from the interior' from the classical chain rule?
   (c) Conclude that for a diffeomorphism $F : \Omega \to \Omega'$ the differential dF_p is an isomorphism for all $p \in \Omega$.

2. (Examples and non-examples of manifolds with corners)
   A subset $X \subset \mathbb{R}^N$ is a (weak) submanifold with corners if it can be covered by charts as in the definition of a (weak) manifold with corners which are immersions into $\mathbb{R}^N$.
   (a) Show that the three-sided infinite pyramid $P_3 = \{\sum_{i=1}^3 t_i w_i \,:\, t_i \geq 0 \,\forall i\}$, where $w_1, w_2, w_3 \in \mathbb{R}^3$ are linearly independent, is a submanifold with corners of $\mathbb{R}^3$.
   (b) Show that the four-sided infinite pyramid $P_4 = \{\sum_{i=1}^4 t_i w_i : t_i \geq 0 \,\forall i\}$, where $w_1, \dots, w_4$ are the vectors $(\pm 1, \pm 1, 1) \in \mathbb{R}^3$, is not a submanifold with corners of $\mathbb{R}^3$.
   (c) Show that the set $\{(x, y) \in \mathbb{R}^2 : |y| \leq x^2\}$ is not a submanifold with corners of $\mathbb{R}^2$.

3. (p-submanifolds and b-maps)
   (a) Show that the half parabola $\{(x, y) : x = y^2,\ y \geq 0\}$ is not a p-submanifold of the half plane $\mathbb{R}_+ \times \mathbb{R}$.
   (b) Let X be a weak mwc and $Y \subset X$, and consider the inclusion map $i : Y \hookrightarrow X$.
   Show that if Y is a p-submanifold then i is a b-map.
   (c) Let X be a weak mwc. Show that the diagonal embedding $X \to X \times X$, $p \mapsto (p, p)$, is a b-map, and that its image is a p-submanifold iff $\partial X = \emptyset$.
   (This shows that the converse in the previous problem does not hold.)

4. (b-maps) Let X, Y be manifold with corners and $F : X \to Y$ smooth.
   (a) What does it mean for F to be a b-map if $\partial X = \emptyset$? What if $\partial Y = \emptyset$?
   (b) In the case $X = Y = \mathbb{R}_+$ find a simple characterization of b-maps.

5. (Important example of a b-map)
   Consider the b-map
   $$F : \mathbb{R}^2_+ \to \mathbb{R}^2_+, \quad F(x, y) = (xy, y)\,.$$
   So if (u, v) are coordinates on the range space then $u = xy$, $v = y$ informally.

(a) Describe what the map does geometrically.[1] Check that F is a diffeomorphism $(0, \infty)^2 \to (0, \infty)^2$.

(b) Calculate $F_*(x\partial_x)$ and $F_*(y\partial_y)$.[2]

(c) Show that any b-vector field W on the range $\mathbb{R}^2_+$ lifts under F to a b-vector field on the domain $\mathbb{R}^2_+$ of F. That is, given $W \in \mathcal{V}_b(\mathbb{R}^2_+)$ there is a unique $V \in \mathcal{V}_b(\mathbb{R}^2_+)$ so that $F_*V = W$.

6. (b-vector fields) Recall that a vector field V on a manifold X defines a map (action)

$$C^\infty(X) \to C^\infty(X), \;\; f \mapsto Vf$$

where Vf is the derivative of f in the direction of V. In coordinates, if $V = \sum_i a_i \partial_{z_i}$ then $Vf = \sum a_i \frac{\partial f}{\partial z_i}$. This clearly remains true for X a mwc and $V \in \mathcal{V}(X)$.

(a) Find a characterization of b-vector fields in terms of their action on functions.

(b) Recall that the Lie bracket of $V, W \in \mathcal{V}(X)$ is defined as the vector field $[V, W] \in \mathcal{V}(X)$ satisfying $[V, W]f = V(Wf) - W(Vf)$ for any $f \in C^\infty(X)$. Show that

$$V, W \in \mathcal{V}_b(X) \Rightarrow [V, W] \in \mathcal{V}_b(X)$$

using (a) or otherwise (e.g. calculation in coordinates).

[1] One idea is to consider images of lines $x = \text{const}$ and $y = \text{const}$, in particular of the boundary hypersurfaces $x = 0$, $y = 0$.

[2] Recall that the push-forward F_*V of a vector field V under a map F is defined as $(F_*V)_{F(p)} = dF_p(V_p)$, if this is well-defined. So part of the claim is that here the push-forwards are well-defined.