Scand. Actuarial J. 1984: 127–129

A Note on Random Time Changes of Markov Chains

By D. Pfeifer

Abstract

We present simple conditions under which Markov time changes are obtained, and give formulae for the resulting transition probabilities.

1. Introduction

Let $\mathbf{N}$ denote the set of positive integers. We consider random subsequences $\{X_{T_n}; n \in \mathbf{N}\}$ of a time-homogeneous Markov chain $\{X_n; n \in \mathbf{N}\}$, defined on a probability space $(\Omega, \mathscr{A}, P)$ with arbitrary state space $(\mathscr{X}, \mathscr{B})$, where $\{T_n; n \in \mathbf{N}\}$ is a strictly increasing sequence of Markov times. Simple conditions are given under which $\{X_{T_n}; n \in \mathbf{N}\}$ again is a Markov chain, and formulae for the resulting transition probabilities are presented. This completes results of Pittenger (1982) who considers similar problems, however restricted to a countable state space. In what follows X^T will denote the Markov chain $\{X_{T+n}; n \in \mathbf{N}\}$ for a Markov time T (cf. Revuz, 1975), and $\sigma(X)$ will denote the σ-algebra generated by the random variable X.

2. Main results

Theorem. *If $T_{n+1} - T_n$ is measurable with respect to $\sigma(X^{T_n})$ for all $n \in \mathbf{N}$, then $\{X_{T_n}; n \in \mathbf{N}\}$ and $\{(T_n, X_{T_n}); n \in \mathbf{N}\}$ both are (possibly non-homogeneous) Markov chains.*

Proof. For any $B \in \mathscr{B}$, $n \in \mathbf{N}$,

$$\{X_{T_{n+1}} \in B\} = \bigcup_{k=1}^{\infty} \{X_{T_n+k} \in B,\ T_{n+1} - T_n = k\} \in \sigma(X^{T_n}), \tag{1}$$

hence by the strong Markov property,

$$\begin{aligned} P(X_{T_{n+1}} \in B | X_{T_1}, \dots, X_{T_n}) &= E[P(X_{T_{n+1}} \in B | \sigma(X_k;\ k \leq T_n)) | X_{T_1}, \dots, X_{T_n}] \\ &= E[P(X_{T_{n+1}} \in B | X_{T_n}) | X_{T_1}, \dots, X_{T_n}] \\ &= P(X_{T_{n+1}} \in B | X_{T_n}) \quad \text{a.s.} \end{aligned} \tag{2}$$

which says that $\{X_{T_n}; n \in \mathbf{N}\}$ is a Markov chain. Replacing X_n by (n, X_n) now also gives the Markov property for $\{(T_n, X_{T_n}); n \in \mathbf{N}\}$.

In fact, the measurability property of the Theorem is equivalent to the existence of measureable N-valued functions $\{f_n;\ n \in \mathbf{N}\}$ such that

$$T_1 = f_1(X_1, X_2, \ldots), \quad T_{n+1} = T_n + f_{n+1}(X^{T_n}), \quad n \in \mathbf{N} \tag{3}$$

(cf. Billingsley, 1979, Problem 13.6). In this setting, the transition probabilities of $\{X_{T_n};\ n \in \mathbf{N}\}$ and $\{(T_n, X_{T_n});\ n \in \mathbf{N}\}$ are readily obtained.

Corollary. *Under the conditions of the Theorem,*

$$P(T_{n+1} = k,\ X_{T_{n+1}} \in B | T_n = m,\ X_{T_n} = x)$$
$$= P(f_{n+1}(X^1) = k-m,\ X_{k-m+1} \in B | X_1 = x) \quad \text{a.s.} \tag{4}$$

$$P(X_{T_{n+1}} \in B | X_{T_n} = x) = \sum_{j=1}^{\infty} P(f_{n+1}(X^1) = j,\ X_{j+1} \in B | X_1 = x) \quad \text{a.s.} \tag{5}$$

for $n \in \mathbf{N}$, $m<k$, $B \in \mathcal{B}$, $x \in \mathcal{X}$. *Also,* $T_1, T_2 - T_1, \ldots, T_{n+1} - T_n$ *are conditionally independent given* $X_{T_1}, \ldots, X_{T_n}$.

Proof. This follows immediately from (3) and the homogeneity assumptions made on $\{X_n;\ n \in \mathbf{N}\}$.

As an example, relations (4) and (5) provide simple expressions for the transition probabilities of the record value sequence of a Markov chain which was investigated by Biondini & Siddiqui (1973). For this purpose, let $\{X_n;\ n \in \mathbf{N}\}$ be real-valued such that $\limsup_{n\to\infty} X_n = \infty$ a.s. Define

$$T_1 = 1, \quad T_{n+1} = \inf\{k > T_n | X_k > X_{T_n}\}$$
$$= T_n + \inf\{k \in \mathbf{N} | X_{T_n + k} > X_{T_n}\}.$$

Then the record times $\{T_n,\ n \in \mathbf{N}\}$ are Markov times, and by the Theorem and (3), the record value sequence $\{X_{T_n};\ n \in \mathbf{N}\}$ as well as $\{(T_n, X_{T_n});\ n \in \mathbf{N}\}$ are Markov chains with

$$P(T_{n+1} = k,\ X_{T_{n+1}} \in B | T_n = m,\ X_{T_n} = x)$$
$$= P(X_2, \ldots, X_{k-m} \leqslant x < X_{k-m+1} \in B | X_1 = x) \quad \text{a.s.} \tag{6}$$

$$P(X_{T_{n+1}} \in B | X_{T_n} = x) = \sum_{j=1}^{\infty} P(X_2, \ldots, X_j \leqslant x < X_{j+1} \in B | X_1 = x) \quad \text{a.s.}$$

for $n \in \mathbf{N}$, $m<k$, B a Borel set, $x \in \mathbf{R}$.

References

Billingsley, P. (1979). *Probability and measure*. Wiley, New York.

Biondini, R. W. & Siddiqui, M. M. (1975). Record values in Markov sequences. Proceedings

of the Summer Research Institute on Statistical Inference for Stochastic Processes, Bloomington, July 31 – August 9, 1975. In *Statistical inference and related topics*, vol. 2, pp. 291–352, Academic Press, New York.
Pittenger, A. O. (1982). Time changes of Markov chains. *Stoch. Proc. Appl.* **13**, 189–199.
Revuz, D. (1975). *Markov chains*. North-Holland Publ. Comp., Amsterdam.

Dietmar Pfeifer
Institut für Statistik und Wirtschaftsmathematik
RWTH Aachen
Wüllnerstr. 3
D-5100 Aachen
West-Germany