Prof. Dr. Daniel Grieser 11.12.2020

Third problem set 'Singular Analysis'
(Mellin transform, regularized integrals and push-forward theorem)

We will talk about these on Wednesday, December 16. Please think about them and present your thoughts. If you want to write up and turn in solutions please do, I will read them!

1. **(Change of variables in regularized integrals)**

   (a) Let $u \in C_0^\infty(\mathbb{R}_+)$. For $c > 0$ define $u_c(x) = u(\frac{x}{c})$. Show that
$$⨍_0^\infty u_c(x)\,\frac{dx}{x} = ⨍_0^\infty u(x)\,\frac{dx}{x} \;+\; u(0)\log c \tag{1}$$
   and for $\alpha > 0$
$$⨍_0^\infty u(x^\alpha)\,\frac{dx}{x} = \frac{1}{\alpha}⨍_0^\infty u(x)\,\frac{dx}{x}$$
   Remark: Thus, substitution $\frac{x}{c} \mapsto x$ produces an extra term (while $x^\alpha \mapsto x$ does not).If u is polyhomogeous then there will be an additional term in (1) for each $x^0 \log^k x$ term in the asymptotics of u.

   (b) Show that
$$⨍_0^\infty e^{-ax}\,\frac{dx}{x} = -\gamma - \log a$$
   for $a > 0$, where $\gamma = -\Gamma'(1) = -\int_0^\infty e^{-x}\log x\,dx = 0.57721\ldots$ is the Euler-Mascheroni constant.

   (c) **(Regularized integral on manifolds with boundary)**
   Let X be a manifold with boundary and ρ a boundary defining function. For an index set E and $\alpha \in \mathcal{A}_0^E(X, |\Omega_{\mathrm{b}}|)$ define
$$⨍_{X,\rho} \alpha := \text{ the } s^0 \text{ term in the Laurent expansion of } (\mathcal{M}_\rho\alpha)(s) := \int_X \rho^s \alpha\,.$$
   Convince yourself that the things done in the lecture (for $X = \mathbb{R}_+$, $\rho = x$) carry over to this case, that is:
   - $\mathcal{M}_\rho\alpha$ is defined in some right half plane and has a meromorphic continuation to $\mathbb{C}$, so that the definition makes sense.
   - The function $v(\varepsilon) = \int_{\rho>\varepsilon}\alpha$ (for $\varepsilon > 0$) is polyhomogeneous on $\mathbb{R}_+$ and
$$⨍_{X,\rho}\alpha = \mathrm{FP}_{\varepsilon=0}\int_{\rho>\varepsilon}\alpha$$
     (where FP – finite part – denotes the coefficient of ε^0 in the polyhomogeneous expansion).

   Important: $⨍_X \alpha$ is not defined invariantly unless $\inf E > 0$. Explain why this follows from (a).
   So $⨍_{X,\rho}\alpha$ depends on the choice of boundary defining function. Show ($m \in \mathbb{N}_0$):
$$\inf E > -m\,, \quad \rho, \tilde\rho \text{ agree to order } m \text{ at } \partial X \implies ⨍_{X,\rho}\alpha = ⨍_{X,\tilde\rho}\alpha$$
   The condition means $\tilde\rho = \rho + O(\rho^{m+1})$. For $m = 1$, e.g. smooth b-densities, the condition is $d\rho = d\tilde\rho$ at ∂X.

2. **(Polyhomogeneity at ∞)**

   (a) Show: If $u \in L^1_{\mathrm{loc}}(\mathbb{R}_>)$ vanishes near 0 and satisfies $u(x) = O(x^M)$ then $\mathcal{M}u(s)$ is defined and holomorphic in the left half plane $\operatorname{Re} s < -M$.

   (b) We make $[0,\infty]$ into a manifold with corners by taking $x \mapsto x^{-1}$ as local chart near ∞. This means that a function is $\mathcal{E}$-smooth for an index family $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_\infty\}$ if $x \mapsto u(x)$ is $\mathcal{E}_0$-smooth near $x = 0$ and $z \mapsto u(z^{-1})$ is $\mathcal{E}_\infty$-smooth near $z = 0$.
   For $u \in \mathcal{A}^{\mathcal{E}}([0,\infty])$ define
   $$\mathcal{M}u = \mathcal{M}(u\chi_{(0,C]}) + \mathcal{M}(u\chi_{[C,\infty)})$$
   for $C > 0$, where χ are the characteristic functions and $\mathcal{M}$ on the right denotes the meromorphic continuations to all of $\mathbb{C}$.
   Show that $\mathcal{M}u$ is independent of the choice of C (and equal to the analogous expression where smooth cutoffs are used instead of χ) and has a meromorphic function with poles determined by $\mathcal{E}_0 \cup (-\mathcal{E}_\infty)$ as in the theorem of the lecture.
   Remark: This is *not* a special case of 1(c).

   (c) Show that
   $$\mathcal{M}(x^z \log^k x) \equiv 0$$
   for any $z \in \mathbb{C}$, $k \in \mathbb{N}_0$.

3. **(Resolving a b-map to a b-fibration, and polyhomogeneity)** Consider the b-map
   $$f : \mathbb{R}^3_+ \to \mathbb{R}^2_+\,, \quad f(x,y,z) = (xy, xz)$$

   (a) Show that f is not a b-fibration.

   (b) Show that f can be 'resolved' to a b-fibration by blowing up $(0,0)$ in the range and the x-axis in the domain. That is, with $X = [\mathbb{R}^3_+, \mathbb{R}_+ \times \{(0,0)\}]$, $Y = [\mathbb{R}^2_+, (0,0)]$ there is a lift $\tilde{f}$ making the diagram
   $$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\beta_Y"] \\ \mathbb{R}^3_+ \arrow[r, "f"] & \mathbb{R}^2_+ \end{tikzcd}$$
   commute, and $\tilde{f}$ is a b-fibration.

   (c) Conclude that if $u \in C_0^\infty(\mathbb{R}^3_+)$ (or polyhomogeneous) and
   $$v(t,t') = \int_0^\infty u(x, \frac{t}{x}, \frac{t'}{x})\,\frac{dx}{x}$$
   then $\beta_Y^* v$ is polyhomogeneous. Find its index sets. Show that v itself is not polyhomogeneous in general.

4. **(Some formulas involving logarithms)** Formulas for phg functions having logarithmic terms often become simpler when written in terms of
   $$L_k(x) = \frac{\log^k x^{-1}}{k!}\,, \quad k \in \mathbb{N}_0,$$
   instead of $\log^k x$. Note that the generating function of the $L_k(x)$ is $\sum_{k=0}^\infty L_k(x) r^k = x^{-r}$. Using this, or otherwise, show:

   (a) $-x\partial_x L_k = L_{k-1}$ (where $L_{-1} := 0$)

   (b) $L_k(xy) = \sum_{m=0}^k L_m(x) L_{k-m}(y)$

   (c) Let $u \in \mathcal{A}_0^E(\mathbb{R}_+)$ for an index set E and $u(x) \sim_{x\to 0} \sum_{(z,k)\in E} a_{z,k}\, x^z L_k(x)$. Then the terms of the Laurent series of $(\mathcal{M}u)(s)$ at $s = -z$ are:

- $a_{z,k}(s+z)^{-k-1}$ for $k \geq 0$
- $I_{z,m}(s+z)^m$ for $m \geq 0$ where

$$I_{z,m} = \fint_0^\infty u(x)x^{-z}L_m(x^{-1})\,\frac{dx}{x} \tag{2}$$

(d) **(General coefficient formula)**
Let E, F be index sets, $u \in \mathcal{A}_0^{E,F}(\mathbb{R}_+^2)$ and

$$\begin{aligned} u(x,y) &\overset{x\to 0}{\sim} \sum_{(z,k)\in E} a_{z,k}(y)x^z L_k(x) \\ u(x,y) &\overset{y\to 0}{\sim} \sum_{(w,l)\in F} b_{w,l}(x)y^w L_l(y) \end{aligned} \tag{3}$$

with corner expansion[1]

$$u(x,y) \overset{x,y\to 0}{\sim} \sum_{(z,k)\in E}\sum_{(w,l)\in F} c_{z,w}^{k,l}\, x^z L_k(x)\ y^w L_l(y) \tag{4}$$

Then for the push-forward under the map $f : \mathbb{R}_+^2 \to \mathbb{R}_+,\ (x,y) \mapsto xy$,

$$v(t)\frac{dt}{t} = f_*(u(x,y)\frac{dx}{x}\frac{dy}{y}), \quad \text{that is} \quad v(t) = \int_0^\infty u(x,\frac{t}{x})\frac{dx}{x} \tag{5}$$

we have

$$\begin{aligned} v(t) \sim & \sum_{\substack{z,k,l:\\(z,k)\in E,(z,l)\in F}} c_{zz}^{kl}\ t^z L_{k+l+1}(t) \\ & + \sum_{(z,k)\in E}\sum_{m=0}^{k}\left(\fint_0^\infty a_{z,k}(y)\,y^{-z}L_m(y^{-1})\,\frac{dy}{y}\right)\ t^z L_{k-m}(t) \\ & + \dots \end{aligned} \tag{6}$$

where the dots mean the same as the previous term, with E, y, a replaced by F, x, b respectively.
Can you find a better (less messy) way to write this?

5. What does the push-forward theorem yield for the map $f : \mathbb{R}_+^3 \to \mathbb{R}_+$, $f(x,y,z) = xyz$? Find coefficient formulas for $f_*\mu$ where $\mu = u(x,y,z)\ \frac{dx}{x}\ \frac{dy}{y}\ \frac{dz}{z}$ with u smooth.

6. Let $\sigma \in C^\infty(\mathbb{R}_+^2)$, with $\sigma(x,\zeta) = 0$ for $x > C$ and having a polyhomogeneous expansion as $\zeta \to \infty$, uniformly in x. Show that

$$v(z) = \int_0^\infty \sigma(x, xz)\,dx$$

has a polyhomogeneous asymptotic expansion as $z \to \infty$. (And find it, if you really like calculating.)

[1] This is short notation for $a_{z,k}(y) \overset{y\to 0}{\sim} \sum_{(w,l)\in F} c_{z,w}^{k,l}\, y^w L_l(y)$ and a similar expansion for $b_{w,l}(x)$ as $x \to 0$, with the same coefficients $c_{z,w}^{k,l}$.