Reflections on a canonical construction principle for multivariate copula models

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Abstract We consider a canonical construction principle for multivariate copula models on the basis of independent standard random variables which is in particular well suited for Monte Carlo Studies.

1. Introduction. There are many approaches to copula modelling in the literature, cf. e.g. the papers listed in the References below. Now for our investigations, let $\mathbf{U} = \{U_k\}_{k \in \mathbb{N}}$ be a sequence of independent standard random variables, i.e. each U_k has a continuous uniform distribution over the interval $[0,1]$. Let further $T_1, \cdots, T_n$, $\mathrm{n} \in \mathbb{N}$ be real continuous functions over $\mathbb{R}^{\mathbb{N}}$ and $V_i = T_i(\mathbf{U})$ for $i = 1, \cdots, n$ with a continuous uniform distribution over $[0,1]$ each. Then $\mathbf{V} = (V_1, \cdots, V_n)$ is a representative of an *n*-dimensional copula.

Note that if $W_i = T_i(\mathbf{U})$ is not immediately uniformly distributed then $V_i = F_i(W_i)$ is so if F_i denotes the c.d.f. of W_i.

2. Particular Cases. Consider the following special cases of a construction as indicated in the Introduction.

Case 1. Let $n = 2$ and $T_1(\mathbf{U}) = U_1$, $T_2(\mathbf{U}) = \alpha U_1 + (1-\alpha) U_2$, $0 < \alpha \le \frac{1}{2}$. it can easily be shown that the c.d.f. F_2 is given by

$$F_2(x,\alpha) = \begin{cases} \dfrac{x^2}{2\alpha(1-\alpha)} & 0 \le x \le \alpha \\ \dfrac{x}{\alpha} - \dfrac{\alpha^2}{2\alpha(1-\alpha)} & \alpha \le x \le 1-\alpha, \\ 1 - \dfrac{(1-x)^2}{2\alpha(1-\alpha)} & 1-\alpha \le x \le 1 \end{cases}$$

The following graphs show 5.000 simulations of $\mathbf{V}$ each, for various values of α.

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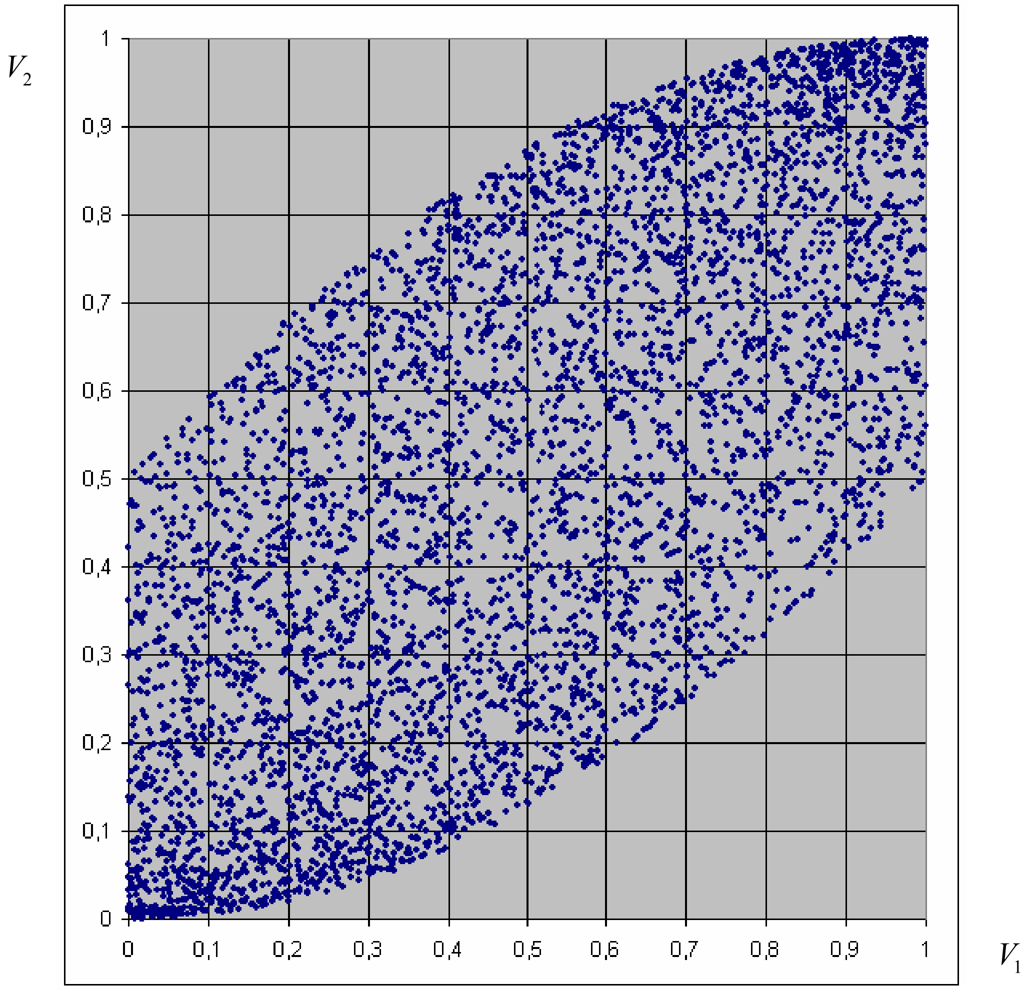
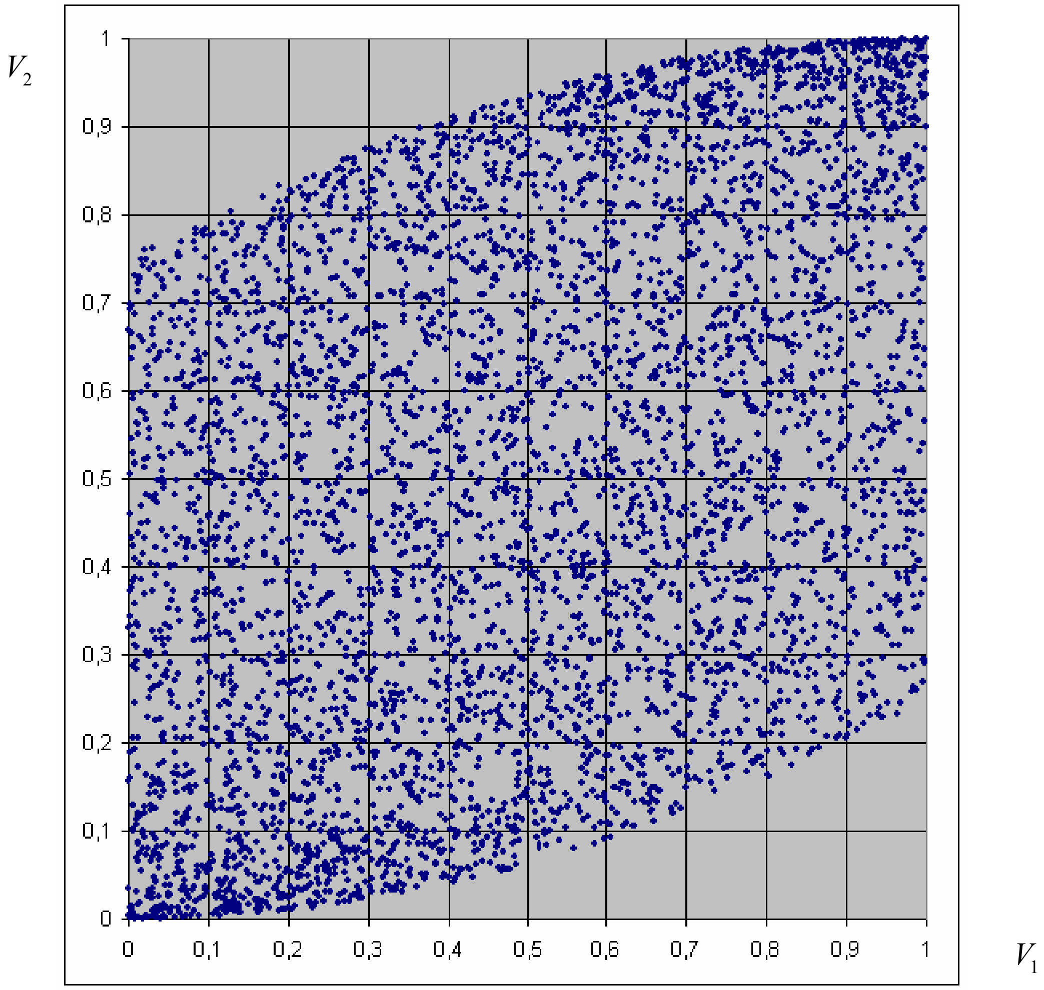
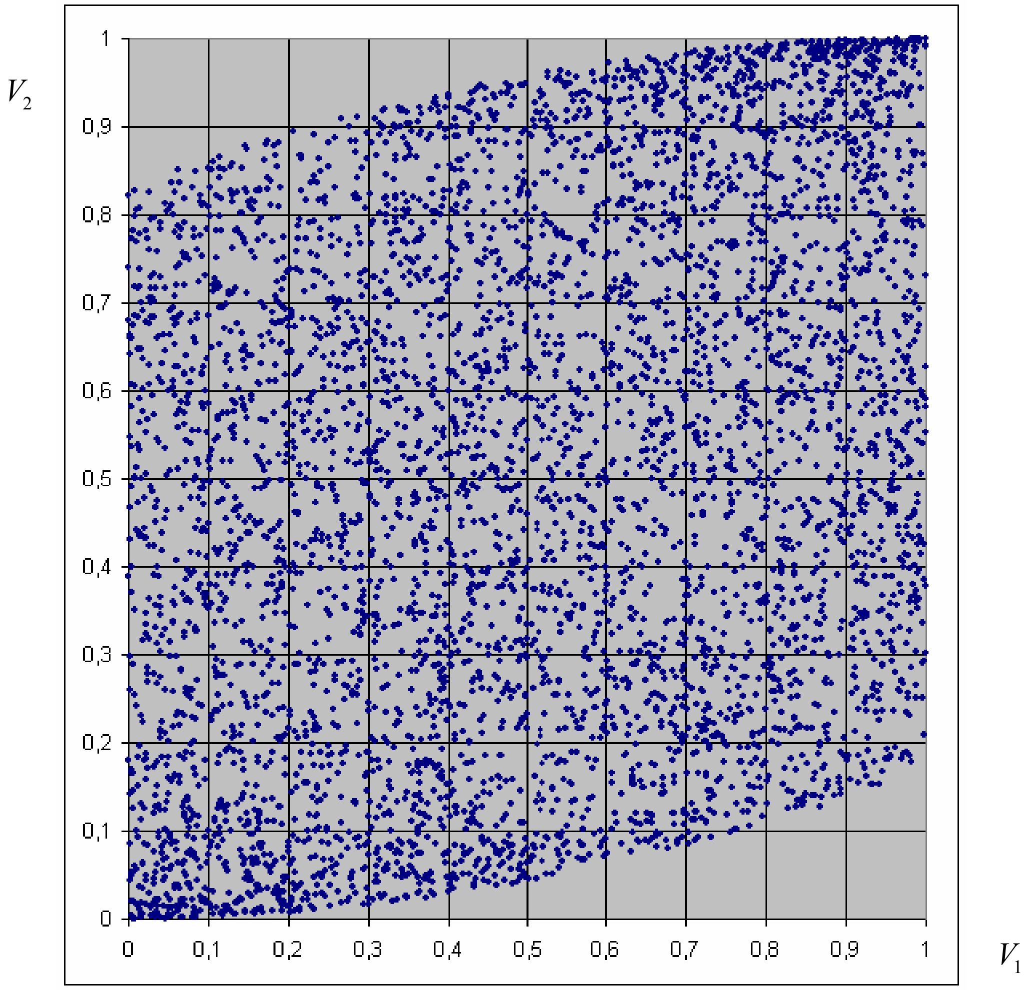


Scatterplot of $\mathbf{V}, \alpha = \frac{1}{2}$



Scatterplot of $\mathbf{V}, \alpha = \frac{1}{3}$



Scatterplot of $\mathbf{V}, \alpha = \frac{1}{4}$

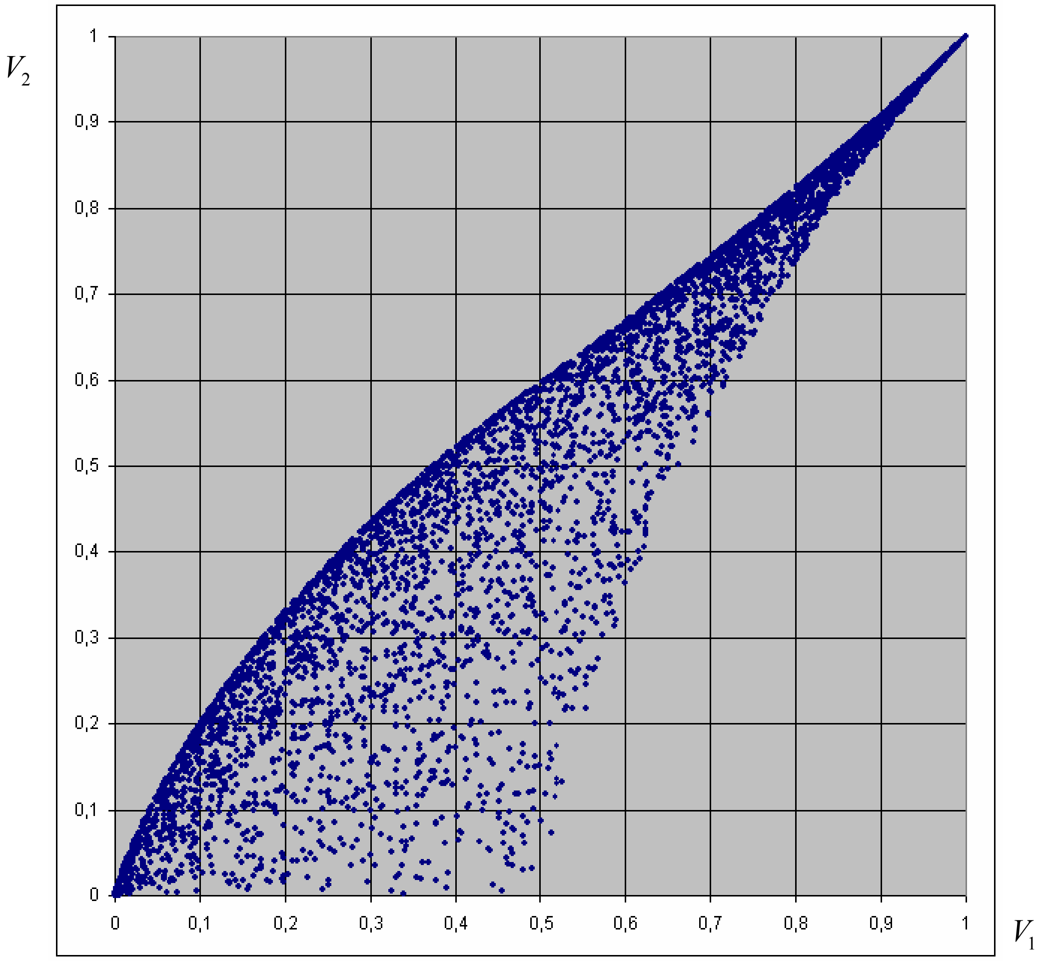
Case 2. Let $n = 2$ and $T_1(\mathbf{U}) = U_1 + U_2$, $T_2(\mathbf{U}) = U_1 \cdot U_2$. It is easy to see that the c.d.f. F_2 is given by

$F_2(x) = (1 - \ln(x)) \cdot x, \; 0 < x \leq 1$ and

$$F_1(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \leq x \leq 1 \\ 1 - 2\left(1 - \dfrac{x}{2}\right)^2, & 1 \leq x \leq 2 \end{cases}$$

(cf. Case 1 for $\alpha = \frac{1}{2}$).

This follows from the observation that $-\ln(T_2(\mathbf{U}))$ represents the sum of two independent standard exponentally distributed random variables, hence is gamma-distributed. The following graph shows 5.000 simulations of $\mathbf{V}$.

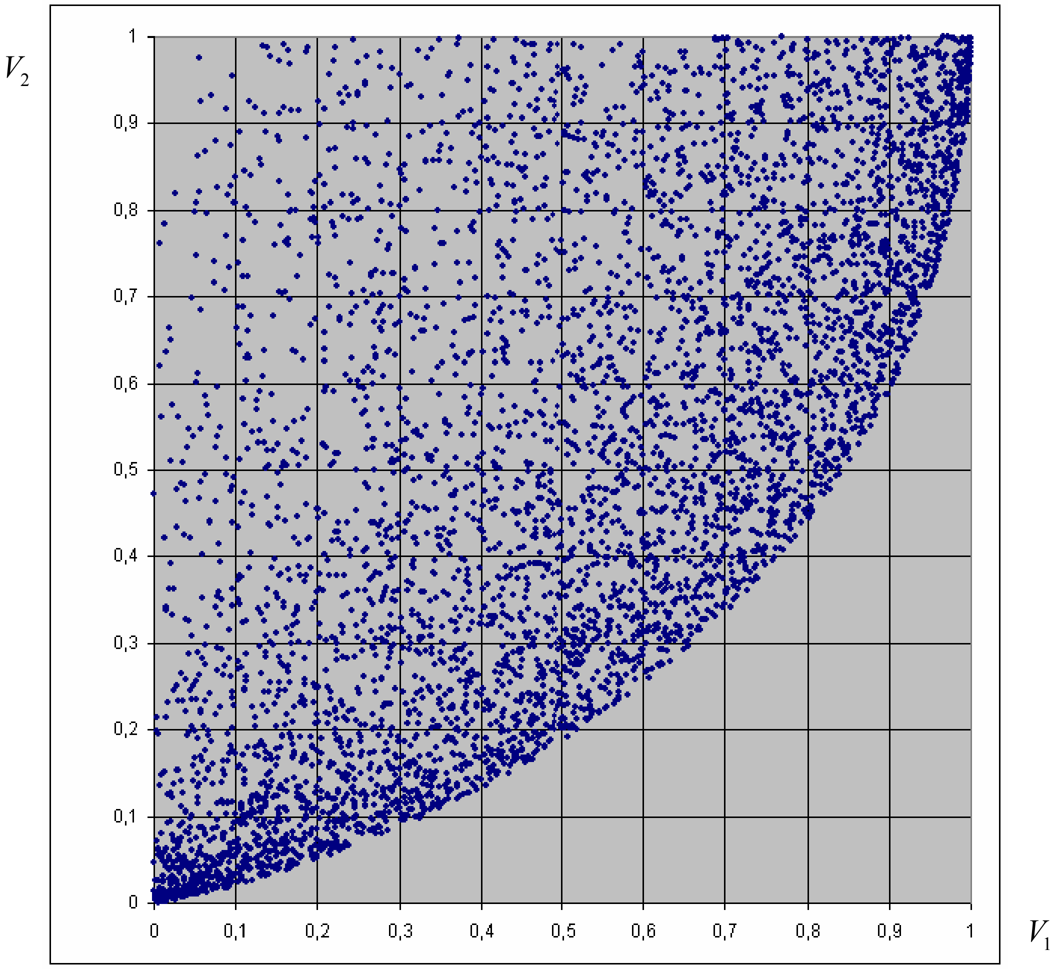


Scatterplot of **V**

Case 3. Let $n = 2$ and $T_1(\mathbf{U}) = U_1 \cdot U_2$, $T_2(\mathbf{U}) = \max(U_1, U_2)$ with the c.d.f. F_2 given by

$F_2(x) = x^2, 0 \le x \le 1$ and $F_1(x) = (1 - \ln(x)) \cdot x$, $0 < x \le 1$ (cf. Case 2).

The following graph shows 5.000 simulations of **V**.



Scatterplot of **V**

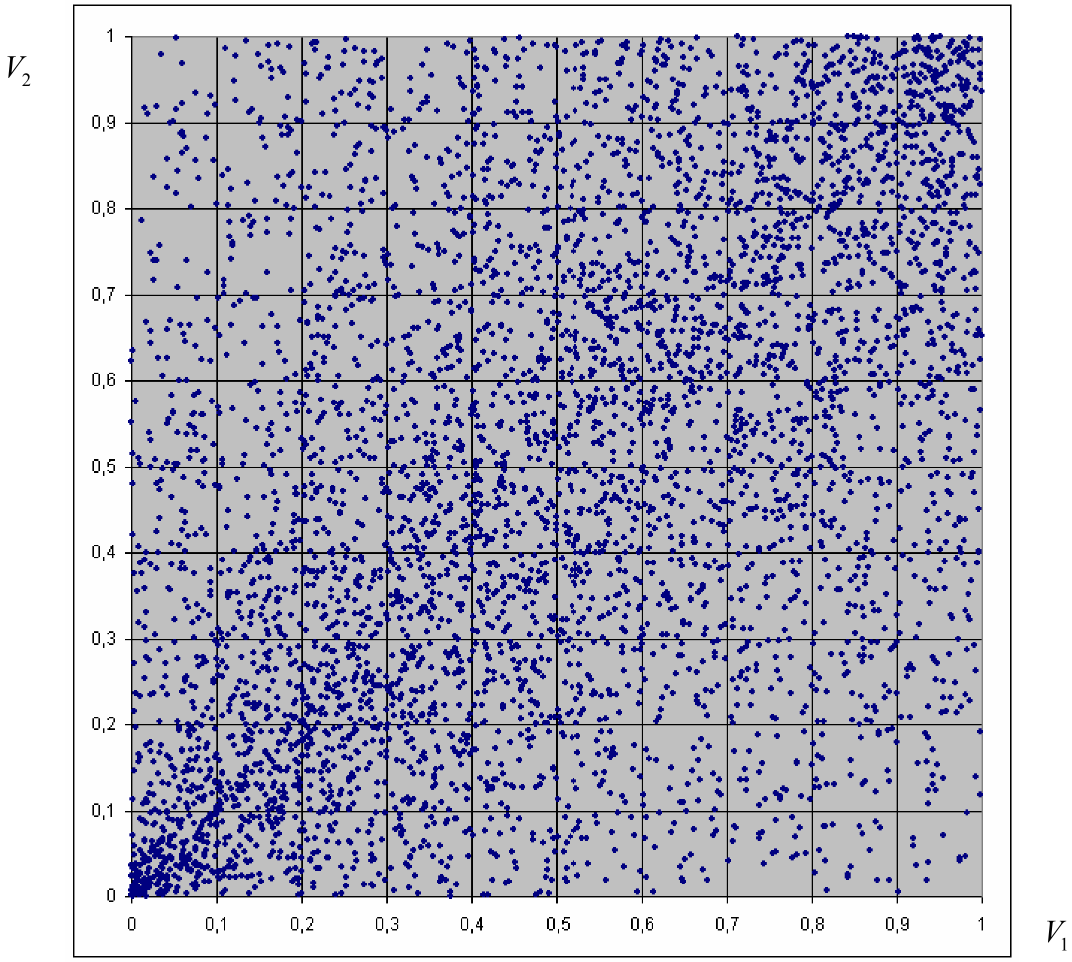
Case 4. Let $n = 3$ and $T_1(\mathbf{U}) = U_1$, $T_2(\mathbf{U}) = (U_1 \cdot U_2)^{U_3}$. Note that $V_2 = T_2(\mathbf{U})$ is already uniformly distributed over $[0,1]$ since for $0 < x < 1$

$$P(V_3 \le x) = P(-\ln(V_3) \ge -\ln(x)) = P\left(-\ln(U_1) - \ln(U_2) \ge \frac{-\ln(x)}{U_3}\right)$$

$$= \int_0^1 \left(1 - \frac{\ln(x)}{w}\right) \cdot x^{1/w} dw = w \cdot x^{1/w} \Big|_{w=0}^{w=1} = x$$

(note that $-\ln(U_1) - \ln(U_2)$ is gamma-distributed).

The following graph shows 5.000 simulations of **V**.



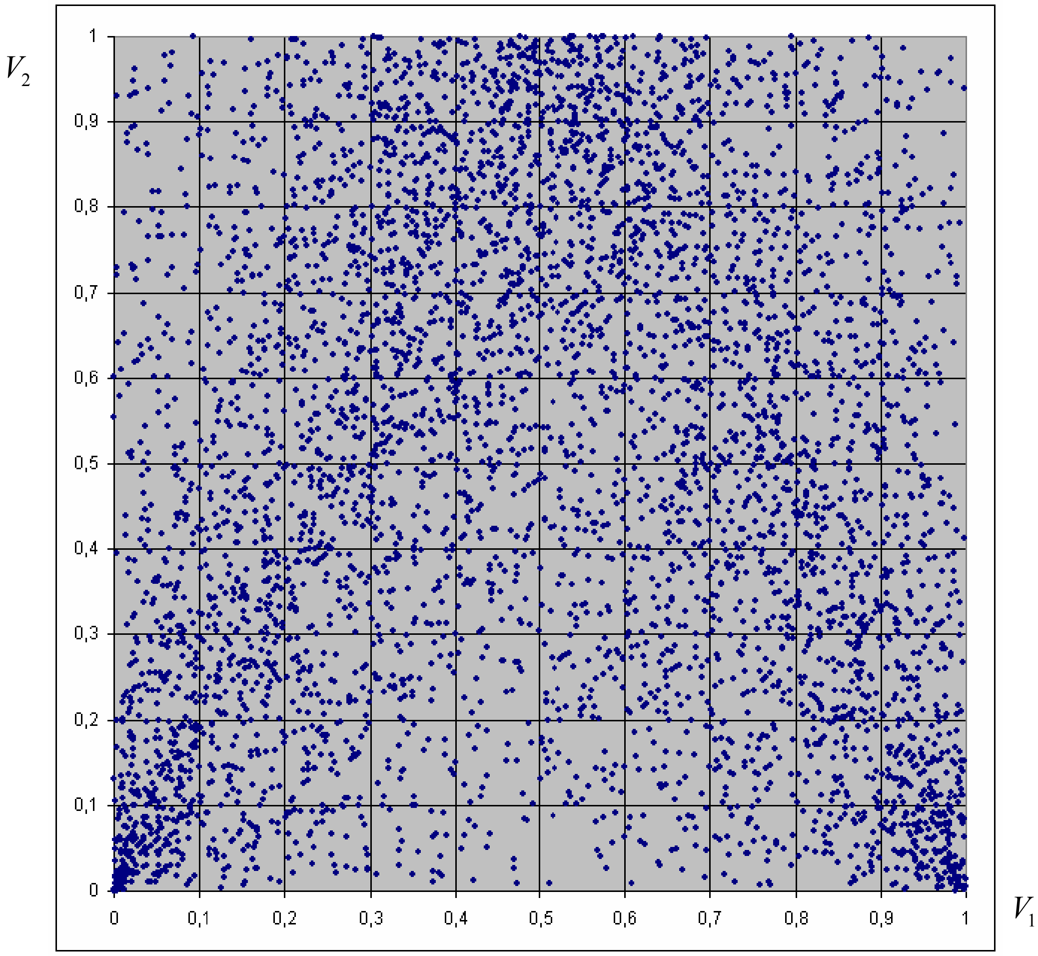
Scatterplot of **V**

Case 5. Let $n = 3$ and $T_1(\mathbf{U}) = \dfrac{U_1}{U_2}$, $T_2(\mathbf{U}) = (U_1 \cdot U_2)^{U_3}$. Note that the c.d.f. F_1 of $T_1(\mathbf{U})$ is given by

$$F_1(x) = \begin{cases} \dfrac{x}{2}, & x \le 1 \\ 1 - \dfrac{1}{2x}, & x \ge 1 \end{cases}$$

while $T_2(\mathbf{U})$ is continuous uniformly distributed over $[0,1]$, cf. Case 4.

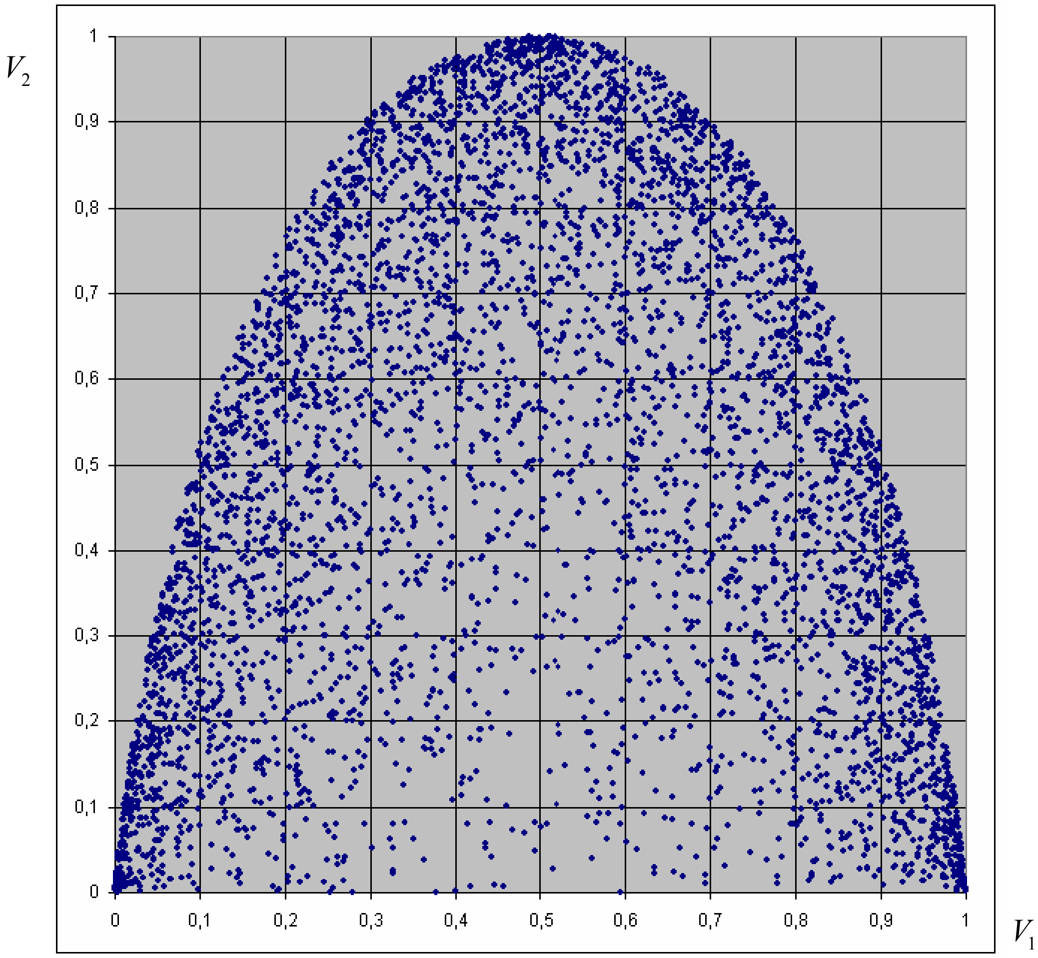
The following graph shows 5.000 simulations of **V**.



Scatterplot of **V**

Case 6. Let $n = 2$ and $T_1(\mathbf{U}) = \dfrac{U_1}{U_2}$, $T_2(\mathbf{U}) = U_1 + U_2$. For the corresponding c.d.f.s, see Cases 5 and 2.

The following graph shows 5.000 simulations of **V**.



Scatterplot of **V**

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