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Generating unfavourable VaR scenarios under Solvency II with patchwork copulas

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Abstract: The central idea of the paper is to present a general simple patchwork construction principle for multivariate copulas that create unfavourable VaR (i.e. Value at Risk) scenarios while maintaining given marginal distributions. This is of particular interest for the construction of Internal Models in the insurance industry under Solvency II in the European Union. Besides this, the Delegated Regulation by the European Commission requires all insurance companies under supervision to consider different risk scenarios in their risk management system for the company's own risk assessment. Since it is unreasonable to assume that the potential worst case scenario will materialize in the company, we think that a modelling of various unfavourable scenarios as described in this paper is likewise appropriate. Our explicit copula approach can be considered as a special case of ordinal sums, which in two dimensions even leads to the technically worst VaR scenario.

Keywords: Solvency II, copulas, patchwork copulas, Bernstein copulas, Monte Carlo methods

MSC: 62H05, 62H12, 62H17, 11K45

1 Introduction

Reasonable VaR-estimates from original data or suitable scenarios for risk management within so-called Internal Models are of particular interest in the insurance industry under Solvency II (see, e.g., [1, 4, 6–8, 21, 31]). In this paper, we propose a simple stochastic Monte Carlo algorithm on patchwork copulas for the generation of VaR scenarios that are suitable for comparison purposes in Internal Models for the calculation of Solvency Capital Requirements (SCR), in particular for the Non-Life Module. Note that in the Standard Formula of Solvency II, there is a formula for the calculation of the non-life premium and reserve risk SCR given by the volume factor

$$\rho_{1-\alpha}(\sigma)_{\mathrm{VaR}} = \frac{\exp\left(k_{1-\alpha}\sqrt{\ln\left(1+\sigma^2\right)}\right)}{\sqrt{1+\sigma^2}} - 1$$

applied to the volume measure (i.e. premium income) of the year considered (see, e.g., [31, p. 324, relation (21.9b)]; cf. also [16, p. 329 ff.]). Here α denotes the risk level (i.e. 0.5% in Solvency II) and $k_{1-\alpha}$ the corresponding $1-\alpha$ quantile of the standard normal distribution. Further, σ denotes the standard deviation of the underlying risk, i.e. the ultimate combined loss ratio, which is assumed to be lognormally distributed with expectation 1≙100% (which is the limit towards certain ruin according to the law of large numbers). However, this formula is questionable from a scientific point of view (see [14]). Note also that this formula was simplified in the Commission Delegated Regulation of the EU [12, Article 115]:

$$\rho_{1-\alpha}(\sigma)_{\mathrm{VaR}} \approx 3\sigma \quad \text{for} \quad \alpha = 0.005.$$

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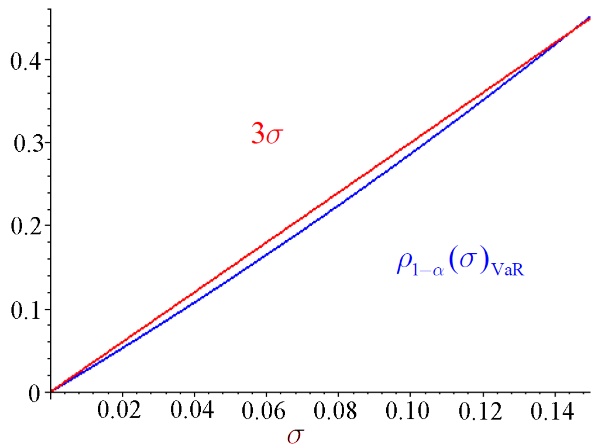

Figure 1: Plot of the Non-Life SCR volume factor $\rho_{1-\alpha}(\sigma)_{\text{VaR}}$ vs. its simplification 3σ.

This is a reasonable conservative approximation as long as $\rho < 0.15$ (see Figure 1).

Another questionable point here is the aggregation to the overall SCR from different module SCR's by correlations in Solvency II (see, e.g., [31]). This has been discussed in detail, e.g., in [24, 29].

Note that no official legislative paper on Solvency II contains a strict mathematical definition of the underlying risk measure Value at Risk, cf. [11, Article 104, L 335/52, No. 4] or the Commission Delegated Regulation of the EU [12, L 12/20 (53)]. The wording used in these documents, however, suggests that "the Value-at-Risk measure with a 99.5% confidence level" is the corresponding lower quantile of the risk distribution.

Note also that the above-mentioned Commission Delegated Regulation [12] concerning the implementation of Solvency II in the EU requires the consideration of risk scenarios in several Articles, in particular in Article 259, L 12/161 on Risk Management Systems saying that insurance and reinsurance undertakings shall, where appropriate, include performance of stress tests and scenario analyses with regard to all relevant risks faced by the undertaking, in their risk-management system. The results of such analyses also have to be reported in the ORSA (Own Risk and Solvency Assessment, see, e.g., [23]) as described in Article 306 of the Commission Delegated Regulation of the EU [12]. In the light of the outlined structural problems with the standard formula above, the ORSA is probably a better instrument to rate the enterprise's risks in a more reliable way. The problem is, however, that the Commission Delegated Regulation does not make any clear statements on how such stress tests or scenario analyses have to be performed.

Article 1 of the Commission Delegated Regulation of the EU [12, L 12/20, No. 2] defines a "scenario analysis" as an analysis of the impact of a combination of adverse events. The Monte Carlo simulation algorithm developed in this paper allows for a mathematically rigorous description how such scenarios can be generated, being flexible enough to cover also extreme situations.

In what follows, we shall focus mainly on the Non-Life Modules under Solvency II. Therefore, we only consider continuous risk distributions. In this case, VaR is simply a lower quantile of the cumulative risk distribution function. For corresponding considerations for the Life and Capital Asset Modules under Solvency II, we refer to [3, 32].

Besides Solvency II aspects, the method proposed in this paper might also be of interest for reinsurance companies for the risk assessment of statistically dependent natural perils like windstorm, hail or flooding triggered by adverse climate conditions.

2 Unfavourable patchwork copulas

Patchwork copulas in the context of risk management have been treated in detail in [1, 5, 15, 24–26, 30], among others. In several of the cited papers the question of an unfavourable, i.e. superadditive VaR estimate for a

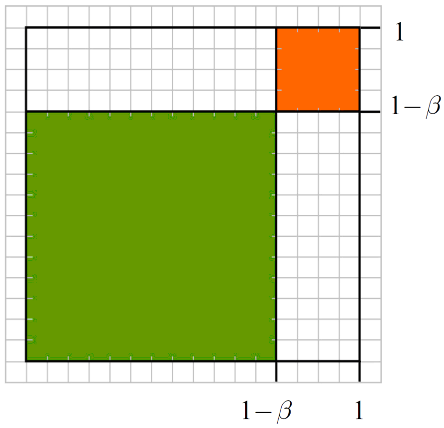

Figure 2: Shape of the support of the underlying patchwork copula **W**.

portfolio of aggregated risks was in particular emphasized, see also [27]. However, the construction of worst VaR scenarios in this context is quite complicated; note that a worst VaR is a supremum of VaR's over the Fréchet class of all possible joint distributions with given marginals (see, e.g., [10, Sections 1.1 and 1.3]). The situation is more simple in the two-dimensional case with identical margins (see [10, Section 2]). A numerical approach to a constructive solution to the general problem is given, e.g., by the rearrangement algorithm (see, e.g., [1, 10, 20]). From a practical point of view, simpler and yet explicit constructions for unfavourable but not necessarily worst VaR estimates by appropriate copula constructions seem to be a useful alternative. In this paper, we describe how such a construction could be performed. We start with an explicit approach in two dimensions, which is later extended to arbitrary dimensions. For better readability, all proofs are shifted to the Appendix.

Lemma 1. *Let, for $d \geq 2$, $d \in \mathbb{N}$, $\mathbf{U} = (U_1, \ldots, U_d)$ and $\mathbf{V} = (V_1, \ldots, V_d)$ be d-dimensional random vectors over $[0,1]^d$ with continuous uniform margins (i.e., **U** and **V** represent d-dimensional copulas). Let further I denote a binomially distributed random variable, independent of **U** and **V**, with $\mathbb{P}(I = 1) = p \in (0,1)$. Then the random vector **W** with components $W_i := IpU_i + (1-I)[p + (1-p)V_i]$ for $1 \leq i \leq d$ also has continuous uniform margins, i.e. **W** represents a d-dimensional copula.*

Note that **W** can be considered as a special case of ordinal sums (cf. [22, Chapter 3.3.2] for the two-dimensional case, and [18, relation (4.31)], [19, Definition 2.1] and [9, Example 2.2.10 and Chapter 3.8] for the multivariate case).

Suppose now that a portfolio of d insurance risks is considered where a mutual probabilistic dependence structure is assumed to be described by **U**. If the d (for simplicity assumed continuous) marginal risk distribution functions are denoted by $F_1, \ldots, F_d$ and by $Q_1, \ldots, Q_d$ their pseudo-inverses (quantile functions), then both random vectors $\big(Q_1(U_1), \ldots, Q_d(U_d)\big)$ and $\big(Q_1(W_1), \ldots, Q_d(W_d)\big)$ represent a risk vector $\mathbf{X} = (X_1, \ldots, X_d)$ with the given marginal distributions. However, w.r.t. to risk aggregation, $\mathbf{X} := \big(Q_1(W_1), \ldots, Q_d(W_d)\big)$ creates in general an unfavourable VaR scenario for $S = \sum_{i=1}^d X_i$, even if p is close to 1 and therefore **U** and **W** differ only marginally. The graph in Figure 2 shows the corresponding support of **W** in two dimensions.

In the sequel, put $p := 1 - \beta$ for $0 < \beta < 1$. Then $\mathbf{W} = I(1-\beta)\mathbf{U} + (1-I)(1-\beta+\beta\mathbf{V})$.

We start with some further preliminary Lemmata.

Lemma 2. *Let $\mathbf{W}_1 := (1-\beta)\mathbf{U}$, $\mathbf{W}_2 := 1-\beta+\beta\mathbf{V}$, $Z_{1i} := Q_i(W_{1i})$ and $Z_{2i} := Q_i(W_{2i})$, $i = 1, 2$. Then there hold*

$$F_{Z_{1i}}(x,\beta) = \begin{cases} \dfrac{F_i(x)}{1-\beta}, & 0 \leq x \leq Q_i(1-\beta), \\ 1, & x \geq Q_i(1-\beta), \end{cases} \quad \text{and} \quad F_{Z_{2i}}(x,\beta) = \begin{cases} 0, & 0 \leq x \leq Q_i(1-\beta), \\ \dfrac{F_i(x)+\beta-1}{\beta}, & x \geq Q_i(1-\beta). \end{cases}$$

Lemma 3. *Assume that f and g are Lebesgue densities of independent random variables X and Y, concentrated on the same finite interval $[0, M]$ with $M > 0$. Then $S := X + Y$ has the density h_1 given by*

$$h_1(x) = \begin{cases} \int_0^x f(x-y)g(y)\,\mathrm{d}y, & 0 \le x \le M, \\ \int_{x-M}^M f(x-y)g(y)\,\mathrm{d}y, & M \le x \le 2M. \end{cases}$$

If f and g are concentrated on the same infinite interval $[M, \infty)$ with $M \ge 0$, then $S := X + Y$ has the density h_2 given by

$$h_2(x) = \int_M^{x-M} f(x-y)g(y)\,\mathrm{d}y, \quad x \ge 2M.$$

In particular, if F and G are the corresponding cdf's pertaining to f and g, respectively, then in either case, $\frac{\mathrm{d}}{\mathrm{d}x} F \star G(x)\big|_{x=2M} = 0$, where $\star$ means convolution.

Lemma 4. *Assume that all $F_i \equiv F$ being equal with quantile function Q, and that* **U** *and* **V** *have independent components each. Denote*

$$\underline{F}(x,\beta) := \begin{cases} \dfrac{F(x)}{1-\beta}, & x \le Q(1-\beta), \\ 1, & x \ge Q(1-\beta), \end{cases} \quad \textit{and} \quad \overline{F}(x,\beta) := \frac{F\big(x + Q(1-\beta)\big) + \beta - 1}{\beta}, \quad x \ge 0.$$

Let further denote $X_i := Q(W_i)$ and $S = \sum_{i=1}^d X_i$. Then we can conclude that

$$F_S(x,\beta) = \begin{cases} (1-\beta)\underline{F}^{d\star}(x,\beta), & x \le dQ(1-\beta), \\ (1-\beta) + \beta\overline{F}^{d\star}\big(x - dQ(1-\beta), \beta\big), & x > dQ(1-\beta), \end{cases}$$

where $\star$ again means convolution. If F has a density f, then correspondingly

$$\underline{f}(x,\beta) := \begin{cases} \dfrac{f(x)}{1-\beta}, & x \le Q(1-\beta), \\ 1, & x \ge Q(1-\beta), \end{cases} \quad \textit{and} \quad \overline{f}(x,\beta) := \frac{f\big(x + Q(1-\beta)\big)}{\beta}, \quad x \ge 0,$$

and

$$f_S(x,\beta) = \begin{cases} (1-\beta)\underline{f}^{d\star}(x,\beta), & x \le dQ(1-\beta), \\ (1-\beta) + \beta\overline{f}^{d\star}\big(x - dQ(1-\beta), \beta\big), & x > dQ(1-\beta). \end{cases}$$

The following examples show the effect of a risk aggregation with an unfavourable VaR scenario for two dimensions in detail.

Example 1 (exponential distributions)**.** Assume that

$$F_1 = F_2 = \begin{cases} 0, & x < 0, \\ 1 - e^{-x}, & x \ge 0. \end{cases}$$

Then

$$F_{Z_{1i}}(x,\beta) = \frac{1-e^{-x}}{1-\beta}, \ 0 \le x \le -\ln(\beta), \ \text{ and } \ F_{Z_{2i}}(x,\beta) = \frac{\beta - e^{-x}}{\beta} = 1 - e^{-x-\ln(\beta)}, \ x \ge -\ln(\beta), \ i = 1, 2.$$

For the corresponding densities, we obtain by differentiation

$$f_{Z_{1i}}(x,\beta) = \begin{cases} \dfrac{e^{-x}}{1-\beta}, & 0 \le x \le -\ln(\beta), \\ 0, & x > -\ln(\beta), \end{cases} \quad \text{and} \quad f_{Z_{2i}}(x,\beta) = \begin{cases} 0, & x < -\ln(\beta), \\ e^{-x-\ln(\beta)}, & x \ge -\ln(\beta), \end{cases} \quad i = 1, 2,$$

and

$$\underline{f}(x,\beta) = \begin{cases} \dfrac{e^{-x}}{1-\beta}, & 0 \le x \le -\ln(\beta), \\ 0, & x > -\ln(\beta), \end{cases} \quad \text{and} \quad \overline{f}(x,\beta) = \begin{cases} 0, & x < 0, \\ e^{-x}, & x \ge 0. \end{cases}$$

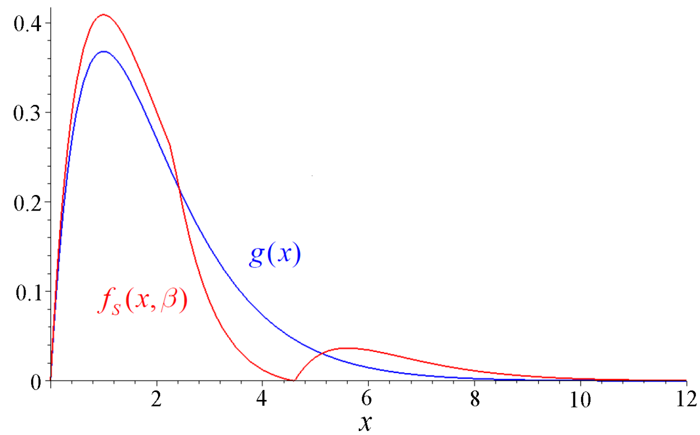

Figure 3: Plots of the densities $f_S(x,\beta)$ for $\beta = 0.1$ and $g(x)$ in Example 1.

By Lemma 4, we obtain the following density f_S of the aggregated risk S:

$$f_S(x,\beta) = \begin{cases} \dfrac{xe^{-x}}{1-\beta}, & 0 \le x \le -\ln(\beta), \\ \dfrac{(-2\ln(\beta) - x)e^{-x}}{1-\beta}, & -\ln(\beta) \le x \le -2\ln(\beta), \\ \dfrac{(x + 2\ln(\beta))e^{-x}}{\beta}, & x \ge -2\ln(\beta), \end{cases}$$

with the corresponding cdf F_S:

$$F_S(x,\beta) = \begin{cases} \dfrac{1-(1+x)e^{-x}}{1-\beta}, & 0 \le x \le -\ln(\beta), \\ \dfrac{1-2\beta+2e^{-x}\ln(\beta)+(1+x)e^{-x}}{1-\beta}, & -\ln(\beta) \le x \le -2\ln(\beta), \\ \dfrac{\beta - 2e^{-x}\ln(\beta) - (1+x)e^{-x}}{\beta}, & x \ge -2\ln(\beta). \end{cases}$$

For the graph in Figure 3, let g denote the density of $T := Q_1(U_1) + Q_2(U_2)$ (independent summands, Gamma distribution). In what follows, let G denote the cdf of $T := Q_1(U_1)+Q_2(U_2)$ (independent summands, Gamma distribution) and H be the cdf of S under the worst VaR scenario (see the graph in Figure 4), i.e. the distribution of **V** corresponds to the lower Fréchet bound or countermonotonicity copula (see, e.g., Remark 3 and the comments after Figure 3 in [10], or [24]). In this case we have

$$H(x,\beta) = \begin{cases} F_S(x), & x \le -2\ln(\beta), \\ 1-\beta, & -2\ln(\beta) \le x \le -2\ln(\beta/2), \\ 1-\beta+\sqrt{\beta^2 - 4e^{-x}}, & x \ge -2\ln(\beta/2). \end{cases}$$

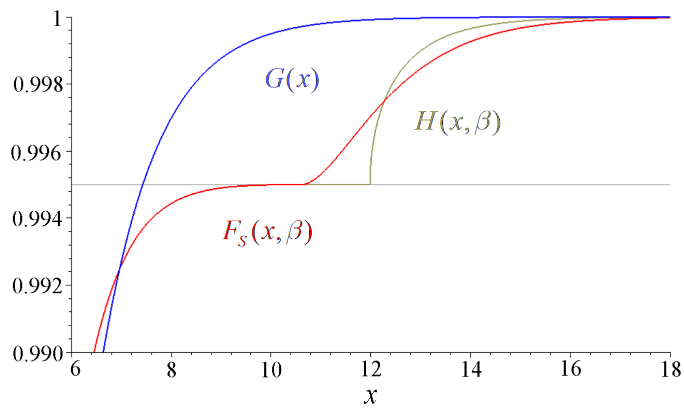

Figure 4: Plots of the cdf's $F_S(x,\beta)$, $G(x)$ and $H(x,\beta)$ for $\beta = 0.005$ in Example 1.

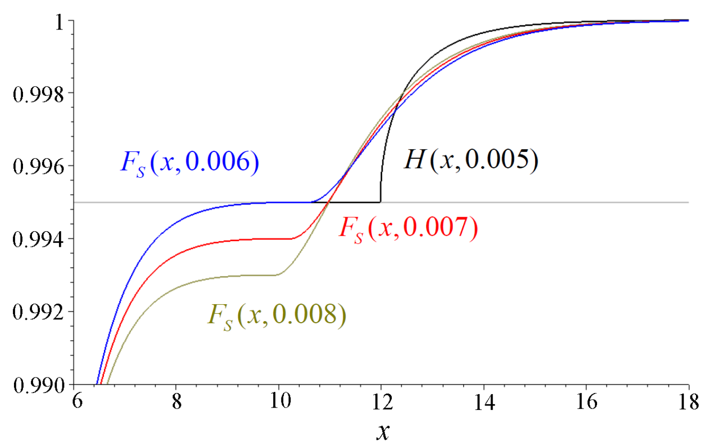

Figure 5: Plots of the cdf's $F_S(x, 0.005+\varepsilon)$ for $\varepsilon \in \{0.001, 0.002, 0.003\}$ and $H(x, 0.005)$ in Example 1.

Note that with the Solvency II standard $\alpha = 0.005$, we get here, for $\beta = \alpha$, $\text{VaR}_\alpha(S) = -2\ln(\beta) = 10.5914 > \text{VaR}_\alpha(T) = 7.4301$, where $\text{VaR}_\alpha(T)$ is the numerical solution to the equation $(1+x)e^{-x} = \alpha$. For the worst VaR scenario, however, we get $\text{wVaR}_\alpha(S) = -2\ln\left(\frac{\beta}{2}\right) = 11.9829$ with $10.5966 = \text{SVaR}_\alpha := \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2) > \text{VaR}_\alpha(S) = 10.5914$. This means that even with the construction for S with $\beta = \alpha$, we still have a (quite small) diversification effect, but not in the worst VaR scenario. This changes, however, if we look at $\text{VaR}_\alpha(S) = 10.9630$ when we replace β by $\alpha + \varepsilon$ in the definition of $\mathbf{W}$ for e.g. $\varepsilon = 0.001$.

The graph in Figure 5 shows the cdf's in the tails for several choices of ε. The graph in Figure 6 shows the values of $Q_S(0.995, \beta) = F_S^{-1}(0.995, \beta)$ in the range $0.0062 \le \beta \le 0.0076$. A numerical calculation shows that for $\alpha = 0.005$ the worst $\text{VaR}_\alpha(S) = 10.9829$ is attained for $\beta = 0.0068$, i.e. $\varepsilon = 0.0018$. Table 1 summarizes the results found for $\alpha = 0.005$.

Table 1: Summarized results for Example 1.

β	0.0050	0.0060	0.0068	0.0070	0.0080
$\text{VaR}_\alpha(S)$	10.5914	10.9630	10.9829	10.9821	10.9618
$\text{VaR}_\alpha(T)$	7.4301	7.4301	7.4301	7.4301	7.4301
$\text{wVaR}_\alpha(S)$	11.9829	11.9829	11.9829	11.9829	11.9829
SVaR_α	10.5966	10.5966	10.5966	10.5966	10.5966

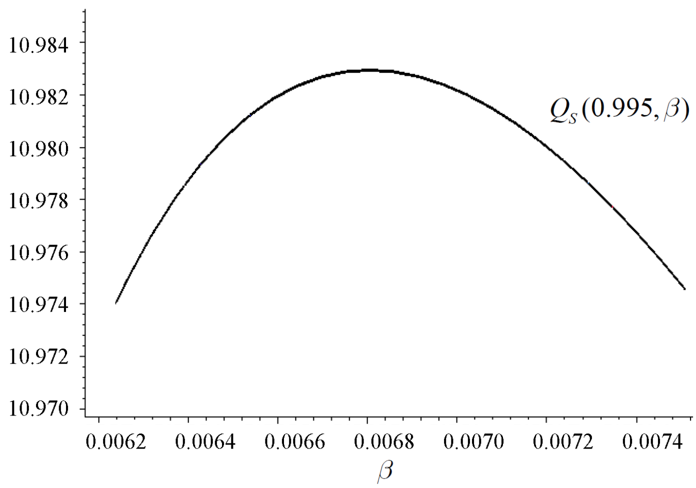

Figure 6: Plot of the parametrized quantile function $Q_S(0.995,\beta)=F_S^{-1}(0.995,\beta)$ in Example 1.

Example 2 (uniform distributions). Assume that

$$F_1=F_2=\begin{cases}0, & x\le 0,\\ x, & 0\le x\le 1,\\ 1, & x\ge 1.\end{cases}$$

Then

$$F_{Z_{1i}}(x,\beta)=\frac{x}{1-\beta},\quad 0\le x\le 1-\beta,\quad \text{and}\quad F_{Z_{2i}}(x,\beta)=\frac{x+\beta-1}{\beta},\quad x\ge 1-\beta,\quad i=1,2.$$

By Lemma 4, we obtain the following density f_S of the aggregated risk S:

$$f_S(x,\beta)=\begin{cases}\frac{x}{1-\beta}, & x\le 1-\beta,\\ \frac{2-2\beta-x}{1-\beta}, & 1-\beta\le x\le 2-2\beta,\\ \frac{x-2+2\beta}{\beta}, & 2-2\beta\le x\le 2-\beta,\\ \frac{2-x}{\beta}, & 2-\beta\le x\le 2,\end{cases}$$

with the corresponding cdf F_S:

$$F_S(x,\beta)=\begin{cases}\frac{x^2}{2(1-\beta)}, & x\le 1-\beta,\\ \frac{4x(1-\beta)-x^2-2(1-\beta)^2}{2(1-\beta)}, & 1-\beta\le x\le 2-2\beta,\\ \frac{4(1-\beta)(1-x)+x^2-2\beta+2\beta^2}{2\beta}, & 2-2\beta\le x\le 2-\beta,\\ \frac{2\beta-4(1-x)-x^2}{2\beta}, & 2-\beta\le x\le 2.\end{cases}$$

In what follows, g is the density of $T:=Q_1(U_1)+Q_2(U_2)$ (independent summands, triangle distribution), see the graph in Figure 7. In the graph in Figure 8, G is the cdf for $T:=Q_1(U_1)+Q_2(U_2)$ (independent summands, triangle distribution) and H is the cdf for S under the worst VaR scenario, i.e. the distribution of $\mathbf{V}$ corresponds to the lower Fréchet bound. In this case we have

$$H(x,\beta)=\begin{cases}F_S(x), & x\le 2-2\beta,\\ 1-\beta, & 2-2\beta\le x<2-\beta,\\ 1, & x\ge 2-\beta.\end{cases}$$

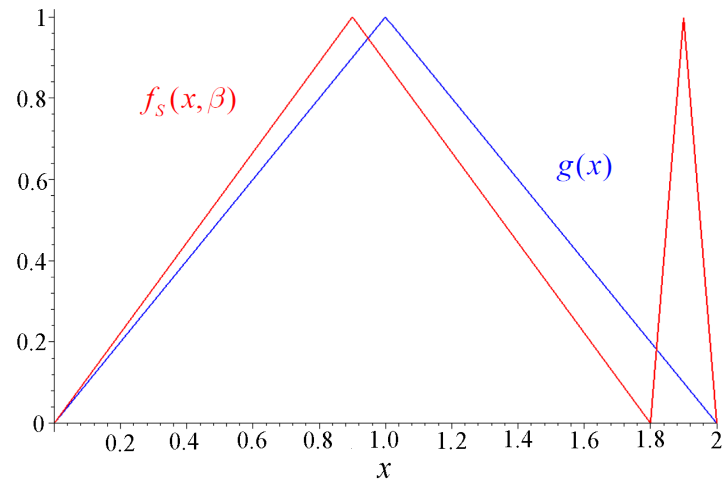

Figure 7: Plots of the densities $f_S(x, \beta)$ for $\beta = 0.1$ and $g(x)$ in Example 2.

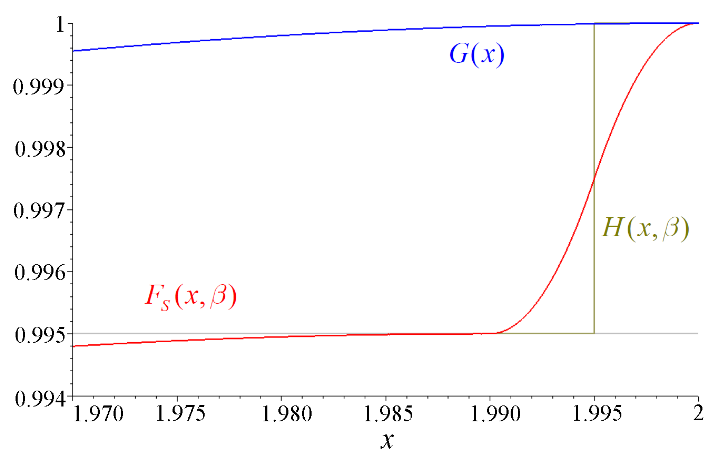

Figure 8: Plots of the cdf's $F_S(x, \beta)$, $G(x)$ and $H(x, \beta)$ for $\beta = 0.005$ in Example 2.

Note that with the Solvency II standard $\alpha = 0.005$, we have here, for $\beta = \alpha$, $\mathrm{VaR}_\alpha(S) = 2 - 2\alpha = 1.9900 = \mathrm{VaR}_\alpha(T)$. For the worst VaR scenario, however, we get here $\mathrm{wVaR}_\alpha(S) = 2 - \alpha = 1.9950 > 1.9900 = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) = \mathrm{SVaR}_\alpha = \mathrm{VaR}_\alpha(S)$. This means that with the construction for S we have no true diversification effect, in contrast to the worst VaR scenario. This changes, however, if we look at $\mathrm{VaR}_\alpha(S) = 1.9910$ when we replace β by $\alpha + \varepsilon$ in the definition of **W** for e.g. $\varepsilon = 0.001$.

The graph in Figure 9 shows the cdf's in the tails for several choices of ε. The graph in Figure 10 shows the values of $Q_S(0.995, \beta) = F_S^{-1}(0.995, \beta)$ in the range $0.0054 \leq \beta \leq 0.0070$. A numerical calculation shows that for $\alpha = 0.005$ the maximal $\mathrm{VaR}_\alpha(S) = 1.9915$ is attained for $\beta = 0.0060$, i.e. $\varepsilon = 0.0010$.

Note that in this example a closed-form representation for $Q_S(u, \beta)$ is given by

$$Q_S(u, \beta) = 2 - 2\beta + \sqrt{2\beta(\beta + u - 1)}, \quad 1 - \beta \leq u \leq 1 - \frac{\beta}{2}.$$

This implies

$$Q_S(1 - \alpha, \beta) = 2 - 2\beta + \sqrt{2\beta(\beta - \alpha)}, \quad \alpha \leq \beta \leq 2\alpha,$$

with its maximum being attained for $\beta_0 = \frac{1+\sqrt{2}}{2}\alpha$ with value $Q_S(1 - \alpha, \beta_0) = 2 - \left(1 + \frac{\sqrt{2}}{2}\right)\alpha$. Note that in contrast the worst VaR here is $\mathrm{wVaR}_\alpha(S^*) = 2 - \alpha$. Table 2 summarizes the results found for $\alpha = 0.005$.

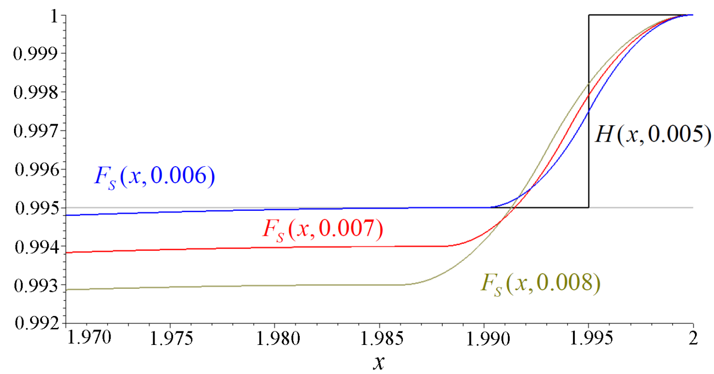

Figure 9: Plots of $F_S(x, 0.005 + \varepsilon)$ for $\varepsilon \in \{0.001, 0.002, 0.003\}$ and $H(x, 0.005)$ in Example 2.

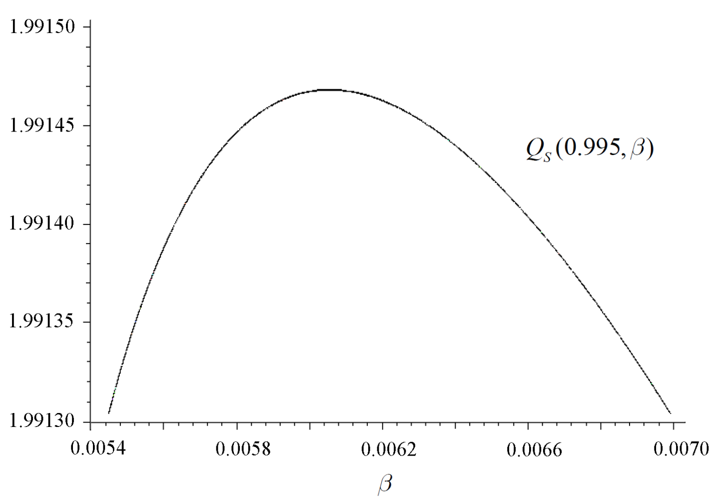

Figure 10: Plot of the parametrized quantile function $Q_S(0.995, \beta) = F_S^{-1}(0.995, \beta)$ in Example 2.

Table 2: Summarized results for Example 2.

β	0.0050	0.0055	0.0060	0.0065	0.0070
$\mathrm{VaR}_\alpha(S)$	1.9900	1.9130	1.9915	1.9914	1.9913
$\mathrm{VaR}_\alpha(T)$	1.9900	1.9900	1.9900	1.9900	1.9900
$\mathrm{wVaR}_\alpha(S)$	1.9950	1.9950	1.9950	1.9950	1.9950
SVaR_α	1.9900	1.9900	1.9900	1.9900	1.9900

Example 3 (Pareto distributions). Assume that

$$F_1 = F_2 = \begin{cases} 0, & x \le 0, \\ \dfrac{x}{1+x}, & x > 0. \end{cases}$$

Then

$$F_{Z_{1i}}(x,\beta) = \frac{x}{(1-\beta)(1+x)}, \quad 0 \le x \le \frac{1}{\beta} - 1, \quad \text{and} \quad F_{Z_{2i}}(x,\beta) = 1 - \frac{1}{\beta(1+x)}, \quad x \ge \frac{1}{\beta} - 1, \quad i = 1,2.$$

For the corresponding densities, we obtain by differentiation

$$f_{Z_{1i}}(x,\beta) = \begin{cases} \dfrac{1}{(1-\beta)(1+x)^2}, & 0 \le x \le \dfrac{1}{\beta} - 1, \\ 0, & x > \dfrac{1}{\beta} - 1, \end{cases} \quad \text{and} \quad f_{Z_{2i}}(x,\beta) = \begin{cases} 0, & x < \dfrac{1}{\beta} - 1, \\ \dfrac{1}{\beta(1+x)^2}, & x \ge \dfrac{1}{\beta} - 1, \end{cases}$$

and

$$\underline{f}(x,\beta) = \begin{cases} \dfrac{1}{(1-\beta)(1+x)^2}, & 0 \le x \le \dfrac{1}{\beta} - 1, \\ 0, & x > \dfrac{1}{\beta} - 1, \end{cases} \quad \text{and} \quad \overline{f}(x,\beta) = \begin{cases} 0, & x < 0, \\ \dfrac{\beta}{(1+\beta x)^2}, & x \ge 0. \end{cases}$$

In order to calculate the density f_S of the aggregated risk S, we need a suitable partial fraction representation of $\underline{f}(x-y)\underline{f}(y)$ and $\overline{f}(x-y)\overline{f}(y)$. Note that in general, we have

$$\frac{1}{(1+x-y)(1+y)} = \frac{1}{2+x}\left(\frac{1}{1+x-y} + \frac{1}{1+y}\right)$$

and

$$\begin{aligned} \frac{1}{(1+x-y)^2(1+y)^2} &= \frac{1}{(2+x)^2}\left(\frac{1}{1+x-y} + \frac{1}{1+y}\right)^2 \\ &= \frac{1}{(2+x)^2}\left(\frac{1}{(1+x-y)^2} + \frac{1}{(1+y)^2} + \frac{2}{2+x}\left(\frac{1}{1+x-y} + \frac{1}{1+y}\right)\right), \end{aligned}$$

from which we obtain, by Lemma 4,

$$F_S(x,\beta) = \begin{cases} \dfrac{x^2 + 2x - 2\ln(1+x)}{(2+x)^2(1-\beta)}, & 0 \le x \le \dfrac{1}{\beta} - 1, \\ \dfrac{(1-2\beta)x^2 + (4-6\beta)x - 4\beta + 4 + 2\ln(\beta x + 2\beta - 1)}{(2+x)^2(1-\beta)}, & \dfrac{1}{\beta} - 1 \le x \le 2\left(\dfrac{1}{\beta} - 1\right), \\ \dfrac{x^2 - 2x + \frac{2}{\beta}\ln(\beta x + 2\beta - 1)}{(2+x)^2}, & x \ge 2\left(\dfrac{1}{\beta} - 1\right). \end{cases}$$

The density $f_S(x)$ follows by differentiation.

In the following, g denotes the density of $T := Q_1(U_1) + Q_2(U_2)$ (independent summands), see the graph in Figure 11. In the graph in Figure 12, G is the cdf of $T := Q_1(U_1) + Q_2(U_2)$ (independent summands) and H is the cdf of S under the worst VaR scenario, i.e. the distribution of $\mathbf{V}$ corresponds again to the lower Fréchet bound. In this case we have

$$H(x,\beta) = \begin{cases} F_S(x), & x \le \dfrac{2}{\beta} - 2, \\ 1 - \beta, & \dfrac{2}{\beta} - 2 \le x \le \dfrac{4}{\beta} - 2, \\ 1 - \beta + \sqrt{\beta^2 - \dfrac{4\beta}{2+x}}, & x \ge \dfrac{4}{\beta} - 2. \end{cases}$$

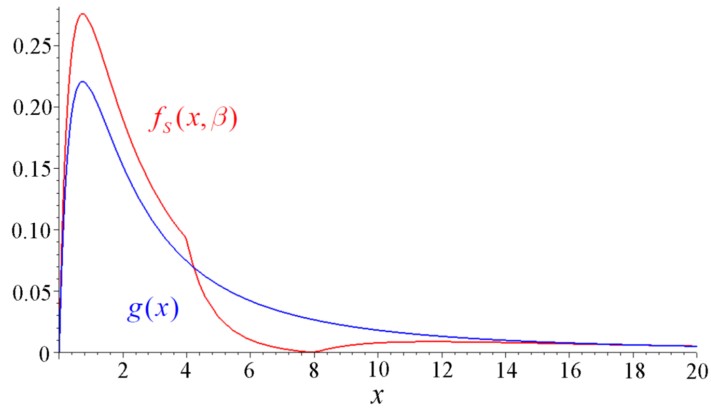

Figure 11: Plots of the densities $f_S(x,\beta)$ for $\beta = 0.1$ and $g(x)$ in Example 3.

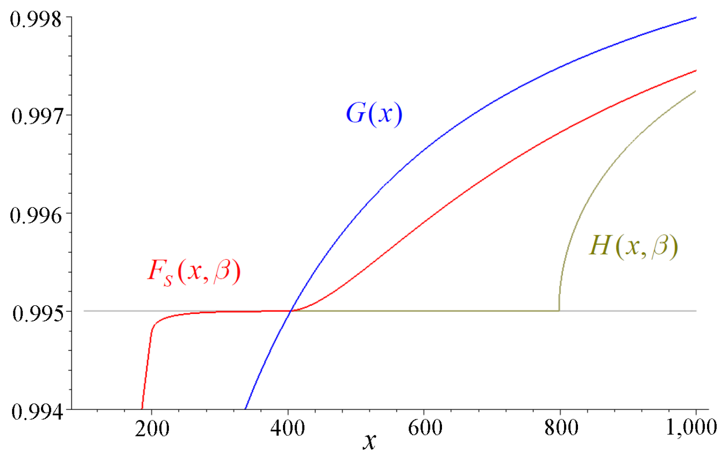

Figure 12: Plots of the cdf's $F_S(x,\beta)$, $G(x)$ and $H(x,\beta)$ for $\beta = 0.005$ in Example 3.

Note that with the Solvency II standard $\alpha = 0.005$, we have here, for $\beta = \alpha$, $\mathrm{VaR}_\alpha(S) = 397.3168 < \mathrm{VaR}_\alpha(T) = 403.9161$. For the worst VaR scenario, however, we get $\mathrm{wVaR}_\alpha(S) = \frac{4}{\beta} - 2 = 798 > 398 = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) = \mathrm{SVaR}_\alpha > \mathrm{VaR}_\alpha(S) = 397.3168$. This means that even with the construction for S we still have a (quite small) diversification effect, but not in the worst VaR scenario, as expected. This changes, however, if we look at $\mathrm{VaR}_\alpha(S) = 488.2116$ when we replace β by $\beta+\varepsilon$ in the definition of $\mathbf{W}$ for, e.g., $\varepsilon = 0.001$.

The graph in Figure 13 shows the cdf's in the tail for several choices of ε. The graph in Figure 14 shows the values of $Q_S(0.995,\beta) = F_S^{-1}(0.995,\beta)$ in the range $0.007 \le \beta \le 0.012$. A numerical calculation shows that for $\alpha = 0.005$ the maximum $\mathrm{VaR}_\alpha(S) = 509.3804$ is attained for $\beta = 0.0089$, i.e. $\varepsilon = 0.0039$. Table 3 summarizes the results found for $\alpha = 0.005$.

Table 3: Summarized results for Example 3.

β	0.0050	0.0070	0.0089	0.0100	0.0110
$\mathrm{VaR}_\alpha(S)$	397.3168	503.2848	509.3804	508.6489	507.0076
$\mathrm{VaR}_\alpha(T)$	403.9161	403.9161	403.9161	403.9161	403.9161
$\mathrm{wVaR}_\alpha(S)$	798.0000	798.0000	798.0000	798.0000	798.0000
SVaR_α	398.0000	398.0000	398.0000	398.0000	398.0000

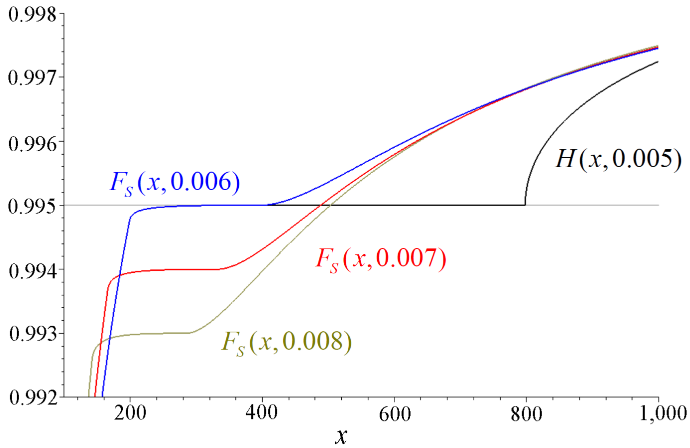

Figure 13: Plots of the cdf's $F_S(x, 0.005 + \varepsilon)$ for $\varepsilon \in \{0.001,\ 0.002,\ 0.003\}$ and $H(x, 0.005)$ in Example 3.

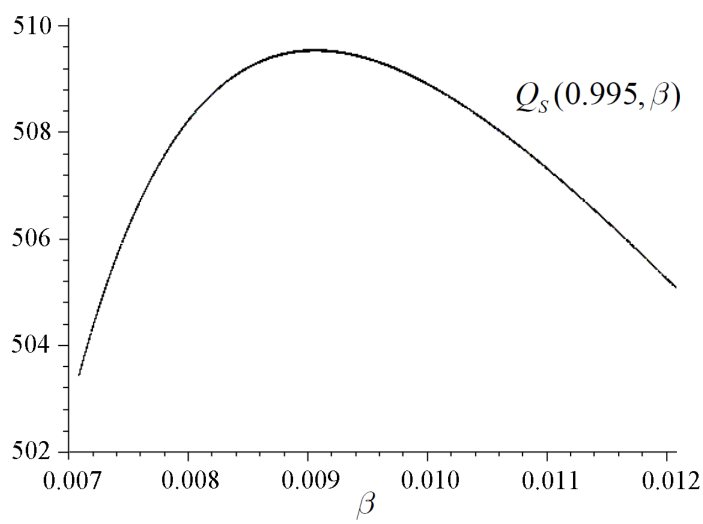

Figure 14: Plot of the parametrized quantile function $Q_S(0.995, \beta) = F_S^{-1}(0.995, \beta)$ in Example 3.

Examples 1–3 show that it is generally possible to obtain unfavourable VaR scenarios by a suitable choice of $\beta = \alpha + \varepsilon$ in the definition of $\mathbf{W}$, i.e. scenarios which lead to an opposite diversification effect in the portfolio and which are sometimes close to the worst VaR scenario.

We continue with a particular construction of $\mathbf{W}$ which allows in general for an unfavourable VaR scenario.

Lemma 5. *For $d \in \mathbb{N}$, $d > 1$, let $\mathbf{I}_d$ denote the d-dimensional unit matrix, $\mathbf{e}_d = (1, \dots, 1)$ the d-dimensional row vector consisting of just ones, and $\mathbf{E}_d = \mathbf{e}_d^{tr}\mathbf{e}_d$ the $d \times d$ matrix with all entries equal to unity. Then $\Sigma_d = (1-r)\mathbf{I}_d + r\mathbf{E}_d$ is a correlation matrix iff $-\frac{1}{d-1} \le r \le 1$. In the general case, the latent roots λ_i of Σ_d are given by $\lambda_1 = 1 + (d-1)r$ and $\lambda_i = 1 - r$, $i = 2, \dots, d$. An orthonormal basis $T_1, \dots, T_d$ of corresponding latent vectors is given by $T_1 = \frac{1}{\sqrt{d}}\mathbf{e}_d^{tr}$ and $T_j = (t_{1j}, \dots, t_{dj})^{tr}$ for $2 \le j \le d$, where*

$$t_{ij} = \begin{cases} -\dfrac{1}{\sqrt{j(j-1)}}, & 1 \le i < j, \\ \sqrt{\dfrac{j-1}{j}}, & j = i, \\ 0, & i > j. \end{cases}$$

Hence Σ_d possesses the spectral decomposition $\Sigma_d = \mathbf{A}\mathbf{A}^{tr}$ with $\mathbf{A} = \mathbf{T}\sqrt{\Delta}$, where $\mathbf{T} = [T_1, \dots, T_d]$ and $\Delta = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$.

Note that there is also an alternative possibility to represent latent roots λ_j^* and normalized latent vectors $T_j^* = (t_{1j}^*, \dots, t_{dj}^*)^{tr}$, $j = 1, \dots, d$, of Σ_d since Σ_d is a particular symmetric Toeplitz matrix for which the latent roots and normalized latent vectors can be expressed via trigonometric functions (see [2, relations (5.89) and (5.90)]). In particular, we can choose

$$\lambda_j^* = 1 + r\sum_{i=1}^{d-1}\cos\Big(\frac{2\pi ij}{d}\Big) = \begin{cases} 1 - r, & j = 1, \dots, d-1, \\ 1 + (d-1)r, & j = d, \end{cases}$$

and

$$t_{ij}^* = \frac{\cos\Big(\frac{2\pi ij}{d}\Big) + \sin\Big(\frac{2\pi ij}{d}\Big)}{\sqrt{d}}, \quad 1 \le i, j \le d.$$

This is due to the fact that the latent roots have multiplicities, hence the linear space spanned by the corresponding latent vectors is $(d-1)$-dimensional, allowing for different representations of the corresponding linear basis. However, for our purposes, the representation in Lemma 5 is more suited.

In what follows we will call a Gaussian copula derived from the correlation matrix $\Sigma_d = \frac{d}{d-1}\mathbf{I}_d - \frac{1}{d-1}\mathbf{E}_d$ for $r = -\frac{1}{d-1}$ a *minimal correlation Gaussian copula*.

Note that the corresponding multivariate normal distribution is degenerated since Σ_d is singular, i.e. a random vector $\mathbf{X}$ with zero mean and correlation matrix Σ_d has the representation $\mathbf{X} = A\mathbf{Y}$, where $\mathbf{Y}$ has a standard multivariate normal distribution with mean zero and variance-covariance matrix $\mathbf{I}_d$. For $d = 2$, the minimal correlation Gaussian copula is identical to the lower Fréchet bound or countermonotonicity copula.

3 A case study

The following example shows the effects of such an approach for the 19-dimensional data set discussed in [26]. Table 4 contains insurance losses from a non-life portfolio of natural perils in $d = 19$ areas in central Europe over a time period of 20 years. The losses are given in million monetary units (MMU).

A statistical analysis of the data shows a good fit to lognormal $\mathcal{LN}(\mu, \sigma)$-distributions for the losses per Area k, $k = 1, \dots, 19$. The parameters μ_k and σ_k for Area k shown in Table 5 were hence estimated from the log data by calculating means and standard deviations.

Table 4: Insurance losses from a Nat Cat portfolio in central Europe.

Year	Area 1	Area 2	Area 3	Area 4	Area 5	Area 6	Area 7	Area 8	Area 9	Area 10
1	23.664	154.664	40.569	14.504	10.468	7.464	22.202	17.682	12.395	18.551
2	1.080	59.545	3.297	1.344	1.859	0.477	6.107	7.196	1.436	3.720
3	21.731	31.049	55.973	5.816	14.869	20.771	3.580	14.509	17.175	87.307
4	28.990	31.052	30.328	4.709	0.717	3.530	6.032	6.512	0.682	3.115
5	53.616	62.027	57.639	1.804	2.073	4.361	46.018	22.612	1.581	11.179
6	29.950	41.722	12.964	1.127	1.063	4.873	6.571	11.966	15.676	24.263
7	3.474	14.429	10.869	0.945	2.198	1.484	4.547	2.556	0.456	1.137
8	10.020	31.283	21.116	1.663	2.153	0.932	25.163	3.222	1.581	5.477
9	5.816	14.804	128.072	0.523	0.324	0.477	3.049	7.791	4.079	7.002
10	170.725	576.767	108.361	41.599	20.253	35.412	126.698	71.079	21.762	64.582
11	21.423	50.595	4.360	0.327	1.566	64.621	5.650	1.258	0.626	3.556
12	6.380	28.316	3.740	0.442	0.736	0.470	3.406	7.859	0.894	3.591
13	124.665	33.359	14.712	0.321	0.975	2.005	3.981	4.769	2.006	1.973
14	20.165	49.948	17.658	0.595	0.548	29.350	6.782	4.873	2.921	6.394
15	78.106	41.681	13.753	0.585	0.259	0.765	7.013	9.426	2.180	3.769
16	11.067	444.712	365.351	99.366	8.856	28.654	10.589	13.621	9.589	19.485
17	6.704	81.895	14.266	0.972	0.519	0.644	8.057	18.071	5.515	13.163
18	15.550	277.643	26.564	0.788	0.225	1.230	26.800	64.538	2.637	80.711
19	10.099	18.815	9.352	2.051	1.089	6.102	2.678	4.064	2.373	2.057
20	8.492	138.708	46.708	3.680	1.132	1.698	165.600	7.926	2.972	5.237

Year	Area 11	Area 12	Area 13	Area 14	Area 15	Area 16	Area 17	Area 18	Area 19
1	1.842	4.100	46.135	14.698	44.441	7.981	35.833	10.689	7.299
2	0.429	1.026	7.469	7.058	4.512	0.762	14.474	9.337	0.740
3	0.209	2.344	22.651	4.117	26.586	3.920	13.804	2.683	3.026
4	0.521	0.696	31.126	1.878	29.423	6.394	18.064	1.201	0.894
5	2.715	1.327	40.156	4.655	104.691	28.579	17.832	1.618	3.402
6	4.832	0.701	16.712	11.852	29.234	7.098	17.866	5.206	5.664
7	0.268	0.580	11.851	2.057	11.605	0.282	16.925	2.082	1.008
8	0.741	0.369	3.814	1.869	8.126	1.032	14.985	1.390	1.703
9	0.524	6.554	5.459	3.007	8.528	1.920	5.638	2.149	2.908
10	9.882	6.401	106.197	44.912	191.809	90.559	154.492	36.626	36.276
11	1.052	8.277	22.564	8.961	19.817	16.437	25.990	2.364	6.434
12	0.136	0.364	28.000	7.574	3.213	1.749	12.735	1.744	0.558
13	1.990	15.176	57.235	23.686	110.035	17.373	7.276	2.494	0.525
14	0.630	0.762	25.897	3.439	8.161	3.327	24.733	2.807	1.618
15	0.770	15.024	36.068	1.613	6.127	8.103	12.596	4.894	0.822
16	0.287	0.464	24.211	38.616	51.889	1.316	173.080	3.557	11.627
17	0.590	2.745	16.124	2.398	20.997	2.515	5.161	2.840	3.002
18	0.245	0.217	12.416	4.972	59.417	3.762	24.603	7.404	19.107
19	0.415	0.351	10.707	2.468	10.673	1.743	27.266	1.368	0.644
20	0.566	0.708	22.646	6.652	14.437	63.692	113.231	7.218	2.548

Table 5: Distributional parameters for fitted lognormal loss distributions

Parameter	Area 1	Area 2	Area 3	Area 4	Area 5	Area 6	Area 7	Area 8	Area 9	Area 10
μ_k	2.806	4.072	3.141	0.638	0.398	1.223	2.321	2.212	1.078	2.106
σ_k	1.216	1.052	1.211	1.569	1.300	1.599	1.198	0.988	1.145	1.253

Parameter	Area 11	Area 12	Area 13	Area 14	Area 15	Area 16	Area 17	Area 18	Area 19
μ_k	−0.323	0.382	3.020	1.749	3.041	1.550	3.070	1.244	0.938
σ_k	1.088	1.335	0.803	1.003	1.122	1.477	0.962	0.858	1.214

As is to be expected, insurance losses in locally adjacent areas show a high degree of stochastic dependence, which can also be seen from the correlation Tables 6 and 7. For a better readability, only two decimal places are displayed.

Table 6: Empirical correlations between original losses in adjacent areas.

Parameter	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16	A17	A18	A19
A1	1	0.46	0.03	0.16	0.47	0.20	0.35	0.49	0.41	0.24	0.78	0.64	0.91	0.63	0.85	0.66	0.30	0.67	0.56
A2	0.46	1	0.64	0.78	0.67	0.36	0.51	0.76	0.57	0.51	0.58	−0.04	0.59	0.84	0.68	0.58	0.87	0.77	0.90
A3	0.03	0.64	1	0.93	0.41	0.26	0.11	0.16	0.33	0.16	0.08	−0.09	0.13	0.64	0.25	0.10	0.74	0.14	0.35
A4	0.16	0.78	0.93	1	0.54	0.36	0.16	0.25	0.43	0.19	0.22	−0.10	0.30	0.79	0.36	0.19	0.84	0.32	0.49
A5	0.47	0.67	0.41	0.54	1	0.41	0.35	0.51	0.84	0.63	0.59	0.02	0.64	0.67	0.59	0.50	0.58	0.71	0.67
A6	0.20	0.36	0.26	0.36	0.41	1	0.07	0.11	0.28	0.19	0.28	0.14	0.31	0.42	0.24	0.27	0.39	0.27	0.40
A7	0.35	0.51	0.11	0.16	0.35	0.07	1	0.44	0.27	0.19	0.48	−0.07	0.46	0.35	0.45	0.91	0.64	0.61	0.49
A8	0.49	0.76	0.16	0.25	0.51	0.11	0.44	1	0.50	0.75	0.61	−0.03	0.54	0.47	0.71	0.53	0.40	0.75	0.90
A9	0.41	0.57	0.33	0.43	0.84	0.28	0.27	0.50	1	0.66	0.68	−0.01	0.52	0.60	0.50	0.41	0.46	0.65	0.63
A10	0.24	0.51	0.16	0.19	0.63	0.19	0.19	0.75	0.66	1	0.33	−0.12	0.27	0.28	0.43	0.24	0.23	0.45	0.65
A11	0.78	0.58	0.08	0.22	0.59	0.28	0.48	0.61	0.68	0.33	1	0.19	0.79	0.65	0.80	0.73	0.43	0.84	0.74
A12	0.64	−0.04	−0.09	−0.10	0.02	0.14	−0.07	−0.03	−0.01	−0.12	0.19	1	0.44	0.21	0.28	0.17	−0.12	0.13	0.03
A13	0.91	0.59	0.13	0.30	0.64	0.31	0.46	0.54	0.52	0.27	0.79	0.44	1	0.71	0.86	0.74	0.47	0.76	0.65
A14	0.63	0.84	0.64	0.79	0.67	0.42	0.35	0.47	0.60	0.28	0.65	0.21	0.71	1	0.74	0.54	0.79	0.68	0.72
A15	0.85	0.68	0.25	0.36	0.59	0.24	0.45	0.71	0.50	0.43	0.80	0.28	0.86	0.74	1	0.69	0.47	0.71	0.75
A16	0.66	0.58	0.10	0.19	0.50	0.27	0.91	0.53	0.41	0.24	0.73	0.17	0.74	0.54	0.69	1	0.63	0.77	0.64
A17	0.30	0.87	0.74	0.84	0.58	0.39	0.64	0.40	0.46	0.23	0.43	−0.12	0.47	0.79	0.47	0.63	1	0.59	0.64
A18	0.67	0.77	0.14	0.32	0.71	0.27	0.61	0.75	0.65	0.45	0.84	0.13	0.76	0.68	0.71	0.77	0.59	1	0.86
A19	0.56	0.90	0.35	0.49	0.67	0.40	0.49	0.90	0.63	0.65	0.74	0.03	0.65	0.72	0.75	0.64	0.64	0.86	1

Table 7: Empirical correlations between log losses in adjacent areas.

Parameter	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16	A17	A18	A19
A1	1	0.27	0.30	0.16	0.17	0.45	0.28	0.32	0.32	0.29	0.67	0.51	0.76	0.34	0.67	0.74	0.18	0.21	0.29
A2	0.27	1	0.48	0.66	0.39	0.37	0.71	0.69	0.52	0.64	0.30	−0.02	0.45	0.66	0.58	0.45	0.73	0.74	0.78
A3	0.30	0.48	1	0.70	0.40	0.31	0.42	0.51	0.58	0.53	0.18	0.07	0.21	0.32	0.54	0.26	0.47	0.21	0.57
A4	0.16	0.66	0.70	1	0.77	0.47	0.46	0.47	0.59	0.49	0.18	−0.13	0.33	0.50	0.47	0.18	0.76	0.43	0.54
A5	0.17	0.39	0.40	0.77	1	0.59	0.30	0.20	0.49	0.39	0.28	0.08	0.35	0.56	0.44	0.16	0.55	0.36	0.41
A6	0.45	0.37	0.31	0.47	0.59	1	0.14	0.01	0.36	0.34	0.33	0.12	0.48	0.46	0.48	0.37	0.59	0.17	0.50
A7	0.28	0.71	0.42	0.46	0.30	0.14	1	0.52	0.27	0.40	0.45	−0.07	0.31	0.31	0.46	0.62	0.63	0.58	0.57
A8	0.32	0.69	0.51	0.47	0.20	0.01	0.52	1	0.64	0.81	0.27	−0.02	0.38	0.35	0.56	0.35	0.28	0.62	0.63
A9	0.32	0.52	0.58	0.59	0.49	0.36	0.27	0.64	1	0.78	0.40	0.19	0.27	0.50	0.44	0.30	0.33	0.57	0.61
A10	0.29	0.64	0.53	0.49	0.39	0.34	0.40	0.81	0.78	1	0.21	−0.02	0.21	0.37	0.52	0.30	0.31	0.53	0.81
A11	0.67	0.30	0.18	0.18	0.28	0.33	0.45	0.27	0.40	0.21	1	0.47	0.49	0.45	0.60	0.67	0.20	0.45	0.39
A12	0.51	−0.02	0.07	−0.13	0.08	0.12	−0.07	−0.02	0.19	−0.02	0.47	1	0.44	0.21	0.24	0.46	−0.23	0.25	0.05
A13	0.76	0.45	0.21	0.33	0.35	0.48	0.31	0.38	0.27	0.21	0.49	0.44	1	0.55	0.60	0.71	0.37	0.39	0.24
A14	0.34	0.66	0.32	0.50	0.56	0.46	0.31	0.35	0.50	0.37	0.45	0.21	0.55	1	0.59	0.43	0.57	0.58	0.53
A15	0.67	0.58	0.54	0.47	0.44	0.48	0.46	0.56	0.44	0.52	0.60	0.24	0.60	0.59	1	0.59	0.36	0.35	0.63
A16	0.74	0.45	0.26	0.18	0.16	0.37	0.62	0.35	0.30	0.30	0.67	0.46	0.71	0.43	0.59	1	0.38	0.43	0.39
A17	0.18	0.73	0.47	0.76	0.55	0.59	0.63	0.28	0.33	0.31	0.20	−0.23	0.37	0.57	0.36	0.38	1	0.52	0.56
A18	0.21	0.74	0.21	0.43	0.36	0.17	0.58	0.62	0.57	0.53	0.45	0.25	0.39	0.58	0.35	0.43	0.52	1	0.60
A19	0.29	0.78	0.57	0.54	0.41	0.50	0.57	0.63	0.61	0.81	0.39	0.05	0.24	0.53	0.63	0.39	0.56	0.60	1

The graph in Figure 15 shows estimated cdf's on a basis of 100,000 Monte Carlo simulations for the aggregated loss using lognormal margins with the parameters from Table 5 with a Bernstein copula representing **U** and a minimal correlation Gaussian copula representing **V**, for various values of p. For comparison purposes, we have also added an estimated cdf for the aggregated loss for a Bernstein copula representing **U** and an upper Fréchet (or comonotonicity) copula representing **V**. Note that the Bernstein copula is here constructed according to [5] on the basis of the ranks of the risk vectors (see also [28, Section 3]).

The plots in Figure 15 for the tail cdf's correspond to a Bernstein copula **U** with a minimal correlation Gaussian copula **V**: $p = 1$ ($F_1(x)$), $p = 0.99$ ($F_2(x)$), $p = 0.994$ ($F_3(x)$), as well as to a Bernstein copula **U** with $p = 0.994$ but different copulas **V**: upper Fréchet bound or comonotonicity copula ($F_4(x)$) and independence copula ($F_5(x)$).

Table 8 shows the estimated risk measures VaR_α for $\alpha = 0.005$ (Solvency II standard) for the various values of p and different types of **V**.

Table 8: Survey over VaR-estimates under different copula models with lognormal margins, in MMU.

p	0.99	0.994	0.994	0.994	1
V	minimal correlation Gaussian	minimal correlation Gaussian	upper Fréchet	independence	—
VaR_α	4,647	5,272	3,976	5,018	2,229

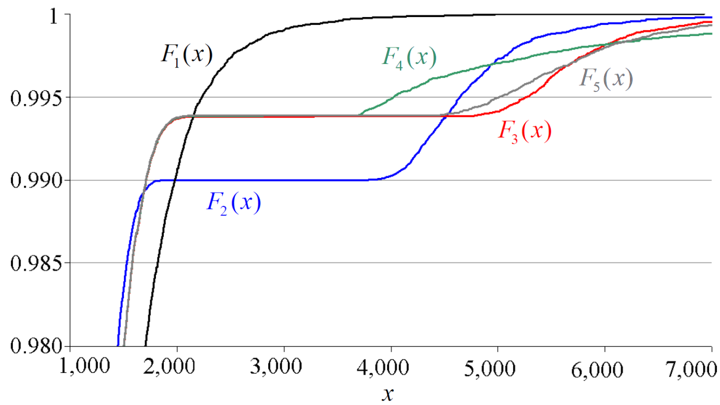

Figure 15: Plots of the estimated cdf's $F_i(x)$, $i = 1, \ldots, 5$, in the tail.

As can clearly be seen, the patchwork construction with the minimal correlation Gaussian copula representing **V** with no tail dependence gives the largest VaR estimate here and is typically larger than the upper Fréchet copula, which has a positive tail dependence. Note that the sum of individual VaR's is given by 3,976 MMU, which means that using the Bernstein copula alone would lead to a diversified portfolio while all other copula models do not.

Finally, it should be pointed out that the effects described here are independent of the particular copula chosen for **U**, i.e. the magnitude of the estimated VaR's under the patchwork construction would remain roughly equal also under an elliptical, an Archimedean, a vine or an adapted Bernstein copula approach for **U** ([28], cf. also the comments after Figure 3 in [10]).

4 Concluding remarks

The patchwork copula construction presented in this paper allows for a simple but yet effective and well-defined way to generate unfavourable VaR scenarios, i.e. scenarios with opposite diversification effects in particular for applications in Solvency II. Such scenario considerations are prescribed by legislative guidelines as, e.g., specified in the Commission Delegated Regulation of the EU [12] (p. L12/6 (16), L12/9 (49), L12/12 (75) or (77), just to mention some). Besides Solvency II, such unfavourable VaR scenario generations could also be of interest in the Basel III framework (e.g., economic scenario generators) or in the reinsurance industry, in particular w.r.t. extreme natural perils.

Although there is theoretically also a method to create worst VaR scenarios by means of the rearrangement algorithm, the latter approach easily becomes numerically cumbersome in high-dimensional portfolios as in our case study, especially, if the risk distributions are not identical (see [10, Section 2.2]). Hence a sub-optimal but easy to implement alternative is of value, in particular, since it seems unlikely that the worst VaR scenario would actually occur in real life portfolios.

The approach discussed in this paper seems, at a first glance, to be related to the recent paper [27]. The essential difference is, however, that the latter paper is not based on an observation-free copula construction for the tails as in the present paper. The algorithm proposed there leads only to stochastic approximations of the underlying distributions by a marginal-wise backwards transformation of the simulated multivariate distribution with the quantile functions of the originally estimated marginal cdf's. This emphasizes the fact that unfavourable VaR estimates cannot perhaps be characterized by the copula structure alone but that the interplay between the dependence structure and the marginal distributions is also essential (see the discussion in [17]). Such a kind of interplay could potentially also be considered in the present approach, allowing non-constant negative pairwise correlations in the matrix Σ_d for the Gaussian copula in Lemma 5.

Note that Value at Risk is not the only risk measure that is used for calculating capital requirements in Europe. For instance, the Swiss Solvency Test uses the Expected Shortfall (ES) $\mathrm{ES}_\alpha(X)$ of risks X as the underlying risk measure (cf. [13]). In accordance with our terminology and under the assumption of a continuous risk distribution, it is defined as

$$\mathrm{ES}_\alpha(X) = \mathbb{E}\big(X \,|\, X > \mathrm{VaR}_\alpha(X)\big), \quad 0 < \alpha < 1.$$

Unfortunately, it is impossible to generate true unfavourable ES scenarios since ES is a coherent (i.e. subadditive) risk measure which in the worst case generates additive risk scenarios if the risks involved follow a comonotone dependence structure (see, e.g., [21, Chapter 7.2]). Note, however, that it is sufficient for the generation of additive ES scenarios to use a dependence structure as in Lemma 1 with the upper Fréchet bound for $\mathbb{V}$, which is a copula in any dimension.

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Appendix: Proofs of Lemmata

Proof of Lemma 2. We have

$$\begin{aligned} F_{Z_{1i}}(x,\beta) &= \mathbb{P}\big(Q_i((1-\beta)U_i) \le x\big) = \mathbb{P}\big((1-\beta)U_i \le F_i(x)\big) \\ &= \mathbb{P}\Big(U_i \le \frac{F_i(x)}{1-\beta}\Big) = \frac{F_i(x)}{1-\beta}, \quad 0 \le x \le Q_i(1-\beta), \end{aligned}$$

and

$$\begin{aligned} F_{Z_{2i}}(x,\beta) &= \mathbb{P}\big(Q_i(1-\beta+\beta V_i) \le x\big) = \mathbb{P}\big(1-\beta+\beta V_i \le F_i(x)\big) \\ &= \mathbb{P}\Big(V_i \le \frac{F_i(x)+\beta-1}{\beta}\Big) = \frac{F_i(x)+\beta-1}{\beta}, \quad x \ge Q_i(1-\beta), \quad i = 1,2. \end{aligned}$$

□

Proof of Lemma 3. In the finite interval case, we have, by the usual convolution formula,

$$h_1(x) = \int\limits_{\substack{0 \le y \le M \\ 0 \le x-y \le M}} f(x-y)g(y)\,\mathrm{d}y = \int\limits_{\max(0,x-M) \le y \le \min(x,M)} f(x-y)g(y)\,\mathrm{d}y.$$

Now for $0 \le x \le M$, we have $\max(0, x-M) = 0$, $\min(x, M) = x$, from which the upper formula in brackets for $h_1(x)$ follows. For $M \le x \le 2M$, we have $\max(0, x-M) = x-M$, $\min(x, M) = M$, from which the lower formula in brackets for $h_1(x)$ follows.

The proof for the infinite interval case is analogous, observing that for $x \ge 2M$, we have

$$h_2(x) = \int\limits_{\substack{M \le y \le x \\ M \le x-y}} f(x-y)g(y)\,\mathrm{d}y = \int\limits_{M \le y \le x-M} f(x-y)g(y)\,\mathrm{d}y.$$

Further, under the conditions made, we have, in either case,

$$\frac{\mathrm{d}}{\mathrm{d}x} F \star G(x)\Big|_{x=2M} = h_1(2M) = h_2(2M) = \int_M^M f(x-y)g(y)\,\mathrm{d}y = 0,$$

as stated. □

Proof of Lemma 4. Let ξ_i and ζ_i be independent random variables with the cdf's $\underline{F}(\bullet,\beta)$ and $\overline{F}(\bullet,\beta)$, respectively. Then $I\xi_i + (1-I)\big(Q(1-\beta)+\zeta_i\big)$ is a stochastic representation of X_i, $i = 1,\ldots,d$, where again I is a binomial random variable with $\mathbb{P}(I=1) = 1-\beta$ and $\mathbb{P}(I=0)=\beta$, independent of $(\mathbf{U},\mathbf{V})$ according to Lemma 2. Hence

$$I\sum_{i=1}^{d}\xi_i + (1-I)\sum_{i=1}^{d}\big(Q(1-\beta)+\zeta_i\big) = I\sum_{i=1}^{d}\xi_i + (1-I)\left(dQ(1-\beta)+\sum_{i=1}^{d}\zeta_i\right)$$

is a stochastic representation of S. Note that the cdf of $\sum_{i=1}^{d}\xi_i$ is $\underline{F}^{d\star}(\bullet,\beta)$ and that of $\sum_{i=1}^{d}\zeta_i$ is $\overline{F}^{d\star}(\bullet,\beta)$, from which the assertion follows. □

Proof of Lemma 5. The proof relies on the following two relations:

a) $\displaystyle\sum_{k=2}^{d}\frac{1}{k(k-1)} = \frac{d-1}{d}$ for all $d \geq 2$;

b) $\displaystyle\frac{i-1}{i} + \sum_{k=1}^{d-i}\frac{1}{(i+k)(i+k-1)} = \frac{d-1}{d}$ for all $d \geq 2$ and $1 \leq i \leq d$.

Clearly a) follows easily by induction. Relation b) follows immediately from a) since

$$\frac{i-1}{i} = \sum_{k=2}^{i}\frac{1}{k(k-1)} \quad\text{and}\quad \sum_{k=1}^{d-i}\frac{1}{(i+k)(i+k-1)} = \sum_{k=i+1}^{d}\frac{1}{k(k-1)}.$$

To prove Lemma 5, we first show that $\mathbf{T}\mathbf{T}^{tr} = \mathbf{I}_d = \mathbf{T}^{tr}\mathbf{T}$. Let $\mathbf{T}\mathbf{T}^{tr} = [b_{ij}]_{i,j=1,\ldots,d}$. For $1 \leq i \leq d$, we obtain, by relation b) above,

$$b_{ii} = \frac{1}{d} + \frac{i-1}{i} + \sum_{k=1}^{d-i}\frac{1}{(i+k)(i+k-1)} = 1.$$

For $1 \leq i,j \leq d$ with $i \neq j$ we get, with $i \vee j := \max(i,j)$, following again relation b),

$$\begin{aligned} b_{ij} &= \frac{1}{d} - \frac{1}{i\vee j} + \sum_{k=i\vee j+1}^{d}\frac{1}{k(k-1)} = \frac{1}{d} - \frac{1}{i\vee j} + \sum_{k=1}^{d-i\vee j}\frac{1}{(k+i\vee j)(k+i\vee j-1)} \\ &= \frac{1}{d} - \frac{1}{i\vee j} + \frac{d-1}{d} - \frac{i\vee j-1}{i\vee j} = 1-1 = 0. \end{aligned}$$

This proves $\mathbf{T}\mathbf{T}^{tr} = \mathbf{I}_d$. On the other hand, let $\mathbf{T}^{tr}\mathbf{T} = [c_{ij}]_{i,j=1,\ldots,d}$. It is obvious that $c_{11} = \frac{1}{d}d = 1$ and for all $2 \leq i \leq d$, $c_{ii} = \frac{1}{i(i-1)}(i-1) + \frac{i-1}{i} = 1$. Next, for all $2 \leq j \leq d$, we obtain

$$c_{1j} = \frac{1}{\sqrt{d}}\left(-\frac{1}{\sqrt{j(j-1)}}(j-1) + \sqrt{\frac{j-1}{j}}\right) = 0,$$

and for all $2 \leq i \leq d$, we get

$$c_{i1} = \frac{1}{\sqrt{d}}\left(-\frac{1}{\sqrt{i(i-1)}}(i-1) + \sqrt{\frac{i-1}{i}}\right) = 0.$$

Finally, for $2 \leq i,j \leq d$ with $i \neq j$, we get

$$c_{ij} = -\frac{1}{\sqrt{(i\vee j)(i\vee j-1)}}\left(-\frac{1}{\sqrt{(i\vee j)(i\vee j-1)}}(i\vee j-1) + \sqrt{\frac{i\vee j-1}{i\vee j}}\right) = 0.$$

This proves $\mathbf{T}^{tr}\mathbf{T} = \mathbf{I}_d$.

Now let $\lambda_1 = 1+(d-1)r$, $\lambda_i = 1-r$, $i = 2,\ldots,d$, and $\Delta_t = \operatorname{diag}(\lambda_1 - t,\ldots,\lambda_d - t)$. A standard computation yields, for $t \in \mathbb{R}$, $\mathbf{T}\Delta_t = [s_{ij}]_{i,j=1,\ldots,d}$, where

$$s_{ij} = \begin{cases} \frac{1+(d-1)r-t}{\sqrt{d}}, & j = 1, \\ -\frac{1-r-t}{\sqrt{j(j-1)}}, & 1 \leq i < j, \\ \sqrt{\frac{j-1}{j}}\,(1-r-t), & 1 < i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{T}\Delta_t\mathbf{T}^{tr} = [d_{ij}]_{i,j=1,\ldots,d}$. From relation a) above it follows that

$$d_{11} = \frac{1+(d-1)r-t}{d} + (1-r-t)\sum_{k=2}^{d}\frac{1}{k(k-1)} = \frac{1+(d-1)r-t}{d} + (1-r-t)\,\frac{d-1}{d} = 1-t,$$

and for $2 \le i \le d$, relation b) gives

$$\begin{aligned} d_{ii} &= \frac{1+(d-1)r-t}{d} + (1-r-t)\left(\frac{i-1}{i} + \sum_{k=1}^{d-i}\frac{1}{(i+k)(i+k-1)}\right) \\ &= \frac{1+(d-1)r-t}{d} + (1-r-t)\,\frac{d-1}{d} = 1-t. \end{aligned}$$

Next, for $2 \le i, j \le d$ with $i \ne j$ we obtain from relation b) above that

$$\begin{aligned} d_{ij} &= \frac{1+(d-1)r-t}{d} - \frac{1-r-t}{i\vee j} + (1-r-t)\left(\sum_{k=1}^{d-i\vee j}\frac{1}{(i\vee j+k)(i\vee j+k-1)}\right) \\ &= \frac{1+(d-1)r-t}{d} - \frac{1-r-t}{i\vee j} + (1-r-t)\left(\frac{d-1}{d} - \frac{i\vee j-1}{i\vee j}\right) = r. \end{aligned}$$

This in turn means $\mathbf{T}\Delta_t\mathbf{T}^{tr} = \Sigma_d - t\mathbf{I}_d$. Consequently, the characteristic polynomial for Σ_d is given by

$$\begin{aligned} \varphi_{\Sigma_d}(t) &= \det(\Sigma_d - t\mathbf{I}_d) = \det(\mathbf{T}\Delta_t\mathbf{T}^{tr}) = \det(\mathbf{T})\ \det(\Delta_t)\ \det(\mathbf{T}^{tr}) \\ &= \det(\mathbf{T})\ \det(\Delta_t)\ \det(\mathbf{T}^{-1}) = \det(\Delta_t) = \prod_{i=1}^{d}(\lambda_i - t). \end{aligned}$$

Hence λ_i, $1 \le i \le d$, are the latent roots of Σ_d. Therefore, Σ_d is a correlation matrix, i.e. positive semidefinite iff $\lambda_i \ge 0$ for all $1 \le i \le d$, i.e. $-\frac{1}{d-1} \le r \le 1$. Thus Lemma 5 is proved. □

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