2.4 Vector fields and b-vector fields

Vector fields are central to both analysis and geometry: in analysis they may be considered as first order partial differential operators, so they are the basic building blocks of all linear partial differential operators. In geometry they can be integrated to provide flows which are useful for many geometric constructions.

Recall that a **vector field** V on a manifold X is an assignment of an element $V_p \in T_pX$ to each $p \in X$ which is smooth in p. A local coordinate system $z = (z_1, \dots, z_n)$ on $U \subset X$ defines a basis $\partial_{z_1}, \dots, \partial_{z_n}$ of T_pX for each $p \in U$, so we can write $V = \sum_{j=1}^n a_j \partial_{z_j}$ on U, with functions a_j on U. Smoothness of V means by definition that the a_j are smooth. Also recall that the **integral curve** of V through $p \in X$ is the curve $\gamma_p : I_p \to X$, defined on the maximal open interval $I_p \subset \mathbb{R}$ containing 0, so that

$$\gamma_p(0) = p\,, \quad \dot\gamma_p(t) = V_{\gamma_p(t)} \text{ for all } t \in I_p\,.$$

If X is compact then $I_p = \mathbb{R}$ for all p. The **flow** of V combines all integral curves: it is the map $\Phi : X \times I \to X, (p,t) \mapsto \gamma_p(t)$ (supposing $I \subset I_p$ for all p). We also write $\Phi_t(p) = \Phi(p,t)$, so $\Phi_t : X \to X$ for $t \in I$. [6]

The definition of smooth vector field extends verbatim to manifolds with corners X. We denote

$$\mathcal{V}(X) = \{\text{smooth vector fields on } X\}\,.$$

Recall that, by definition, a manifold with corners has local product structure near any boundary point – one of our guiding principles. It will be important to know that such a product structure exists globally near any boundary face. Vector fields are a useful tool for proving this.

Proposition 2.4.1 (Product neighborhoods of faces)**.** *Let X be a manifold with corners and F a compact face of X of codimension k. Then there is a diffeomorphism*

$$U \to F \times [0,1)^k\,, \quad U \textit{ an open neighborhood of } F$$

so that each $p \in F \subset U$ is mapped to $(p,0)$.

Proof. We assume $k = 1$, the general case then follows by induction. For any $p \in F$ choose local coordinates x, y on a neighborhood U_p, so that $F = \{x_1 = 0\}$ in U_p. Let $V^{(p)} = \partial_{x_1}$ on U_p. Note that $V^{(p)}$ is pointing strictly inward with respect to F and is tangent to all other boundary hypersurfaces. The open cover $(U_p)_{p \in F}$ of F has a finite subcover, and using a partition of unity we obtain a vector field V with the same properties defined in a neighborhood of F. The flow of V starting at F is defined up to some positive time $\varepsilon > 0$, and then the flow of εV defines the (inverse of the) desired diffeomorphism. $\square$

Those vector fields which are tangent to the boundary play a central role in singular analysis.

[6] Sometimes the map Φ_t is denoted e^{tV}.

Definition 2.4.2. *A **b-vector field** on a manifold with corners X is a smooth vector field on X which at each $p \in \partial X$ is tangent to all boundary hypersurfaces containing p. We denote*

$$\mathcal{V}_{\mathrm{b}}(X) = \{\textit{b-vector fields on } X\}\,.$$

Lemma 2.4.3. *A smooth vector field $V \in \mathcal{V}(X)$ is a b-vector field if and only if in any local coordinate system (x, y) it has the form*

$$V = \sum_{i=1}^{k} a_i\, x_i \partial_{x_i} + \sum_{j=1}^{n-k} b_j\, \partial_{y_j}\,, \quad a_i, b_j \ \textit{smooth}, \tag{2.8}$$

and the a_i, b_j are uniquely determined by V.

Thus, the ∂_{x_i} always occur in the combination $x_i \partial_{x_i}$.

Proof. Let V be a b-vector field. Since V is smooth, it can be written $V = \sum_{i=1}^{k} A_i\, \partial_{x_i} + \sum_{j=1}^{n-k} b_j\, \partial_{y_j}$ with A_i, b_j smooth. If $p = (x, y)$ is in the coordinate patch and its x_i-coordinate is zero then it lies on the boundary hypersurface $H_i = \{x_i = 0\}$, so V is tangent to H_i at p, so $A_i(p) = 0$. Therefore, we have $A_i(x, y) = 0$ whenever $x_i = 0$. Taylor's theorem implies that $A_i = x_i a_i$ with a_i smooth, so V has the form (2.8).

Conversely, each vector field of this form is clearly a b-vector field, and uniqueness is obvious. □

2.5 Blow up

2.6 The b-tangent bundle

This section is a bit abstract and can be skipped at first reading. However, it will be needed when we talk about symbols of (pseudo-)differential operators, at the latest.

Recall that the tangent bundle TX over a manifold with corners is the vector bundle whose fibre over $p \in X$ is T_pX, the tangent space at p. By definition a vector field V on X is a section of TX:

$$\mathcal{V}(X) = C^\infty(X, TX)\,.$$

In coordinates (x, y) a local basis of TX is $\partial_{x_1}, \ldots, \partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_{n-k}}$, that is, the smooth vector fields on the coordinate patch are the expressions

$$V = \sum_{i=1}^{k} a_i\, \partial_{x_i} + \sum_{j=1}^{n-k} b_j\, \partial_{y_j}\,, \quad a_i, b_j \ \text{smooth}, \tag{2.9}$$

and the a_i, b_j are uniquely determined by V. The fact that local bases exist is expressed by saying that $\mathcal{V}(X)$ is a locally free $C^\infty(X)$-module (of rank n).

Rather than starting with TX and defining $\mathcal{V}(X)$ from it, we can start with $\mathcal{V}(X)$ and define T_pX by

$$T_pX = \mathcal{V}(X)/I_p\mathcal{V}(X)\,, \quad I_p = \{u \in C^\infty(X,\mathbb{R}) :\ u(p) = 0\}\,. \tag{2.10}$$

This is analogous to thinking of the value at p of a function $u \in C^\infty(X,\mathbb{R})$ as its equivalence class in $C^\infty(X,\mathbb{R})/I_p$.[7] In this sense the equivalence class of (2.9) is $\sum_{i=1}^k a_i(p)\,\partial_{x_i} + \sum_{j=1}^{n-k} b_j(p)\,\partial_{y_j}$. The equivalence classes of the ∂_{x_i}, ∂_{y_j} form a basis of T_pX.

The Serre-Swan Theorem says that if we define T_pX by (2.10) then the T_pX fit together naturally to form a vector bundle, whose space of sections is naturally $\mathcal{V}(X)$; also, the analogous statement holds for any locally free $C^\infty(X)$-module in place of $\mathcal{V}(X)$.[8]

We now apply this to $\mathcal{V}_\mathrm{b}(X)$ in place of $\mathcal{V}(X)$. By Lemma 2.4.3 $\mathcal{V}_\mathrm{b}(X)$ is a locally free $C^\infty(X)$-module, with local basis

$$x_1\partial_{x_1}, \dots, x_k\partial_{x_k},\ \partial_{y_1}, \dots, \partial_{y_{n-k}}\,.$$

Definition 2.6.1. *Let X be a manifold with corners. The **b-tangent bundle** of X, denoted ${}^bTX$, is the vector bundle over X whose space of sections is $\mathcal{V}_\mathrm{b}(X)$, in the sense of the Serre-Swan theorem as explained above.*

This is a vector bundle of rank $n = \dim X$. In coordinates (x,y)

$$x_1\partial_{x_1}, \dots, x_k\partial_{x_k},\ \partial_{y_1}, \dots, \partial_{y_{n-k}} \text{ are a basis of } {}^bT_pX\,.$$

It is important to understand that this holds at all points p in the coordinate patch, so $x_i\partial_{x_i}$ is non-zero *as an element of* ${}^bT_pX$ even if $x_i = 0$. It helps to remember ${}^bT_pX = \mathcal{V}_\mathrm{b}(X)/I_p\mathcal{V}_\mathrm{b}(X)$ to understand this.

Since a b-vector field is also just a smooth vector field, there is natural map $\mathcal{V}_\mathrm{b}(X) \to \mathcal{V}(X)$, and taking quotients we obtain a vector bundle map

$$\iota : {}^bTX \to TX$$

which is sometimes called the **anchor map** for ${}^bTX$. It reinterprets a b-tangent vector as a 'usual' tangent vector. If p is an interior point then $\iota_p : {}^bT_pX \to T_pX$ is an isomorphism. However, if p is a boundary point then it is not: if (x,y) are coordinates centered at p (so all $x_i = 0$ at p) then

$$\ker \iota_p = \mathrm{span}\{x_1\partial_{x_1}, \dots, x_k\partial_{x_k}\}\,.$$

This space is called the **b-normal space** at p.

[7] The identification of the equivalence class with the function value is given by the evaluation homomorphism $C^\infty(X,\mathbb{R}) \to \mathbb{R}$, $u \mapsto u(p)$. Since this is surjective and has kernel I_p, it defines an isomorphism $C^\infty(X,\mathbb{R})/I_p \to \mathbb{R}$.

[8] This is actually not hard to prove. Exercise!