For which complex numbers z is z^z real?

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Abstract We consider the problem for which complex numbers $z = a + bi,\ a, b \in \mathbb{R}$ the exponent z^z is a real number. This might be an interesting question for school mathematics.

Introduction and main results.

Complex numbers are, in former times, traditionally treated at the end of high school mathematics in Germany, especially Euler's theorem

$$e^{ix} = \cos(x) + i\sin(x),\ \ \mathrm{x} \in \mathbb{R}.$$

This leads to the conclusion

$$\cos(ix) = \frac{\mathrm{e}^x + e^{-x}}{2},\ \ x \in \mathbb{R},$$

i.e. the cosinus of a purely imaginary argument is real! (cf. Reidt/Wolff/Athen (1967), p.331).

In a similar way, it can be shown that

$$i^i = \left(\exp\left(i\frac{\pi}{2}\right)\right)^i = \exp\left(i^2\frac{\pi}{2}\right) = \exp\left(-\frac{\pi}{2}\right) \in \mathbb{R}.$$

This leads to the more general question for which complex numbers $z = a + bi,\ a, b \in \mathbb{R}$ the exponent z^z is a real number. For simplicity, we start our analysis for $a, b > 0$. In this case, by the polar coordinate transformation (cf. e.g. Reidt/Wolff/Athen (1967), p.236),

$$z = a + bi = re^{i\varphi} \text{ with } r = \sqrt{a^2 + b^2} \text{ and } a = r \cdot \cos(\varphi),\ b = r \cdot \sin(\varphi), \text{ hence } \varphi = \arctan\left(\frac{b}{a}\right).$$

So it follows that

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$$z^z = re^{i\varphi z} = \exp\big((\ln(r)+i\varphi)\cdot(a+bi)\big) = \exp\big((a\ln(r)-\varphi b)+(a\varphi+b\ln(r))i\big).$$

A sufficient condition to make this expression real is

$a\varphi + b\ln(r) = 0$ or $r = \exp\left(-\frac{a\varphi}{b}\right)$ with $\frac{a}{b} = \cot(\varphi)$. Hence we get $r = \exp(-\varphi\cdot\cot(\varphi))$ with $a = a(\varphi) := \exp(-\varphi\cdot\cot(\varphi))\cdot\cos(\varphi)$, $b = b(\varphi) := \exp(-\varphi\cdot\cot(\varphi))\cdot\sin(\varphi)$.

Here is a plot of $[a(\varphi), b(\varphi)]$, $0 \le \varphi \le \frac{\pi}{2}$:

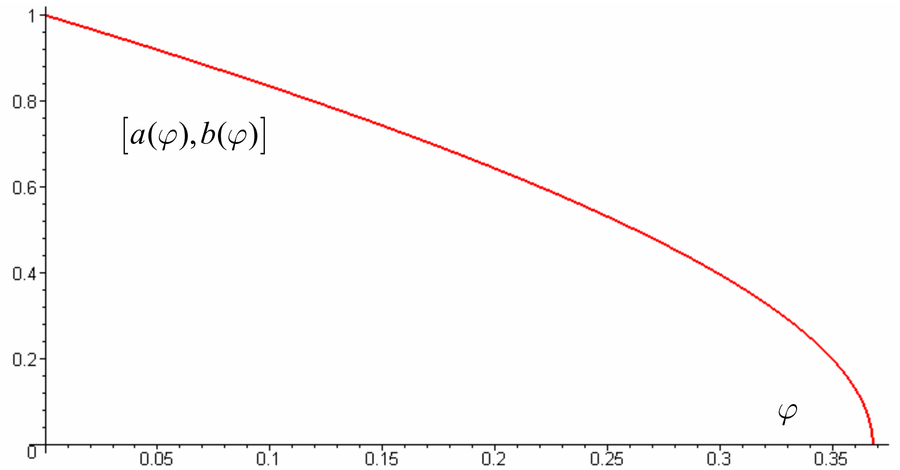

Fig.1

It follows that

$$z^z = re^{i\varphi z} = \exp(a\ln(r)-\varphi b) = \exp\big(-\varphi\exp(-\varphi\cdot\cot(\varphi))\cdot(\cos(\varphi)\cdot\cot(\varphi)+\sin(\varphi))\big)$$
$$= \exp\left(-\frac{\varphi}{\sin(\varphi)}\exp(-\varphi\cdot\cot(\varphi))\right) := f(\varphi)$$

Note that $\cot\left(\frac{\pi}{2}\right) = 0$, $\sin\left(\frac{\pi}{2}\right) = 1$, so that $i^i = \exp\left(-\frac{\pi}{2}\right)$ is reobtained from the last relation.

It can further be shown that the above relations are also generally valid for $0 < \varphi < \pi$ and $\pi < \varphi < 2\pi$. For $\varphi = \pi$, we can choose $a = -r$, $b = 0$ for arbitrary $r \in \mathbb{N}$ which means that $z^z = (-r)^{-r} = \frac{1}{(-r)^r}$ is also real.

Here is a plot of $f(\varphi), 0 \le \varphi \le \pi$:

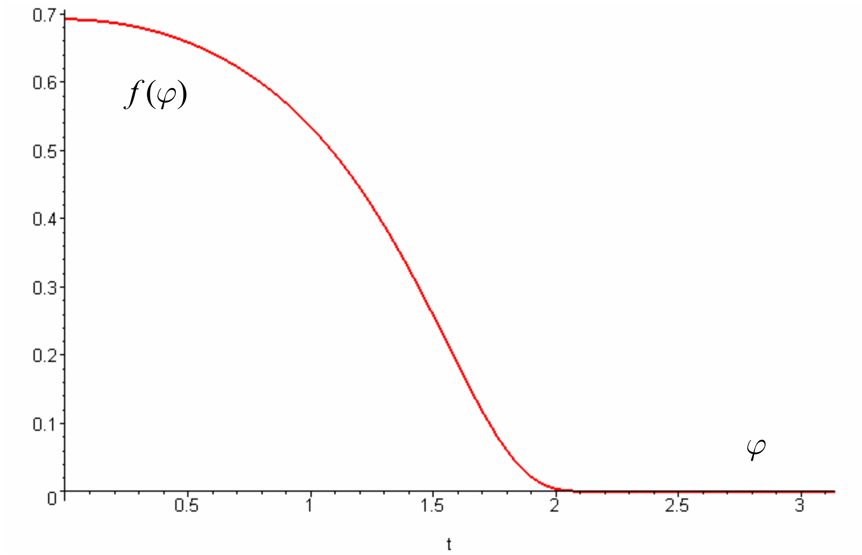

Fig.2

Some numerical examples:

φ	0	$\frac{\pi}{4}$	1	$\frac{\pi}{2}$	2	$\frac{3\pi}{4}$
a	$e^{-1} = 0,3678...$	$0,3223\ldots$	$0,2843\ldots$	0	$-1,0393\ldots$	$-7,4604\ldots$
b	0	$0,3223\ldots$	$0,4427\ldots$	1	$2,2710\ldots$	$7,4604\ldots$
$(a+bi)^{(a+bi)}$	$\exp\left(-e^{-1}\right) = 0,6922...$	$0,6026\ldots$	$0,5350\ldots$	$0,2078\ldots$	$0,0041\ldots$	$0,5390\ldots\cdot 10^{-15}$

Reference

F. Reidt, G. Wolff, H. Athen et al.: Elemente der Mathematik. Mathematisches Unterrichtswerk für höhere Lehranstalten. Oberstufe Band 3. 4.verbesserte Auflage, Hermann Schroedel Verlag KG, Hannover und Verlag Ferdinand Schöningh, Paderborn (1967)