Description of the EXCEL files for the simulation of Partition-of-Unity copulas

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The ZIP-file **PU-Copulas EXCEL.zip** contains the following 34 EXCEL files:

Binomial-Copula.xls
Binomial-Copula_diag.xls
Binomial-Copula_rook.xls
Binomial-Copula_UF.xls
Binomial-Copula_LF.xls
Binomial-Copula_SingMix.xls
Binomial-Copula_triangle.xls
Binomial-Copula_triangle_diag.xls
Binomial-Copula_triangle_antidiag.xls

NB-Copula.xls
NB-Copula_diag.xls
NB-Copula_rook.xls
NB-Copula_UF.xls
NB-Copula_LF.xls
NB-Copula_SingMix.xls
NB-Copula_triangle.xls
NB-Copula_triangle_diag.xls
NB-Copula_triangle_antidiag.xls

Poisson-Copula.xls
Poisson-Copula_diag.xls
Poisson-Copula_rook.xls
Poisson-Copula_UF.xls
Poisson-Copula_LF.xls
Poisson-Copula_SingMix.xls
Poisson-Copula_triangle.xls
Poisson-Copula_triangle_diag.xls
Poisson-Copula_triangle_antidiag.xls

Gamma-Copula_rook.xls
Gamma-Copula_UF.xls
Gamma-Copula_LF.xls
Gamma-Copula_SingMix.xls
Gamma-Copula_triangle.xls
Gamma-Copula_triangle_diag.xls
Gamma-Copula_triangle_antidiag.xls

The files **Binomial-Copula.xls, Binomial-Copula_diag.xls, NB-Copula.xls, NB-Copula_diag.xls, Poisson-Copula.xls** and **Poisson-Copula_diag.xls** refer to the paper

[1] New copulas based on general partitions-of-unity and their applications to risk management (with H. Awoumlac Tsatedem, A. Mändle and C. Girschig). Dependence Modeling (2016), 123 – 140. http://dx.doi.org/10.1515/demo-2016-0006

They implement the simulation of Binomial, Negative Binomial and Poisson copulas where the copula driver is a generalized Bernstein copula for arbitrary choices of the constant *K*, see Appendix. The other files refer to the papers

[2] New copulas based on general partitions-of-unity and their applications to risk management (part II) (with A. Mändle and O. Ragulina). Dependence Modeling (2017), 246 – 255. https://doi.org/10.1515/demo-2017-0014

[3] New copulas based on general partitions-of-unity – the continuous case (part III) (with A. Mändle, C. Girschig and O. Ragulina). Dependence Modeling (2019), 181 – 201. https://doi.org/10.1515/demo-2019-0009

They implement the simulation of Partition-of-Unity copulas (PU-copulas) with special drivers, which are the rook copula driver, the upper and lower Fréchet-bound drivers (UF and LF drivers), the triangle and singular mixture copula drivers (see Appendix). In all EXCEL files, the data from the paper

[4] From Bernstein polynomials to Bernstein copulas (with C. Cottin), Journal of Applied Functional Analysis (2014), 277 – 288 www.staff.uni-oldenburg.de/dietmar.pfeifer/Publ/P107.pdf

are considered, as in the papers [1] , [2] and [3] cited above.

All EXCEL files contain the worksheets *data*, *emp cop* and *simulation*. In *data*, the data from the paper [4] are listed together with the corresponding ranks, which are the basis for the empirical copula presented in *emp cop*. In *simulation*, 5000 simulated random vectors with the specified copulas are presented, together with a graphical representation of the empirical copula (white circles). Here also the parameters for the copulas can be individually chosen in the first line. For comparison, the generalized Bernstein copula is also displayed in the EXEL files for the Negative Binomial and Poisson copulas.

For the special driver-dependent copulas, there is an additional worksheet called *rnd*. This sheet is necessary in order to simulate the corresponding drivers.

The EXCEL sheets can be easily used to simulate PU-copulas for alternative data sources. For this purpose, the new data should be pasted in data with an update of column A. Afterwards, the worksheets *emp cop* and *simulation* have to be adjusted. In particular, cell J1 in *emp cop* has to contain the number of data used. An extension to data sets of higher dimensions is also possible.

Note that the EXCEL files were built with a German version of EXCEL under WINDOWS. This means that you eventually have to change the files to your country settings. The execution mode is "manual", i.e. you have to rerun the files with the key F9.

Appendix: generalized Bernstein and other finite Partition-of-Unity copulas

Note that all usual finite Partition-of-Unity copulas such as the rook and Bernstein copulas can be generalized to further finite partition-of-unity copulas in the following way.

"Simple" finite Partition-of-Unity copulas are based on a family of functions $\{\phi(m,k,\bullet) | 0 \le k \le m-1,\ m \in \mathbb{N}\}$ (called partition of unity) with the following properties:

$$\int_0^1 \phi(m,k,u)\,du = \frac{1}{m} \quad \text{for } k = 0,\cdots,m-1$$

$$\sum_{k=0}^{m-1} \phi(m,k,\bullet) = 1 \quad \text{for } m \in \mathbb{N}.$$

Now for $d \in \mathbb{N}$ let $\mathbf{U} = (U_1,\cdots,U_d)$ be a random vector whose components U_i follow a discrete uniform distribution over the set $T := \{0,1,\cdots,m-1\}$ with $m \in \mathbb{N}$ for $i = 1,\cdots,d$. Let further denote

$$p_m(k_1,\cdots,k_d) := P\left(\bigcap_{i=1}^{d} \{U_i = k_i\}\right) \text{ for all } (k_1,\cdots,k_d) \in T^d$$

the joint probabilities of $\mathbf{U}$ (forming a *d*-dimensional contingency table) and

$$I_{k_1,\cdots,k_d} := \bigtimes_{j=1}^{d} \left(\frac{k_j}{m}, \frac{k_j+1}{m}\right] \text{ for } (k_1,\cdots,k_d) \in T^d$$

giving all possible subcubes of $(0,1]^d$ with edge length $1/m$. The pertaining Partition-of-Unity copula density c_m is then given by

$$c_m := m^d \sum_{k_1=0}^{m-1} \cdots \sum_{k_d=0}^{m-1} p_m(k_1,\cdots,k_d)\, \mathbb{1}_{I_{k_1,\cdots,k_d}}.$$

Remark: Rook, Bernstein and Triangle copulas are special cases of a partition of unity copula where

Rook copula: $\phi(m,k,u) = \mathbb{1}_{\left[\frac{k}{m},\frac{k+1}{m}\right]}(u)$

Bernstein copula: $\phi(m,k,u) = \binom{m-1}{k} u^k (1-u)^{m-1-k}$

Triangle copula: $\phi(m,k,u) = \max\left(0; 1 - \left| um - k - \frac{1}{2} \right|\right) + \max\left(0; \frac{1}{2} - mu^{\max(0;1-k)}(1-u)^{\max(0;2-m+k)}\right)$

for $0 \le u \le 1$ and $0 \le k \le m-1$, $m \in \mathbb{N}$.

Now every partition of unity $\{\phi(m,k,\bullet) | 0 \le k \le m-1,\ m \in \mathbb{N}\}$ generates a new (sub-)partition of unity $\{\phi_K(m,k,\bullet) | 0 \le k \le m-1,\ m \in \mathbb{N}\}$ for any fixed $K \in \mathbb{N}$ via

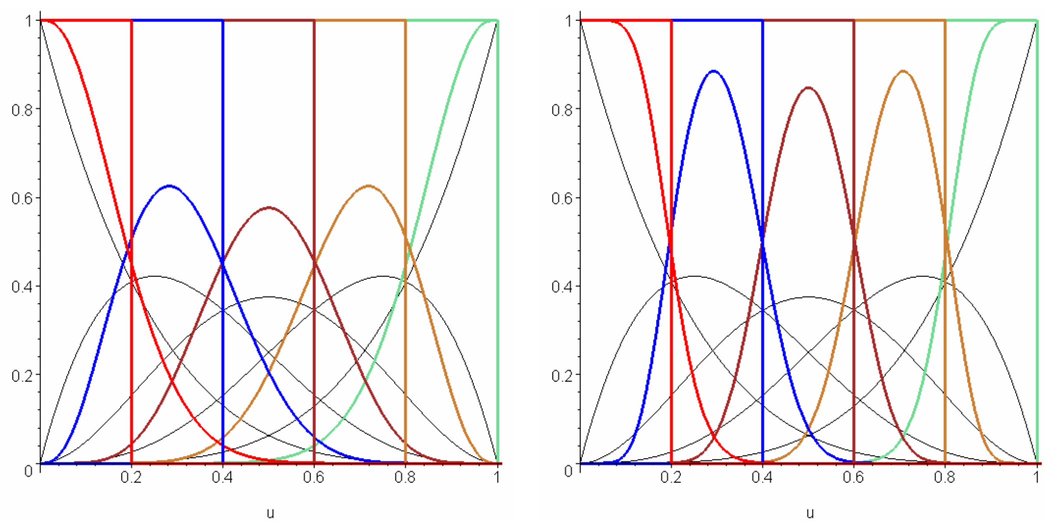
$$\phi_K(m,k,\bullet) := \sum_{j=0}^{K-1} \phi(K \cdot m, K \cdot k + j, \bullet) \text{ for } k = 0, \cdots, m-1$$

since

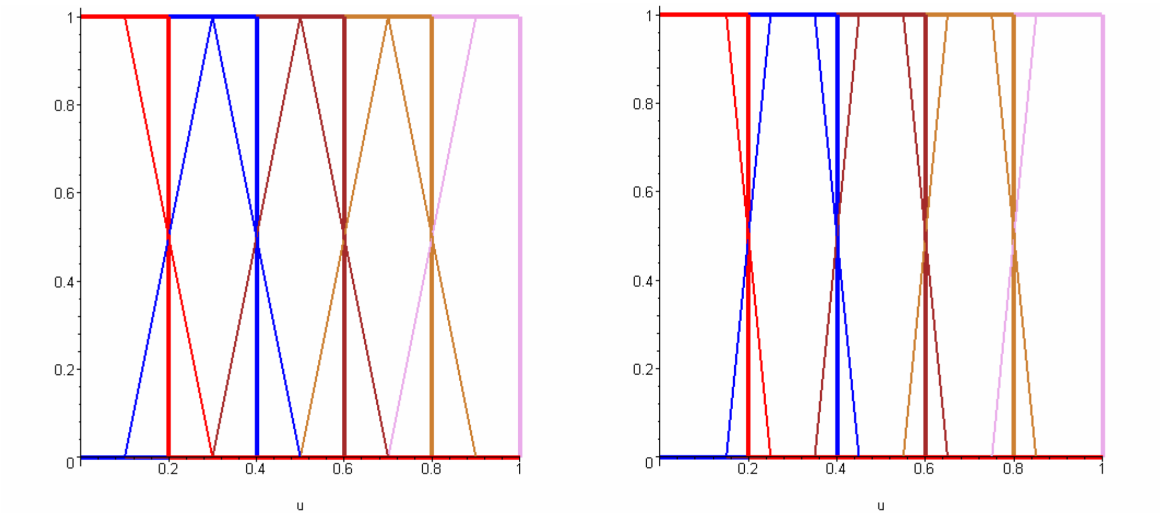
$$\int_0^1 \phi_K(m,k,u)\,du = \sum_{j=0}^{K-1} \int_0^1 \phi(K \cdot m, K \cdot k + j, u)\,du = \sum_{j=0}^{K-1} \frac{1}{K \cdot m} = \frac{1}{m},\ k = 0, \cdots, m-1$$

$$\sum_{k=0}^{m-1} \phi_K(m,k,\bullet) = \sum_{j=0}^{K-1} \sum_{k=0}^{m-1} \phi(K \cdot m, K \cdot k + j, \bullet) = \sum_{i=0}^{K \cdot m} \phi(K \cdot m, i, \bullet) = 1,\ m \in \mathbb{N}$$

(cf. [5]). Visualization of the resulting smoothing effect for Bernstein and triangle copulas (one-dimensional):



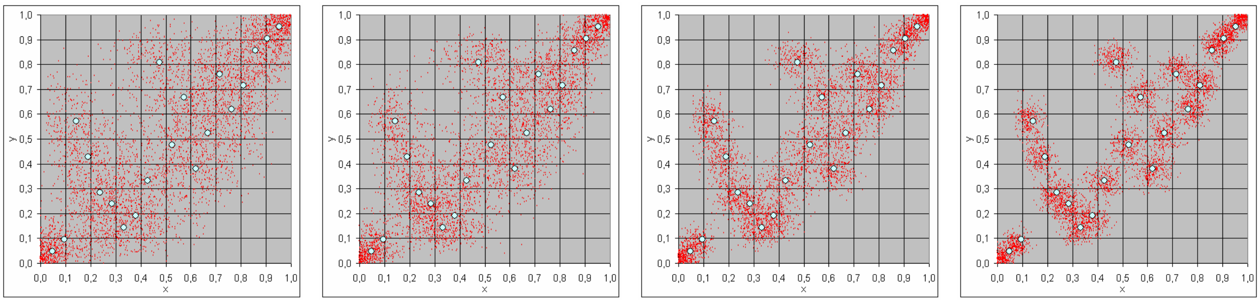
$K = 3$ $\quad m = 5 \quad$ $K = 10$



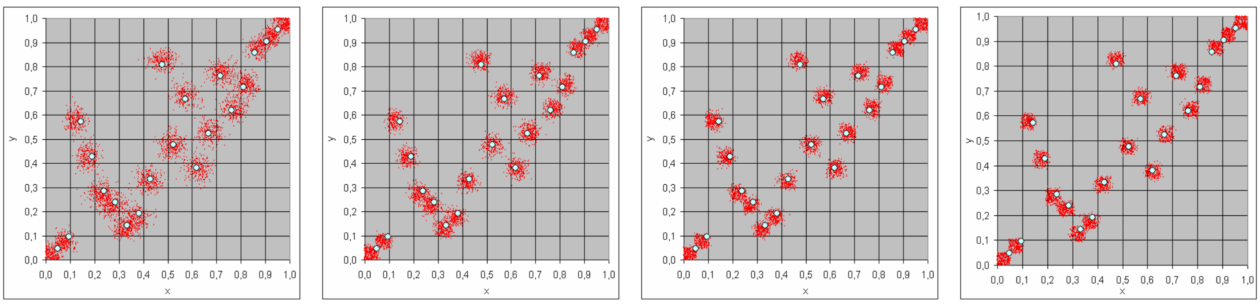
$K = 1$ $m = 5$ $K = 2$

Triangle

In **Binomial-Copula.xls, NB-Copula.xls** and **Poisson-Copula.xls** the generalized Bernstein-copula (as a driver) is generated by *independent* sub-partitions of unity for arbitrary choices of *K*. The following graphs show some examples. Note that the case *K* = 1 corresponds to the ordinary Bernstein copula.



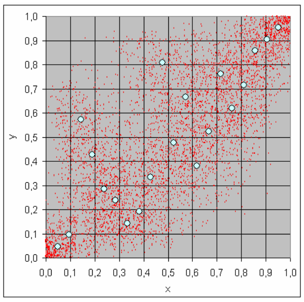
K = 1 *K* = 2 *K* = 5 *K* = 10



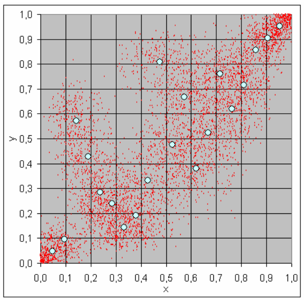
K = 20 *K* = 50 *K* = 100 *K* = 200

Note that the limiting copula driver here is the rook copula driver.

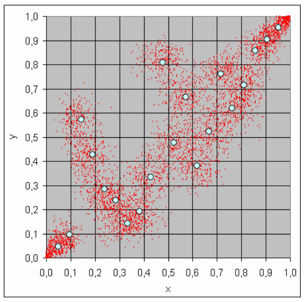
In **Binomial-Copula_diag.xls, NB-Copula_diag.xls** and **Poisson-Copula_diag.xls** the generalized Bernstein-copula (as a driver) is generated by *dependent* sub-partitions for arbitrary choices of *K*. The following graphs show some examples.



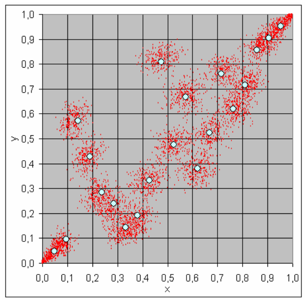
$K = 1$



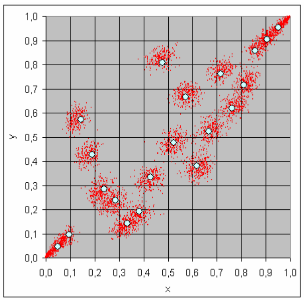
$K = 2$



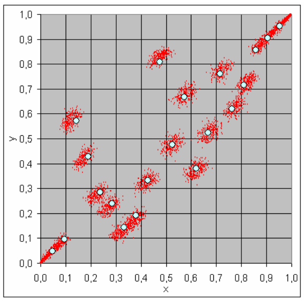
$K = 5$



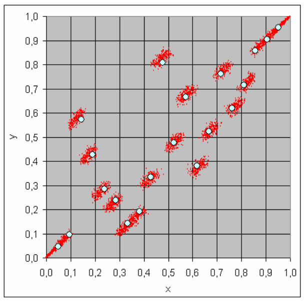
$K = 10$



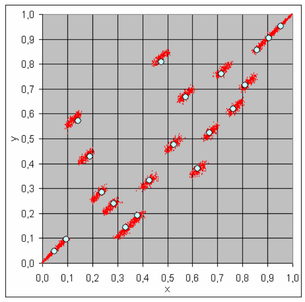
$K = 20$



$K = 50$



$K = 100$

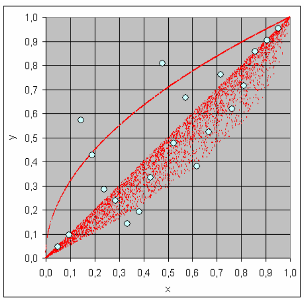


$K = 200$

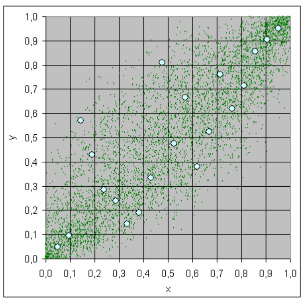
Note that the limiting copula driver here is the upper Fréchet bound driver.

Finally, a particular copula driver used in the examples is a special Singular Mixture copula, cf. [6]. For the data above, the singular mixture copula with the following input was used:

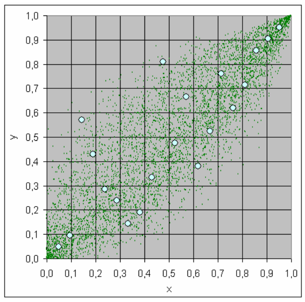
$F(x) = x^2$ and $G_\alpha(x) = \dfrac{x - \alpha x^2}{1-\alpha}$ for $0 < x < 1$, where α is uniformly distributed over $\left[0, \dfrac{1}{2}\right]$.



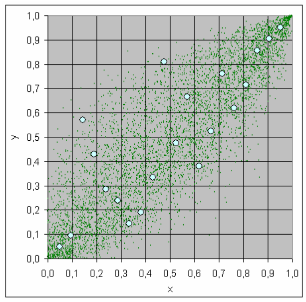
Sing. mixture copula



Binomial copula



Neg. binomial copula



Poisson copula

References:

[5] D. Pfeifer, D. Straßburger, J, Philipps: Modelling and simulation of dependence structures in nonlife insurance with Bernstein copulas. Paper presented on the occasion of the International ASTIN Colloquium, June 1 – 4, 2009, Helsinki.

http://www.staff.uni-oldenburg.de/dietmar.pfeifer/Publ/P97.pdf

[6] D. Pfeifer, D. Lauterbach: Chapter 8: Singular Mixture Copulas. In: P. Jaworski et al. (eds.), *Copulae in Mathematical and Quantitative Finance*, Lecture Notes in Statistics 213, 165 – 175.

https://www.springerprofessional.de/singular-mixture-copulas/4036784