Chapter 8
Singular Mixture Copulas

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Abstract We present a new family of copulas—the *Singular Mixture Copulas*. We begin with the construction of singular copulas whose supports lie on the graphs of two given quantile functions. These copulas are then mixed with respect to a continuous distribution resulting in a nonsingular parametric copula.

The Singular Mixture Copulas we construct have a Lebesgue density and in special cases even a closed form representation. Moreover, they have positive lower and upper tail dependence. Because Singular Mixture Copulas are mixtures of "simple" singular copulas, they can be simulated easily.

8.1 Introduction

Copulas provide an effective and versatile tool for modeling multivariate stochastic dependence. Since their introduction by Sklar in 1959 (see [11]) there have been intense developments in both the copula theory and their applications, see, e.g., [1, 5–7, 9, 10, 12].

In [10] several geometric methods of constructing copulas are presented. One approach deals with the construction of singular copulas whose supports lie in a given set. Another approach mixes an infinite family of copulas with respect to a mixing distribution. We present a new family of copulas—the *Singular Mixture Copulas*. These copulas result from a combination of the above-mentioned methods. In Sect. 8.2 we construct singular copulas whose supports lie on the graphs of two given quantile functions. These copulas are then mixed with respect to a continuous distribution resulting in an absolutely continuous parametric copula (Sect. 8.3).

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P. Jaworski et al. (eds.), *Copulae in Mathematical and Quantitative Finance*, Lecture Notes in Statistics 213, DOI 10.1007/978-3-642-35407-6_8,

As mixing distribution we particularly use a generalized beta distribution, i.e., a linear transformation of a beta distribution. Section 8.4 summarizes the results and gives an outlook on some extensions of this approach.

8.2 Singular Copulas

Let F be a continuous distribution function on $[0, 1]$ and let α be some constant in $]0, 1[$. Then there exists a continuous function G such that

$$\alpha F(x) + (1-\alpha)G(x) = x \tag{8.1}$$

for all $x \in [0, 1]$. The function G is given by

$$G(x) = \frac{x - \alpha F(x)}{1-\alpha}. \tag{8.2}$$

In general, G is not necessarily a distribution function. However, we are interested in exactly this case.

Let us assume for a moment that G is also a distribution function. Let X be a random variable with a continuous uniform distribution on $[0, 1]$, and let I be a random variable, independent of X, with a binomial $B(1, \alpha)$-distribution. Define the random variable Y via

$$Y := I \cdot F^{-1}(X) + (1-I) \cdot G^{-1}(X). \tag{8.3}$$

Easy calculations show that Y follows a continuous uniform distribution on $[0, 1]$. As a consequence the distribution function of (X, Y) is a certain singular copula. So with two distribution functions F and G satisfying (8.1) we can construct singular copulas. Those copulas are given by

$$\begin{aligned} C_{XY}(x, y) &= \mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x, I \cdot F^{-1}(X) + (1-I) \cdot G^{-1}(X) \le y) \\ &= \mathbb{P}(I = 1)\mathbb{P}(X \le x, X \le F(y)) + \mathbb{P}(I = 0)\mathbb{P}(X \le x, X \le G(y)) \\ &= \alpha \min(x, F(y)) + (1-\alpha)\min(x, G(y)). \end{aligned}$$

As mentioned above, G is not necessarily a distribution function, so we have to make assumptions on F to guarantee that G is also a distribution function.

Lemma 8.1. *Let F be a differentiable distribution function on $[0, 1]$. Then the function G given by (8.2) is a differentiable distribution function on $[0, 1]$ if and only if $F'(x) \le \frac{1}{\alpha}$ for all $x \in [0, 1]$.*

Proof. From $F(0) = 0$ and $F(1) = 1$ it follows immediately that $G(0) = 0$ and $G(1) = 1$. From (8.2) we have

$$G'(x) = \frac{1-\alpha F'(x)}{1-\alpha}, \tag{8.4}$$

so that $G'(x) \geq 0 \Leftrightarrow F'(x) \leq \frac{1}{\alpha}$, which completes the proof. □

In a more general approach we can formulate the following theorem which follows from Lemma 8.1 and the construction discussed above.

Theorem 8.1. *Let F be a differentiable function on $[0,1]$ and let X be a random variable with a continuous uniform distribution on $[0,1]$. Then the distribution function of (X,Y) with Y given by (8.3) and G given by (8.2) is a copula if and only if*

(i) $F(0) = 0$ *and* $F(1) = 1$,
(ii) $0 \leq F'(x) \leq \frac{1}{\alpha}$ *for all* $x \in [0,1]$.

We denote the class of functions that fulfill the properties (i) and (ii) in Theorem 8.1 by $\mathscr{F}_\alpha$, i.e.,

$$\mathscr{F}_\alpha := \{F : [0,1] \to [0,1] \mid F(0) = 0, F(1) = 1, 0 \leq F'(x) \leq \tfrac{1}{\alpha}\}.$$

Lemma 8.2.

1. *Let F and G be two functions in $\mathscr{F}_\alpha$, then $F \cdot G$ is in $\mathscr{F}_{\frac{\alpha}{2}}$.*
2. *Let F and G be two functions in $\mathscr{F}_\alpha$ and $\theta \in [0,1]$, then $\theta F + (1-\theta)G$ is in $\mathscr{F}_\alpha$.*
3. *Let $F_1, F_2, \ldots$ be functions in $\mathscr{F}_\alpha$ with $\lim_{n\to\infty} F_n = F$, where the convergence is uniform, then F is in $\mathscr{F}_\alpha$.*
4. *Let α and β be some constants in $[0,1]$ with $\alpha \leq \beta$, then $\mathscr{F}_\beta \subseteq \mathscr{F}_\alpha$.*
5. *Let F be a function in $\mathscr{F}_\alpha$, then G given by (8.2) is an element of $\mathscr{F}_{1-\alpha}$.*

Proof. The proof is straightforward. □

Example 8.1. Let F be a rational function given by $F(x) = (ax+b)/(cx+d)$. For which coefficients is F an element of $\mathscr{F}_\alpha$? From $F(0) \stackrel{!}{=} 0$ it follows that $b = 0$ and from $F(1) \stackrel{!}{=} 1$ it follows that $a = c + d$. Consequently, without loss of generality F can be written as $F(x) = ((c+1)x)/(cx+1)$. From the conditions on F' it follows that F is in $\mathscr{F}_\alpha$ if and only if $c \in [\alpha - 1, \frac{1}{\alpha} - 1]$.

Example 8.2. Let F be a quadratic function given by $F(x) = a_2x^2 + a_1x + a_0$. For which coefficients is F an element of $\mathscr{F}_\alpha$? From $F(0) \stackrel{!}{=} 0$ it follows that $a_0 = 0$ and from $F(1) \stackrel{!}{=} 1$ it follows that $a_2 + a_1 = 1$. As a consequence we have $F'(x) = 2a_2x + 1 - a_2$. In order to satisfy $F'(x) \geq 0$ the coefficient a_2 has to be in $[-1,1]$. To fulfill the condition $F'(x) \leq \frac{1}{\alpha}$ easy calculations show that a_2 has to be an element of $[1 - \frac{1}{\alpha}, \frac{1}{\alpha} - 1]$. Altogether we can conclude that the quadratic function F given by $F(x) = ax^2 + (1-a)x$ is in $\mathscr{F}_\alpha$ if and only if $a \in [\max(-1, 1 - \frac{1}{\alpha}), \min(1, \frac{1}{\alpha} - 1)]$. Figure 8.1 shows the functions F given by $F(x) = x^2$ and G given by (8.2) for different values of α.

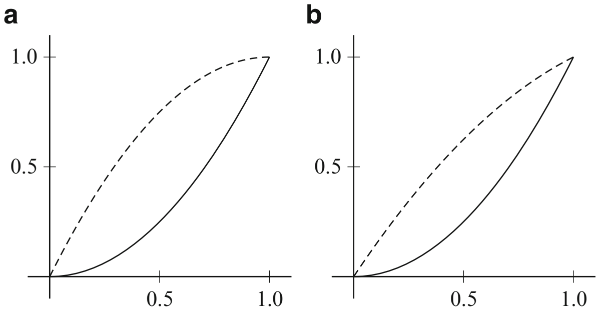

Fig. 8.1 F (*solid*), given by $F(x) = x^2$ and G, given by (8.2) for different values of α. (**a**) $\alpha = \frac{1}{2}$. (**b**) $\alpha = \frac{1}{3}$

Remark 8.1. The copula C_{XY} is a special case of the construction presented in [2] for the choice of $f_1 = f_2 = id_{[0,1]}$, $g_1 = F$, $g_2 = G$, $A(u,v) = B(u,v) = \min(u,v)$ and $H(x,y) = \alpha x + (1-\alpha)y$. In this setting (8.1) can be obtained from Theorems 1 and 2 of [2].

In [3, 4] a copula $K_{\delta,\lambda}$ is presented that is given as follows

$$K_{\delta,\lambda}(x,y) = \min\{x, y, \lambda\delta(x) + (1-\lambda)\delta(y)\}, \tag{8.5}$$

where δ is the diagonal section of a copula and λ is a constant that lies in an interval that is dependent on δ. They also show that $K_{\delta,\lambda}$ has a diagonal section equal to δ. Although the definitions of the copula $K_{\delta,\lambda}$ and C_{XY} might seem similar, they are not identical.

Remark 8.2. The copula $K_{\delta,\lambda}$ and the copula C_{XY} are essentially different.

Proof. Without loss of generality let $F(y) \leq y$ for all $y \in [0,1]$, then $G(y) \geq y$ for all $y \in [0,1]$. Choose x, y in a way that $y < x < G(y)$ holds. If $K_{\delta,\lambda}$ were equal to C_{XY}, then the diagonal δ of $K_{\delta,\lambda}$ would be given by $\delta(x) = C_{XY}(x,x) = \alpha F(x) + (1-\alpha)x$. Consequently, the following equations would hold

$$C_{XY}(x,y) = \alpha F(y) + (1-\alpha)x,$$
$$K_{\delta,\lambda}(x,y) = \min\{y, \alpha\lambda F(x) + \lambda(1-\alpha)x + (1-\lambda)\alpha F(y) + (1-\alpha)(1-\lambda)y\}.$$

Obviously, the equation $y = \alpha F(y) + (1-\alpha)x$ does not hold for arbitrary x, y with $y < x < G(y)$, so it must hold

$$\alpha F(x) + (1-\alpha)x = \alpha\lambda F(x) + \lambda(1-\alpha)x + (1-\lambda)\alpha F(y) + (1-\alpha)(1-\lambda)y \tag{8.6}$$

in order to satisfy $K_{\delta,\lambda} = C_{XY}$. Equation 8.6 is equivalent to

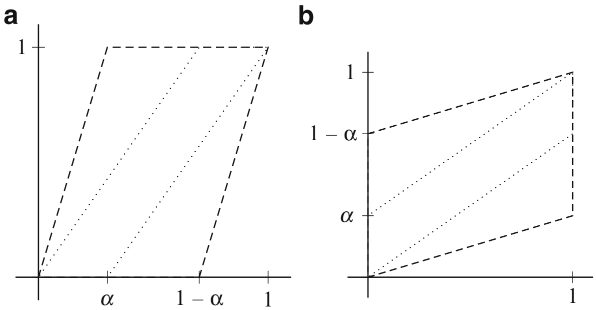

Fig. 8.2 (**a**) Borders for F and G. (**b**) Borders for the copula

$$\alpha\lambda(F(y) - F(x)) + (1-\lambda)(1-\alpha)(x-y) = 0, \tag{8.7}$$

which can be written as

$$\lambda\left(1 + \frac{(1-\alpha)(x-y)}{\alpha(F(x)-F(y))}\right) = \frac{(1-\alpha)(x-y)}{\alpha(F(x)-F(y))}, \tag{8.8}$$

since $F(x) \neq F(y)$ due to (8.7) and $y < x$. From (8.8) we can conclude[1] that

$$\lambda = \frac{1}{1 + \frac{\alpha(F(x)-F(y))}{(1-\alpha)(x-y)}}. \tag{8.9}$$

Since the right-hand side of the last equation is not constant for any nonlinear function F the statement follows. □

The support of the constructed copula C_{XY} always lies on the graphs of the functions F^{-1} and G^{-1}. Given a fixed α due to the restrictions on F (and G) there are points in $[0,1]^2$ which cannot be part of the support of the copula, regardless of which function $F \in \mathscr{F}_\alpha$ is chosen. Part (a) of Fig. 8.2 shows the borders in which the graphs of F (dashed line) and G (dotted line) have to lie. Having the borders of F and G it is easy to calculate the borders in which the support of the copula has to lie (see part (b) of Fig. 8.2). The function F has to fulfill the condition $F'(x) \leq \frac{1}{\alpha}$ for all $x \in [0,1]$, as a consequence points in the triangle $(1-\alpha,0)(1,1)(1,0)$ or the triangle $(0,0)(0,1)(\alpha,1)$ cannot lie on the graph of F. Analogously, the borders for G can be obtained.

[1]The term in brackets on the left-hand side of (8.8) is unequal to zero because otherwise it would follow that $0 = -1$.

8.3 Singular Mixture Copulas

In this section we construct the convex sums (see [10]) of the singular copulas presented in Sect. 8.2. We start with a description of the general construction and subsequently consider specific mixing distributions.

8.3.1 General Construction

Consider a family $\{F_\omega\} \subset \mathscr{F}_\alpha$ of distribution functions, then for a fixed ω we can construct a singular copula $\check{C}_\omega$ using F_ω and G_ω given by

$$G_\omega(y) = \frac{y - \alpha F_\omega(y)}{1-\alpha}.$$

The copula $\check{C}_\omega$ is the distribution function of the random vector (X, Y) where X is uniformly distributed on $[0, 1]$ and Y is given by

$$Y := I \cdot F_\omega^{-1}(X) + (1 - I) \cdot G_\omega^{-1}(X),$$

with $I \sim \mathscr{B}(1, \alpha)$. If Ω is a real-valued random variable and $F_\omega \in \mathscr{F}_\alpha$ for all observations ω of Ω, then the convex sum of $\{\check{C}_\omega\}$ is given by

$$\begin{aligned}\dot{C}(x, y) &= \int \check{C}_\omega(x, y)\mathbb{P}^\Omega(d\omega)\\ &= \alpha \int \min(x, F_\omega(y))\mathbb{P}^\Omega(d\omega) + (1-\alpha)\int \min(x, G_\omega(y))\mathbb{P}^\Omega(d\omega).\end{aligned}$$

Especially, consider the family of distribution functions F_ω given by $F_\omega(y) = \omega y^2 + (1-\omega)y$ with $\omega \in [-1, 1]$. Let $0 < \alpha \le \frac{1}{2}$, then F_ω is an element of $\mathscr{F}_\alpha$ for all $\omega \in [-1, 1]$ (see Example 8.2). Let Ω be a random variable with values in $[-1, 1]$, then the Singular Mixture Copula resulting from the family $\{F_\omega\}_{\omega\in[-1,1]}$ is given by

$$C_\alpha(x, y) = \mathbb{P}(X \le x, Y \le y)$$
$$= \begin{cases} x & ,(x, y) \in A_1,\\ x + \alpha\left((x-y)\left(F_\Omega(\beta) - 1\right) + (y^2 - y)\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right) & ,(x, y) \in A_2,\\ \alpha\left((x-y)F_\Omega(\beta) + y + (y^2-y)\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right) & \\ +(1-\alpha)\left(x + (y-x)F_\Omega(b)\right) + \alpha(y - y^2)\int_{-1}^b \omega\mathbb{P}^\Omega(d\omega) & ,(x, y) \in A_3,\\ \alpha(x-y)F_\Omega(\beta) + y + \alpha(y-y^2)\int_{-1}^\beta \omega\mathbb{P}^\Omega(d\omega) & ,(x, y) \in A_4,\\ y & ,(x, y) \in A_5,\end{cases}$$

where $\beta = \frac{x-y}{y^2-y}$, $b = \beta\frac{\alpha-1}{\alpha}$ and

$$A_1 = \left\{(x,y) \in [0,1]^2 \,|x < y^2\right\},$$
$$A_2 = \left\{(x,y) \in [0,1]^2 \,|y^2 \leq x < \frac{-\alpha}{1-\alpha}(y-y^2) + y\right\},$$
$$A_3 = \left\{(x,y) \in [0,1]^2 \,|\frac{-\alpha}{1-\alpha}(y-y^2) + y \leq x < \frac{\alpha}{1-\alpha}(y-y^2) + y\right\},$$
$$A_4 = \left\{(x,y) \in [0,1]^2 \,|\frac{\alpha}{1-\alpha}(y-y^2) + y \leq x < 2y - y^2\right\},$$
$$A_5 = \left\{(x,y) \in [0,1]^2 \,|2y - y^2 \leq x\right\}.$$

The density of the copula is given by

$$c_\alpha(x,y) = \begin{cases} 0 & ,(x,y) \in A_1, \\ \alpha f_\Omega(\beta)\frac{y^2-2xy+x}{(y^2-y)^2} & ,(x,y) \in A_2, \\ \frac{y^2-2xy+x}{(y^2-y)^2}\left(\alpha f_\Omega(\beta) + \frac{(1-\alpha)^2}{\alpha} f_\Omega(b)\right) & ,(x,y) \in A_3, \\ \alpha f_\Omega(\beta)\frac{y^2-2xy+x}{(y^2-y)^2} & ,(x,y) \in A_4, \\ 0 & ,(x,y) \in A_5. \end{cases}$$

Remark 8.3. For $\alpha > \frac{1}{2}$ it is possible to change the distribution of Ω in such a way that one receives the same copulas as for $\alpha < \frac{1}{2}$, so we restrict our investigation to the case $\alpha \leq \frac{1}{2}$.

Theorem 8.2. *The copula C_α has upper and lower tail dependence given by*

$$\lambda_U = 1 - \alpha\left(\int_0^1 \omega\mathbb{P}^\Omega(d\omega) - \int_{-1}^0 \omega\mathbb{P}^\Omega(d\omega)\right) = \lambda_L.$$

Proof. The proof is straightforward. □

Since Singular Mixture Copulas are convex sums of the singular copulas mentioned in Sect. 8.2 the borders described in part (b) of Fig. 8.2 are also valid for Singular Mixture Copulas. Moreover, because Singular Mixture Copulas are absolutely continuous we are able to compare the area of the copula's support with the area of the unit square. From the discussion in Sect. 8.2 we know that the support cannot lie in the triangles $(0,0)(0,1)(\alpha,1)$ and $(1-\alpha,0)(1,0)(1,1)$. Consequently, the parallelogram in which the support of the Singular Mixture Copula can lie has an area of $\max(\alpha, 1-\alpha)$.

In the special case where F_ω is a quadratic function for every ω the support of the Singular Mixture Copula is bounded by the inverses of the functions $F_{-1}(x) = 2x - x^2$ and $F_1(x) = x^2$, respectively. Here the support has an area of $\frac{1}{3}$.

8.3.2 Special Cases

In the above-mentioned construction the mixing distribution has to be concentrated on a finite interval. Therefore a generalized beta distribution, viz. a linear transformation of a beta distribution, provides a reasonable choice as a mixing distribution. Moreover, the beta distribution is very flexible so the resulting Singular Mixture Copulas should also show this flexibility.

Figures 8.3 and 8.4 show scatter plots of simulated Singular Mixture Copulas with a generalized beta distribution as mixing distribution.

Theorem 8.3. *Let $C_{\alpha,p,q}$ denote a Singular Mixture Copula with a Beta$(-1,1,p,q)$ mixing distribution. Then the survival copula of $C_{\alpha,p,q}$ is given by $\hat{C}_{\alpha,p,q} = C_{\alpha,q,p}$.*

Proof. The proof is straightforward. □

Another possible mixing distribution is a uniform distribution on the interval $[-\theta,\theta]$ with $\theta \leq 1$. The choice $\theta = 1$ would be a special case of the above-mentioned generalized beta distribution. Here, the copula, which we will denote with $C_{\alpha,\theta}$, and its density have a closed form representation and the upper and lower tail dependence coefficients can be determined. See [8] for the proofs of this subsection. The copula $C_{\alpha,\theta}$ is given by

$$C_{\alpha,\theta}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$
$$= \begin{cases} x & ,(x,y) \in A_1, \\ \frac{\alpha}{4\theta}\left(\frac{(x-y)^2}{y^2-y} + 2\theta(x+y) + \theta^2(y^2-y)\right) + (1-\alpha)x & ,(x,y) \in A_2, \\ \frac{1}{2}\left(\frac{(x-y)^2}{2\theta(y^2-y)}\left(\frac{(1-\alpha)^2}{\alpha} + \alpha\right) + x + (1-\alpha\theta)y + \alpha\theta y^2\right) & ,(x,y) \in A_3, \\ \frac{\alpha}{4\theta}\left(\frac{(x-y)^2}{y^2-y} + 2\theta(x+y) + \theta^2(y^2-y)\right) + (1-\alpha)y & ,(x,y) \in A_4, \\ y & ,(x,y) \in A_5, \end{cases}$$

where

$$A_1 = \left\{(x,y) \in [0,1]^2 \,|x \leq -\theta(y-y^2) + y\right\},$$
$$A_2 = \left\{(x,y) \in [0,1]^2 \,| -\theta(y-y^2) + y < x < -\theta\frac{\alpha}{1-\alpha}(y-y^2) + y\right\},$$
$$A_3 = \left\{(x,y) \in [0,1]^2 \,| -\theta\frac{\alpha}{1-\alpha}(y-y^2) + y < x < \theta\frac{\alpha}{1-\alpha}(y-y^2) + y\right\},$$
$$A_4 = \left\{(x,y) \in [0,1]^2 \,|\theta\frac{\alpha}{1-\alpha}(y-y^2) + y < x < \theta(y-y^2) + y\right\},$$
$$A_5 = \left\{(x,y) \in [0,1]^2 \,|x \geq \theta(y-y^2) + y\right\}.$$

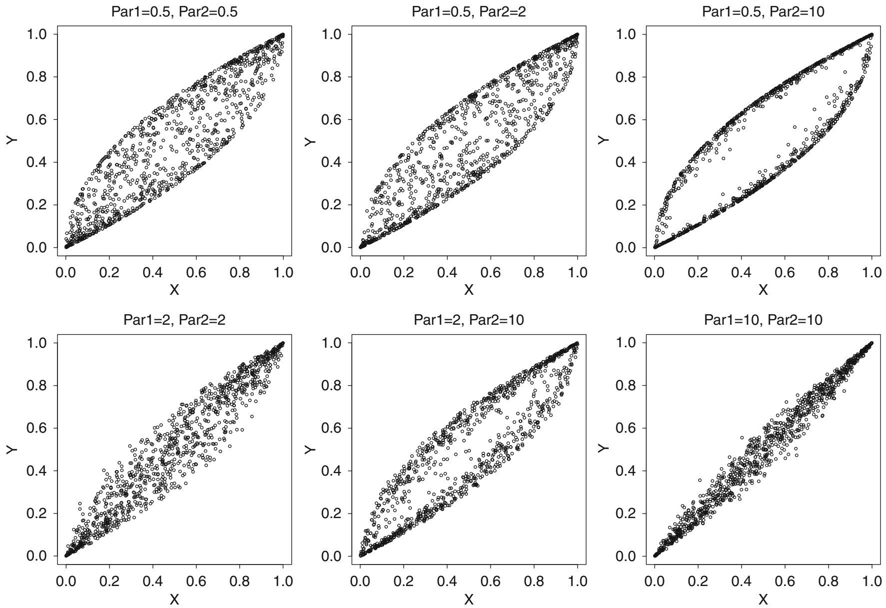

Fig. 8.3 Scatter plots of simulated points from a Singular Mixture Copula with generalized beta mixing distribution for $\alpha = 0.5$ and different shape parameters

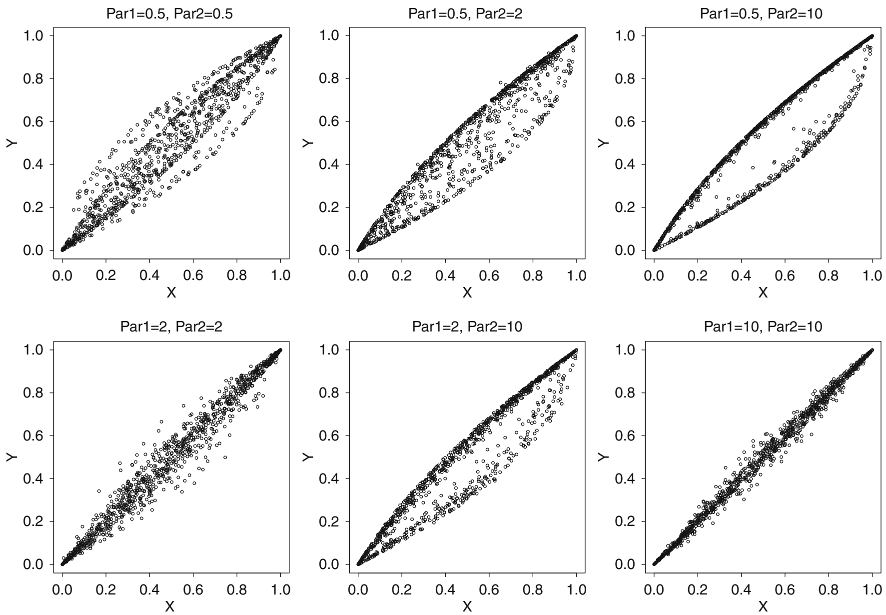

Fig. 8.4 Scatter plots of simulated points from a Singular Mixture Copula with generalized beta mixing distribution for $\alpha = 0.3$ and different shape parameters

The density of this copula is given by

$$c_{\alpha,\theta}(x,y)=\begin{cases}\frac{\alpha}{2\theta}\frac{y^2-2yx+x}{(y^2-y)^2} & ,(x,y)\in A_2,\\ \frac{1}{2\theta}\left(\frac{(1-\alpha)^2}{\alpha}+\alpha\right)\frac{y^2-2yx+x}{(y^2-y)^2} & ,(x,y)\in A_3,\\ \frac{\alpha}{2\theta}\frac{y^2-2yx+x}{(y^2-y)^2} & ,(x,y)\in A_4,\\ 0 & ,\ \text{otherwise.}\end{cases}$$

Theorem 8.4. *The copula $C_{\alpha,\theta}$ has upper and lower tail dependence given by*

$$\lambda_U=1-\frac{\alpha\theta}{2}=\lambda_L.$$

Theorem 8.5. *The copula $C_{\alpha,\theta}$ is radially symmetric, i.e., $C_{\alpha,\theta}=\hat{C}_{\alpha,\theta}$.*

Theorem 8.6. *The concordance measures Kendall's tau and Spearman's rho for the copula $C_{\alpha,\theta}$ are given by*

$$\tau_{\alpha,\theta}=1-\alpha\theta\frac{1+4(\alpha-1)^2}{9(1-\alpha)}\ \textit{and}\ \rho_{\alpha,\theta}=1-\frac{\alpha\theta^2}{15(1-\alpha)},$$

respectively.

Corollary 8.1. *Kendall's tau for the copula $C_{\alpha,\theta}$ lies in the interval $[\frac{7}{9},1]$, Spearman's rho for the copula $C_{\alpha,\theta}$ lies in the interval $[\frac{14}{15},1]$.*

8.4 Concluding Remarks

In this paper we presented a method for the construction of nonsingular copulas by mixing a family of singular copulas. We also showed how the constructed singular copulas differ from similar constructions in the literature. These copulas can be used to model strongly dependent random variables (see [8]).

In the future we want to investigate generalizations of the presented method, e.g., one could replace the quadratic functions in the definition of the singular copulas with other functions or use other mixing distributions.

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