Chapter 18
Correlation, Tail Dependence and Diversification

Dietmar Pfeifer

18.1 Introduction

What is frequently abbreviated as Solvency II is perhaps the most challenging legislative adventure in the European Union (besides Basel II/III for the banking sector) in the last decade. It is a fundamentally new, risk driven approach towards a harmonization of financial regulation for insurance and reinsurance companies writing business in the European Union. One of the major aims of the Solvency II framework is a customer protection limiting the yearly ruin probability of the company to at most 0.5 % by requiring sufficient economic capital. The calculation of this so called Solvency Capital Requirement (SCR) is based on a complicated mathematical and statistical framework derived from an economic balance sheet approach (for more details, see, e.g., Buckham et al. 2011; Cruz 2009; Doff 2007 or Sandström 2006). An essential aspect in the SCR calculation here is the notion of diversification, which aims at a reduction of the overall capital requirement by "distributing" risk in an appropriate way. There are several definitions and explanations of this term, some of which are presented in the sequel.

> "Although it is an old idea, the measurement and allocation of diversification in portfolios of asset and/or liability risks is a difficult problem, which has so far found many answers. The diversification effect of a portfolio of risks is the difference between the sum of the risk measures of stand-alone risks in the portfolio and the risk measure of all risks in the portfolio taken together, which is typically non-negative, at least for positive dependent risks."
>
> [Hürlimann (2009a, p. 325)]

> "Diversification arises when different activities complement each other, in the field of both return and risk. [. . .] The diversification effect is calculated by using correlation factors. Correlations are statistical measures assessing the extend to which events could occur simultaneously. [. . .] A correlation factor of 1 implies that certain events will always occur

D. Pfeifer (✉)
Institut für Mathematik, Carl von Ossietzky Universität Oldenburg, 26111 Oldenburg, Germany
e-mail: dietmar.pfeifer@uni-oldenburg.de

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simultaneously. Hence, there is no diversification effect and two risks identically add up. Risk managers tend to say that such risks are perfectly correlated (i.e., they have a high correlation factor), meaning that these two risks do not actually diversify at all. A correlation factor of 0 implies that diversification effects are present and a certain diversification benefit holds."

[Doff (2007, p. 167f.)]

"By diversifiable we mean that if a risk category can be subdivided into risk classes and the risk charge of the total risk is not higher than the sum of the risk charges of each subrisk, then we have the effect of diversification. [...] This effect can be measured as the difference between the sum of several capital charges and the total capital charge when dependency between them is taken into account."

[Sandström (2006, p. 188)]

"In order to promote good risk management and align regulatory capital requirements with industry practices, the Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases. [...] That economic capital should be calculated on the basis of the true risk profile of those undertakings, taking account of the impact of possible risk-mitigation techniques, as well as diversification effects. [...] Diversification effects means the reduction in the risk exposure of insurance and reinsurance undertakings and groups related to the diversification of their business, resulting from the fact that the adverse outcome from one risk can be offset by a more favourable outcome from another risk, where those risks are not fully correlated. The Basic Solvency Capital Requirement shall comprise individual risk modules, which are aggregated [...] The correlation coefficients for the aggregation of the risk modules [...], shall result in an overall Solvency Capital Requirement [...] Where appropriate, diversification effects shall be taken into account in the design of each risk module."

[Official Journal of the European Union (2009, (64) p. 7; (37) p. 24; Article 104, p. 52)]

One central idea that is common to all of these explanations is that a small, zero or even negative correlation between risks implies a diversification effect, while a large correlation or positive dependence implies the opposite. This is, however, largely based on a naïve understanding of the relationship between correlation and dependence which is not at all justified from a rigorous statistical point of view (see, e.g., Mari and Kotz 2001). This fact has also been emphasized by McNeil et al. (2005) in Chaps. 5 and 6 of their monograph, and in part also by Artzner et al. (1999).

A better way to tackle the understanding of a diversification effect is to replace the notion of correlation by the notion of copulas which describe the dependence structure between risks completely (see, e.g., Nelsen 2006, for a sophisticated survey). With respect to the "dangerousness" of joint risks, tail dependence is often used as a characteristic quantity (see, e.g., McNeil et al. 2005, Sect. 5.2.3). In case of a positive upper coefficient of tail dependence, it is likely that extreme events will occur more frequently simultaneously, just in the spirit of Doff's explanation of diversification above. This might suggest that risks with positive upper tail dependence are less exposed to diversification than those with zero upper tail dependence. However, a more sophisticated analysis shows that this is also not true in general.

The aim of this chapter is twofold:

Firstly, to show that the notion of correlation is completely disjoint from the notion of diversification under the risk measure VaR used in the Solvency II directive, i.e., we shall show that a state of no diversification between risks can be achieved with almost arbitrary positive and negative correlation coefficients, especially with the same marginal risk distributions.

And secondly, that a state of no diversification between risks can also be achieved with a zero tail dependence coefficient, or even worse, with a partial countermonotonic dependence structure, in particular for risks being lognormally distributed which is a basic assumption in the Pillar One standard model of Solvency II.

18.2 A Short Review of Risk Measures

In this section, we shall only focus on risk measures for non-negative risks since these are the essential quantities in insurance, and are also the fundamentals of the SCR calculation under Solvency II. We follow a simplified setup as in Sandström (2006), Sect. 7.4 which is formally slightly different from the approach in Artzner et al. (1999) or McNeil et al. (2005, Chap. 6).

Definition 18.1 Let $\mathcal{X}$ Let be a suitable set of non-negative random variables X on a probability space $(\Omega, \mathcal{A}, P)$. A risk measure R on $\mathcal{X}$ is a mapping $\mathcal{X} \to \mathbb{R}^+$ with the following properties:

$$P^X = P^Y \Rightarrow R(X) = R(Y) \quad \forall X, Y \in \mathcal{X}, \tag{18.1}$$

i.e., the risk measure depends only on the distribution of the risk X;

$$R(cX) = cR(X) \quad \forall X \in \mathcal{X} \quad \forall c \geq 0,$$

i.e., the risk measure is *scale-invariant*;

$$R(X + c) = R(X) + c \quad \forall X \in \mathcal{X} \quad \forall c \geq 0, \tag{18.2}$$

i.e., the risk measure is *translation-invariant*;

$$R(X) \leq R(Y) \quad \forall X, Y \in \mathcal{X},\ X \leq Y, \tag{18.3}$$

i.e., the risk measure is *monotone*.

The risk measure is called *coherent*, if it additionally has the subadditivity property:

$$R(X + Y) \leq R(X) + R(Y) \quad \forall X, Y \in \mathcal{X}. \tag{18.4}$$

This last property is the crucial point: it induces a diversification effect for *arbitrary* non-negative risks $X_1, \ldots, X_n$ (dependent or not) since it follows by induction that coherent risk measures have the property

$$R\left(\sum_{k=1}^{n} X_k\right) \leq \sum_{k=1}^{n} R(X_k) \quad \forall n \in \mathbb{N}.$$

In what follows we shall use the term "(risk) *concentration* effect" as opposite to "*diversification* effect", characterized by the converse inequality

$$\exists (X, Y) \in \mathcal{X} \times \mathcal{X} : R(X+Y) > R(X) + R(Y).$$

Example 18.1 The popular standard deviation principle *SDP* which is sometimes used for tariffing in insurance is defined as

$$\mathrm{SDP}(X) = E(X) + \gamma\sqrt{\mathrm{Var}(X)} \quad \text{for a fixed } \gamma > 0 \text{ and } X \in \mathcal{X} = \mathcal{L}_+^2(\Omega, \mathcal{A}, P),$$

the set of non-negative square-integrable random variables on $(\Omega, \mathcal{A}, P)$. Obviously, SDP fulfils the properties (18.1) to (18.2) and (18.4); the latter because of

$$\begin{aligned}\mathrm{SDP}(X+Y) &= E(X) + E(Y) \\ &\quad + \gamma\sqrt{\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\rho(X,Y)\sqrt{\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}} \\ &\le E(X) + E(Y) + \gamma\sqrt{\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\sqrt{\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}} \\ &= E(X) + E(Y) + \gamma\sqrt{\left(\sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)}\right)^2} \\ &= \mathrm{SDP}(X) + \mathrm{SDP}(Y) \end{aligned} \tag{18.5}$$

for all $X, Y \in \mathcal{X}$. Here $\rho(X,Y) = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$ denotes the correlation between X and Y. However, *SDP* does in general not fulfil property (18.3) and is hence not a proper risk measure, as can be seen as follows: Let Z be a random variable binomially distributed over $\{0, 1\}$ with $P(Z=1) = 1 - P(Z=0) = p$, and $1/(1+\gamma^2) < p < 1$. Consider $X := 2Z$ and $Y := 1 + Z$. Then $X \le Y$, but $R(X) = 2p + 2\gamma\sqrt{p(1-p)} > 1 + p + \gamma\sqrt{p(1-p)} = R(Y)$.

Example 18.2 The risk measure used in Basel II/III and Solvency II is the Value-at-Risk VaR, being defined as a (typically high) quantile of the risk distribution:

$$\mathrm{VaR}_\alpha(X) := Q_X(1-\alpha) \quad \text{for } X \in \mathcal{X} \text{ and } 0 < \alpha < 1,$$

where Q_X denotes the quantile function

$$Q_X(u) := \inf\{x \in \mathbb{R} \mid P(X \le x) \ge u\} \quad \text{for } 0 < u < 1.$$

Value-at-Risk is a proper risk measure, but not coherent in general. This topic will be discussed in more detail in the next section (for a more general discussion, see e.g., McNeil et al. 2005, Sect. 6.1.2).

The "smallest" coherent risk measure above VaR is the expected shortfall (ES), which is in general defined as

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\left\{E(X \cdot \mathbb{I}_{\{X \ge \mathrm{VaR}_\alpha(X)\}}) + \mathrm{VaR}_\alpha(X)\left[\alpha - P\left(X \ge \mathrm{VaR}_\alpha(X)\right)\right]\right\}$$

for $X \in \mathcal{X}$ and $0 < \alpha < 1$, where $\mathbb{I}_A$ denotes the indicator random variable of some event (measurable set) A. In case that $P(X \geq \mathrm{VaR}_\alpha(X)) = \alpha$, this formula simplifies to

$$\mathrm{ES}_\alpha(X) = E\big(X \mid X \geq \mathrm{VaR}_\alpha(X)\big) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)\,du$$

(see McNeil et al. 2005, Definition 2.15 and Remark 2.17); note that the role of α and $1-\alpha$ are interchanged there). A more thorough discussion on the relationship between VaR and ES (and other coherent risk measures) in connexion with Wang's distortion measures can be found in Hürlimann (2004). Expected shortfall is the risk measure which is used in the Swiss Solvency Test (SST), see, e.g., Sandström (2006, Sect. 6.8) or Cruz (2009, Chap. 17).

18.3 A Short Review of Copulas

The copula approach allows for a separate treatment of the margins of joint risks and the dependence structure between them. The name "copula" goes back to Abe Sklar in 1959 who used it as a function which couples a joint distribution function with its univariate margins. For an extensive survey, see, e.g., Nelsen (2006).

Definition 18.2 A copula (in n dimensions) is a function C defined on the unit cube $[0,1]^n$ with the following properties:

1. the range of C is the unit interval $[0,1]$;
2. $C(\mathbf{u})$ is zero for all $\mathbf{u} = (u_1,\ldots,u_n)$ in $[0,1]^n$ for which at least one coordinate is zero;
3. $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except the k-th one;
4. C is n-increasing in the sense that for every $\mathbf{a} \leq \mathbf{b}$ in $[0,1]^n$ the volume assigned by C to the subinterval $[\mathbf{a},\mathbf{b}] = [a_1,b_1]\times\cdots\times[a_n,b_n]$ is nonnegative.

A copula can alternatively be characterized as a multivariate distribution function with univariate marginal distribution functions that belong to a continuous uniform distribution over the unit interval $[0,1]$.

It can be shown that every copula is bounded by the so-called Fréchet–Hoeffding bounds, i.e.,

$$\begin{aligned} C_*(\mathbf{u}) &:= \max(u_1+\cdots+u_n-n+1,0) \leq C(u_1,\ldots,u_n) \\ &\leq C^*(\mathbf{u}) := \min(u_1,\ldots,u_n). \end{aligned}$$

The upper Fréchet–Hoeffding bound C^* is a copula itself for any dimension; however, the lower Fréchet–Hoeffding bound C_* is a copula in two dimensions only. If X is any real random variable, then the random vector $\mathbf{X} = (X,X,\ldots,X)$ with n components possesses the upper Fréchet–Hoeffding bound C^* as copula, while the random vector $\mathbf{X} = (X, -X)$ with two components possesses the lower Fréchet–Hoeffding bound C_* as copula. Random variables who have C^* or C_*, respectively, as copula are also called *comonotone* or *countermonotone*, respectively. An important and well-studied copula is the independence copula, given by $C(\mathbf{u}) = \prod_{i=1}^n u_i$.

The following theorem due to Sklar justifies the role of copulas as dependence functions:

Proposition 18.1 *Let H denote some n-dimensional distribution function with marginal distribution functions $F_1, \dots, F_n$. Then there exists a copula C such that for all real $(x_1, \dots, x_n)$,*

$$H(x_1, \dots, x_n) = C\big(F_1(x_1), \dots, F_n(x_n)\big).$$

If all the marginal distribution functions are continuous, then the copula is unique. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1}, \dots, F_n^{-1}$ the generalized inverses of the marginal distribution functions (or quantile functions), then for every $(u_1, \dots, u_n)$ in the unit cube,

$$C(u_1, \dots, u_n) = H\big(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)\big).$$

For a proof, see (Nelsen 2006, Theorem 2.10.9) and the references given therein. The above theorem shows that copulas remain invariant under strictly monotone transformations of the same kind of the underlying random variables (either increasing or decreasing).

The following result shows the relationship between correlation and copulas.

Proposition 18.2 *Let (X, Y) be a bivariate random vector with a copula C and marginal distribution functions F and G such that $E(|X|) < \infty$, $E(|Y|) < \infty$ and $E(|XY|) < \infty$. Then the covariance between X and Y can be expressed in the following way:*

$$\operatorname{Cov}(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \big[C\big(F(x), G(y)\big) - F(x)G(y)\big]\, dx\, dy.$$

For a proof see, e.g., McNeil et al. (2005, Lemma 5.24).

18.4 Correlation and Diversification

Before going into deeper details, we start with an illustrative example showing essentially that risk concentration under VaR can occur with almost all positive and negative correlation coefficients, even if the marginal distributions remain fixed. According to the Solvency II standard, we choose $\alpha = 0.005$ for simplicity here, but the example can be reformulated accordingly with any other value of $0 < \alpha < 1$.

Table 18.1 Joint distribution of risks

$P(X=x, Y=y)$		x			$P(Y=y)$	$P(Y \leq y)$
		0	50	100		
y	0	β	$0.440-\beta$	0.000	0.440	0.440
	40	$0.554-\beta$	β	0.001	0.555	**0.995**
	50	0.000	0.001	0.004	0.005	1.000
$P(X=x)$		0.554	0.441	0.005		
$P(X \leq x)$		0.554	**0.995**	1.000		

Table 18.2 Moments and correlations

$E(X)$	$E(Y)$	$\sigma(X)$	$\sigma(Y)$	$\rho(\beta)=\rho(X,Y)$
22.550	22.450	25.377	19.912	$-0.9494+3.9579\beta$

Table 18.3 Distribution of aggregate risk

s	0	40	50	90	100	140	150
$P(S=s)$	β	$0.554-\beta$	$0.440-\beta$	β	0.001	0.001	0.004
$P(S \leq s)$	β	0.554	$0.994-\beta$	0.994	**0.995**	0.996	1.000

Example 18.3 Let the joint distribution of the non-negative risks X and Y be given by Table 18.1, with $0 \leq \beta \leq 0.440$, giving $\text{VaR}_\alpha(X) = 50$, $\text{VaR}_\alpha(Y) = 40$.

For the moments of X and Y, we obtain the values in Table 18.2 (with σ denoting the standard deviation). This shows that the range of possible risk correlations is the interval $[-0.9494; 0.7921]$, with a zero correlation being attained for $\beta = 0.2399$.

Table 18.3 shows the distribution of the aggregated risk $S = X + Y$.

We thus obtain a risk concentration due to $\text{VaR}_\alpha(S) = 100 > 90 = \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$, independent of the parameter β and hence also independent of the possible correlations between X and Y.

A closer look to the joint distribution of X and Y shows that the reason for this perhaps unexpected result is the fact that although one can have a "diversification effect" in the central body of the distribution, where a fraction of little less than $1-\alpha$ of the risk pairs are located, the essential "concentration effect", however, is caused by a joint occurrence of very high losses, with a fraction of α of all risk pairs.

The following result is related to the consideration of "worst VaR scenarios" as in McNeil et al. (2005, Sect. 6.2).

Proposition 18.3 *Let X and Y be non-negative risks with cumulative distribution functions F_X and F_Y, respectively, which are continuous and strictly increasing on* *their support. Denote, for a fixed* $\alpha \in (0,1)$,

$$Q^*(\alpha,\delta) := \min\big\{Q_X(u) + Q_Y(2-\alpha-\delta-u) \mid 1-\alpha-\delta \le u \le 1\big\}$$
$$\text{for } 0 \le \delta < 1-\alpha.$$

Then there exists a sufficiently small $\epsilon \in (0, 1-\alpha)$ *with the property*

$$Q^*(\alpha,\epsilon) > Q_X(1-\alpha) + Q_Y(1-\alpha) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y). \tag{18.6}$$

Assume further that the random vector (U, V) *has a copula* C *as joint distribution function with the properties*

$$\begin{aligned} V < 1-\alpha-\epsilon \iff U < 1-\alpha-\epsilon \quad \text{and} \\ V = 2-\alpha-\epsilon-U \iff U \ge 1-\alpha-\epsilon. \end{aligned} \tag{18.7}$$

If we define

$$X^* := Q_X(U), \quad Y^* := Q_Y(V), \quad S^* := X^* + Y^*,$$

then the random vector (X^*, Y^*) *has the same marginal distributions as* (X, Y), *and it holds*

$$\mathrm{VaR}_\alpha\big(X^* + Y^*\big) \ge Q^*(\alpha,\epsilon) > \mathrm{VaR}_\alpha\big(X^*\big) + \mathrm{VaR}_\alpha\big(Y^*\big) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y),$$

i.e., there is a risk concentration effect. Moreover, under the assumption (18.7), *the correlation* $\rho(X^*, Y^*)$ *is minimal if* $V = 1-\alpha-\epsilon-U$ *for* $U < 1-\alpha-\epsilon$ *(lower extremal copula* $\underline{C}$*) and maximal if* $V = U$ *for* $U < 1-\alpha-\epsilon$ *(upper extremal copula* $\overline{C}$*).*

Proof By assumption, the (non-negative) quantile functions Q_X and Q_Y are continuous and strictly increasing over the interval [0, 1] (with a possibly infinite value at the point 1), so that

$$\begin{aligned} &Q^*(\alpha, 0) \\ &\quad = \min\big\{Q_X(u) + Q_Y(2-\alpha-u) \mid 1-\alpha \le u \le 1\big\} > Q_X(1-\alpha) + Q_Y(1-\alpha), \end{aligned}$$

the minimum being actually attained. Since by the continuity assumptions above, $Q^*(\alpha,\epsilon)$ is continuous in ϵ and decreasing when ϵ is increasing, relation (18.6) follows.

The copula construction above now implies that

$$\begin{aligned} &P\big(S^* \le s\big) \le 1-\alpha-\epsilon \\ &\quad \text{for } s \le Q_X(1-\alpha-\epsilon) + Q_Y(1-\alpha-\epsilon) = \mathrm{VaR}_{\alpha+\epsilon}(X) + \mathrm{VaR}_{\alpha+\epsilon}(Y), \\ &P\big(S^* \le s\big) = 1-\alpha-\epsilon \quad \text{for } \mathrm{VaR}_{\alpha+\epsilon}(X) + \mathrm{VaR}_{\alpha+\epsilon}(Y) < s < Q^*(\alpha,\epsilon), \\ &P\big(S^* \le s\big) \ge 1-\alpha-\epsilon \quad \text{for } s \ge Q^*(\alpha,\epsilon). \end{aligned} \tag{18.8}$$

Table 18.4 Examples of risk measures and correlations for various values of σ

σ	$\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(Y)$	$\mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$	$\mathrm{VaR}_\alpha(X^* + Y^*)$	$\rho_{\min}(X^*, Y^*)$	$\rho_{\max}(X^*, Y^*)$
0.1	1.2873	2.5746	2.6205	−0.8719	0.9976
0.2	1.6408	3.2816	3.3994	−0.8212	0.9969
0.3	2.0704	4.1408	4.3661	−0.7503	0.9951
0.4	2.5866	5.1732	5.5520	−0.6620	0.9920
0.5	3.1992	6.3984	6.9901	−0.5598	0.9873
0.6	3.9177	7.8354	8.7134	−0.4480	0.9802
0.7	4.7497	9.4994	10.7537	−0.3310	0.9700
0.8	5.7011	11.4022	13.1401	−0.2136	0.9556
0.9	6.7750	13.5500	15.8969	−0.1002	0.9362
1.0	7.9712	15.9424	19.0412	0.0050	0.9108
1.5	15.4675	30.9350	40.4257	0.3127	0.6839
2.0	23.3748	46.7496	66.8923	0.2723	0.3794
2.5	27.5107	55.0214	86.2673	0.1399	0.1637
3.0	25.2162	50.4324	86.7034	0.0565	0.0611

Relation (18.8) in turn implies that $\mathrm{VaR}_\alpha(S^*) = \mathrm{VaR}_\alpha(X^* + Y^*) \geq Q^*(\alpha, \epsilon)$ which proves the first part of Proposition 18.3, due to relation (18.6).

The remainder part follows from Theorem 5.25 in McNeil et al. (2005) when looking at the conditional distribution of (X^*, Y^*) given the event $\{U < 1 - \alpha - \epsilon\}$. □

Note that both types of copulas that provide the extreme values for the correlations, $\underline{C}$ and $\overline{C}$, are of the type "shuffles of M", see Nelsen (2006, Sect. 3.2.3).

In the following example, we shall show some consequences of Proposition 18.3 in the case of lognormally distributed risks, which are of special importance for Pillar One under Solvency II, see, e.g., Hürlimann (2009a, 2009b).

Example 18.4 To keep things simple and comparable with Solvency II specifications, we shall assume that X and Y follow the same lognormal distribution $\mathcal{LN}(\mu, \sigma)$ with $\mu \in \mathbb{R}$, $\sigma > 0$ and $E(X) = E(Y) = 1$ which corresponds to the case $\mu = -\sigma^2/2$. Table 18.4 shows all relevant numerical results for the extreme copulas $\underline{C}$ and $\overline{C}$ in Proposition 18.3, especially the maximal range of correlations induced by them. According to the Solvency II standard, we choose $\alpha = 0.005$ (and $\epsilon = 0.001$, which will be sufficient here).

Note that the bottom graph in Fig. 18.1 resembles the graph in Fig. 5.8 in McNeil et al. (2005).

The graph in Fig. 18.2 shows parts of the two cumulative distribution functions under the extreme copulas $\underline{C}$ and $\overline{C}$ for $S^* := X^* + Y^*$ in the case $\sigma = 1$. Note that especially for smaller values of σ (which is typical for the calculation of the SCR in

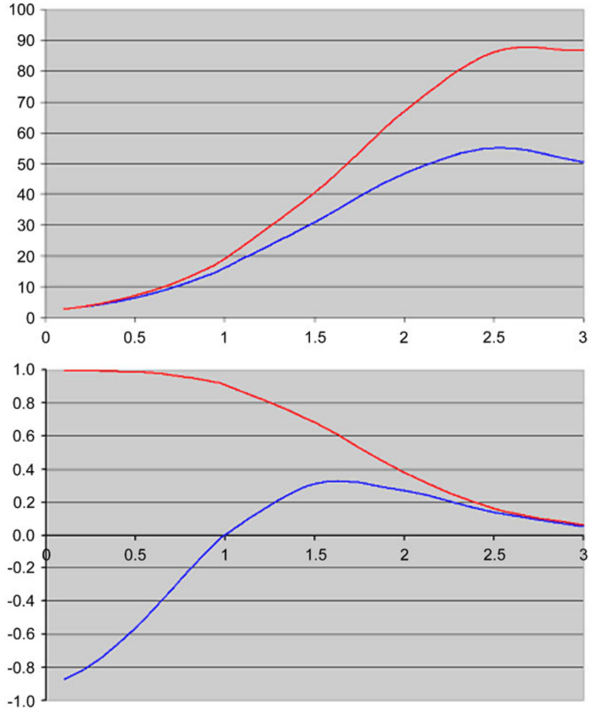

Fig. 18.1 *Top*: Graph of $\mathrm{VaR}_\alpha(X^* + Y^*)$ (*red*) and $\mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$ (*blue*) as functions of σ; *bottom*: graph of $\rho_{\max}(X^*, Y^*)$ (*red*) and $\rho_{\min}(X^*, Y^*)$ (*blue*) as functions of σ

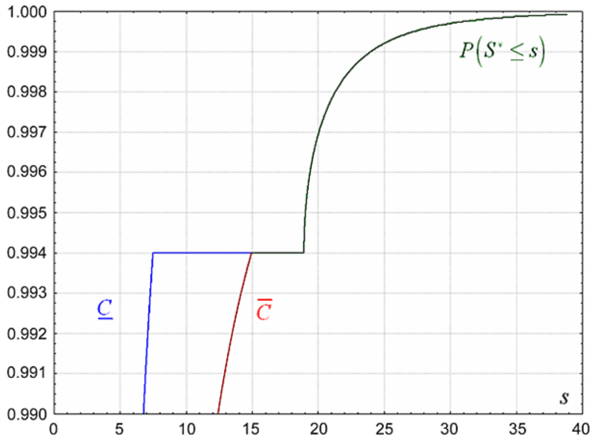

Fig. 18.2 Graph of cumulative distribution functions for extreme copulas for $S^* = X^* + Y^*$ with $\sigma = 1$

the non-life risk module of Solvency II) the range of possible negative and positive correlations between the risks is quite large, with the same significant discrepancy between the Value at Risk of the aggregated risks and the sum of individual Values at Risk.

Note also that any correlation value ρ of $\rho(X^*, Y^*)$ between $\rho_{\min}(X^*, Y^*)$ and $\rho_{\max}(X^*, Y^*)$ can be achieved by a proper mixture of the extreme copulas $\underline{C}$ and $\overline{C}$

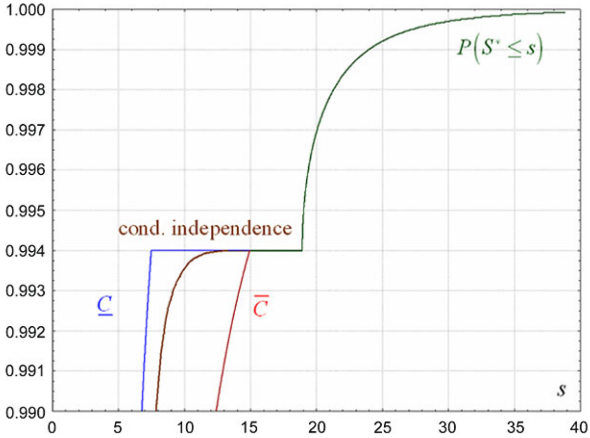

Fig. 18.3 Graph of cumulative distribution functions for extreme copulas and conditional independence

namely for the copula

$$C(p) = \lambda \underline{C} + (1-\lambda)\overline{C} \quad \text{with } \lambda = \frac{\rho_{\max}(X^*, Y^*) - \rho}{\rho_{\max}(X^*, Y^*) - \rho_{\min}(X^*, Y^*)}.$$

This is a direct consequence from Proposition 18.2, for example.

There are, of course, also other possibilities to achieve appropriate intermediary values for the correlation, for instance if U and V are conditionally independent given the event $\{U < 1 - \alpha - \epsilon\}$. The graph in Fig. 18.3 adds a part of the cumulative distribution function of $S^* := X^* + Y^*$ for this case to the graph in Fig. 18.2. The correlation between X^* and Y^* is here given by $\rho(X^*, Y^*) = 0.3132$.

18.5 Tail Dependence and Diversification

As in the case of a large positive correlation between risks, it might be intuitively tempting to assume that a positive upper tail dependence would have a positive impact on risk concentration, too. But this is not true here either. In this section, we shall show that a risk concentration effect can occur with and without tail dependence, while the marginal distributions remain unchanged.

Note first that the copula construction of Proposition 18.3 implies no upper tail dependence since, by the continuity assumption for the marginal distributions made there (see, e.g., McNeil et al. 2005, Sect. 5.2.3),

$$\lambda_u = \lim_{u \uparrow 1} \frac{P(U > u, V > u)}{1 - u} = 0$$

because for $1-(\alpha+\epsilon)/2 < u \le 1$, we have $2-\alpha-\epsilon-u < u$ and hence, for these u,

$$\begin{aligned} P(U > u, V > u) &= P(U > u, 2-\alpha-\epsilon-U > u) \\ &= P(u < U < 2-\alpha-\epsilon-u) = P(\emptyset) = 0. \end{aligned}$$

The following proposition shows that we can incorporate an upper tail dependence in the construction of Proposition 18.3 without essentially loosing the central result.

Proposition 18.4 *Assume that the conditions of Proposition* 18.3 *hold, with the following modification of the copula construction in* (18.7):

$$V < 1-\alpha-\epsilon \iff U < 1-\alpha-\epsilon \quad \text{and}$$
$$V = \begin{cases} 2-\alpha-\epsilon-\gamma-U : 1-\alpha-\epsilon \le U < 1-\gamma \\ U : 1-\gamma \le U \le 1 \end{cases}$$

with some non-negative $\gamma < \alpha$. *Then, for sufficiently small* ϵ *and* γ, *we still have*

$$\begin{aligned} &\min\{Q_X(u) + Q_Y(2-\alpha-\epsilon-\gamma-u) \mid 1-\alpha-\epsilon \le u \le 1-\gamma\} \\ &> Q_X(1-\alpha) + Q_Y(1-\alpha) \end{aligned}$$

and hence again a risk concentration effect, i.e., $\mathrm{VaR}_\alpha(X^*+Y^*) > \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$. *Moreover, under this copula construction, the correlation* $\rho(X^*, Y^*)$ *is again minimal if* $V = 1-\alpha-\epsilon-U$ *for* $U < 1-\alpha-\epsilon$ *(lower extremal copula* $\underline{C}_\gamma$*) and maximal if* $V = U$ *for* $U < 1-\alpha-\epsilon$ *(upper extremal copula* $\overline{C}_\gamma$*). Further, the risks are in all cases upper tail dependent with*

$$\lambda_u = \lim_{u \uparrow 1} \frac{P(U > u, V > u)}{1-u} = 1. \tag{18.9}$$

Proof The first two parts follow along the lines of the proof of Proposition 18.3. For the last part, observe that we have $U = V \iff 1-\gamma \le U \le 1$ which implies $P(U > u, V > u) = 1-u$ for $1-\gamma \le u \le 1$ and hence (18.9). □

Example 18.5 If we choose $\gamma = 0.0005$, we get the extension of the results in Example 18.4, under the same initial conditions (see Table 18.5). The graph in Fig. 18.4 shows an extension of the graph in Fig. 18.3 for the extreme copulas $\underline{C}_\gamma$ and $\overline{C}_\gamma$ for $S^* := X^* + Y^*$ in the case $\sigma = 1$. The case $\gamma > 0$ corresponds to upper tail dependence, the case $\gamma = 0$ correspond to the former situation with no upper tail dependence.

18.6 Conclusions

As the preceding analysis has shown, neither the notion of correlation nor the notion of tail dependence as such has in general a direct impact on diversification under the

Table 18.5 Examples of risk measures and correlations for various values of σ

σ	$\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(Y)$	$\mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$	$\mathrm{VaR}_\alpha(X^* + Y^*)$	$\rho_{\min}(X^*, Y^*)$	$\rho_{\max}(X^*, Y^*)$
0.1	1.2873	2.5746	2.6134	−0.8710	0.9993
0.2	1.6408	3.2816	3.3811	−0.8193	0.9988
0.3	2.0704	4.1408	4.3308	−0.7471	0.9981
0.4	2.5866	5.1732	5.4923	−0.6568	0.9969
0.5	3.1992	6.3984	6.8962	−0.5515	0.9953
0.6	3.9177	7.8354	8.5730	−0.4349	0.9929
0.7	4.7497	9.4994	10.5516	−0.3107	0.9974
0.8	5.7011	11.4022	12.8581	−0.1830	0.9964
0.9	6.7750	13.5500	15.5133	−0.0553	0.9951
1.0	7.9712	15.9424	18.5310	0.0691	0.9744
1.5	15.4675	30.9350	38.8061	0.5658	0.9366
2.0	23.3748	46.7496	63.3300	0.8154	0.9224
2.5	27.5107	55.0214	80.5429	0.9185	0.9423
3.0	25.2162	50.4324	79.8272	0.9636	0.9909

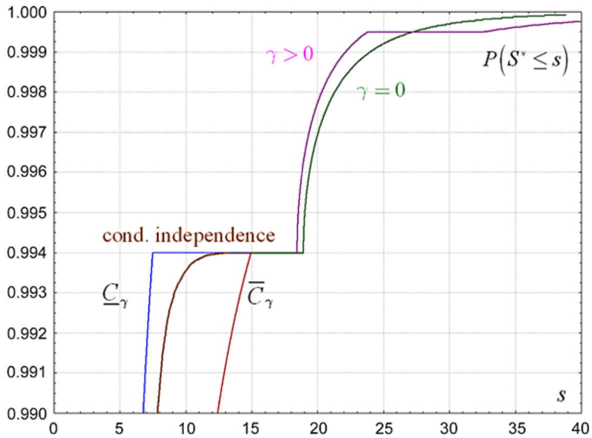

Fig. 18.4 Graph of cumulative distribution functions for extreme copulas and conditional independence, with and without upper tail dependence

risk measure Value at Risk. This means that any attempt to implement such concepts into a simple Pillar One standard model under Solvency II for the purpose of a reduction of the Solvency Capital Requirement in case of a diversification effect cannot be justified by mathematical reasoning. We can perhaps summarize the consequences of this insight in a slight modification of a statement in McNeil et al. (2005, p. 205):

> The concept of diversification is meaningless unless applied in the context of a well-defined joint model. Any interpretation of diversification in the absence of such a model should be avoided.

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