LIMIT LAWS FOR INTER-RECORD TIMES FROM NON-HOMOGENEOUS RECORD VALUES

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ABSTRACT

Starting from a (non-) homogeneous record value sequence from stochastically decreasing distributions possible limit laws for the resulting inter-record times are investigated. Conditions are given under which in the limit normal and Smirnov-type extreme value distributions are obtained.

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1. INTRODUCTION

In this paper we investigate some aspects of the non-homogeneous record process introduced in [3] which arises from the classical case (see [1], [5]) by possible changes of the underlying distributions after every record event. Let $\{X_{00},\, X_{nk}; n,k \geq 1\}$ be a family of independent random variables (r.v.'s) on a probability space $(\Omega, \mathcal{A}, P)$ with F_n being the cumulative distribution function (c.d.f) of the $X_{nk},\, n \geq 0.$ The sequence $\{\Delta_n;\, n \geq 0\}$ of *inter-record times* is recursively defined by

(1.1) $\Delta_0 = 0,\ \Delta_{n+1} = \min\left\{k;\, X_{n+1,k} > X_{n,\Delta_n}\right\}$
with $\min(\varnothing) \equiv X_{n,\infty} \equiv \infty.$

The sequence $\{R_n;\, n \geq 0\}$ of *record values* is defined by

(1.2) $R_n = X_{n,\Delta_n}$

(for measurability and other structural properties see [3]).

It is well known that in the continuous homogeneous case, i.e. all $F_n \equiv F$ where F is a fixed continouos c.d.f., $\log \Delta_n$ is asymptotically normally distributed ([1], [5]). In this paper we investigate limit laws for $\log \Delta_n$ in the case of distributions of the form

(1.3) $F_n = 1-(1-F)^{\lambda_n}$

where F again is a fixed continuous c.d.f. and $\{\lambda_n; n \geq 0\}$ is a non-decreasing sequence of positive real numbers. As has been pointed out in [3], this corresponds to a shock model with increasing safety since in this case, the sequence of inter-record times will be stochastically increasing. (Note that the homogeneous case is obtained if all $\lambda_n \equiv 1$.)

2. MAIN RESULTS

Let us for the moment assume that $\{F_n;\, n \ge 0\}$ only is a non-decreasing sequence of c.d.f.'s with common right end ξ (which may be ∞), and that all F_n are continuous at ξ. Then the following statement holds.

THEOREM 2.1. *With* $G_n = -\log(1-F_n),\ n \ge 0,$ *we have*

$$(2.1)\qquad \limsup_{n\to\infty} \left| \frac{\log \Delta_n - G_n(R_{n-1})}{\log n} \right| = 1 \ \ a.s.$$

PROOF. We will first show that (2.1) holds conditionally given the record sequence $\{R_n;\, n \ge 0\}$ which is non-degenerate under the conditions above (see [3]). But then, $\{\Delta_n;\, n \ge 0\}$ is a conditionally independent sequence given $\{R_n;\, n \ge 0\}$ with

$$(2.2)\qquad P\big(\Delta_n > m \,\big|\, \{R_k;\, k \ge 0\}\big) = F_n^m(R_{n-1}) \ \ a.s., \ \ n \ge 1,\ m \ge 0.$$

In order to show the conditional version of (2.1), it is sufficient to prove that $\sum_{n=1}^{\infty}\big(1-F_n(R_{n-1})\big) < \infty \ \ a.s.;$ the desired result will then follow by [5], Lemma 2. But for all *n*,

$$(2.3)\qquad \int_s^{\infty} (1-F_n(t))\,P_n(dt) \le \frac{1}{2}(1-F_n(s))^2,\ s \in \mathbb{R},$$

where P_n denotes the probability measure corresponding to F_n. Also, for $n \ge 1$,

$$(2.4)\qquad g_n(s) = \int_{(-\infty,x)} \frac{1}{1-F_n(y)}\,P^{R_{n-1}}(dy),\ x \in \mathbb{R}$$

is a P_n-density of R_n by [3], (3.2), hence by (2.3),

$$(2.5)\qquad E\big(1-F_n(R_n)\big) = \iint_{s>t} \frac{1-F_n(t)}{1-F_n(s)}\,P_n(dt)\,P^{R_{n-1}}(ds) \le \frac{1}{2}\int \big(1-F_n(s)\big)P^{R_{n-1}}(ds)$$
$$= \frac{1}{2}E\big(1-F_n(R_{n-1})\big),\ \ n \ge 1.$$

By the monotonicity of $\{F_n;\, n \ge 0\}$ and repeated use of (2.5) we thus have

$$(2.6)\qquad E\big(1-F_n(R_{n-1})\big) \le \frac{1}{2^n},\ n \ge 1, \text{ hence}$$

(2.7) $\sum_{n=1}^{\infty} E\left(1-F_n\left(R_{n-1}\right)\right) \le 1$, implying $\sum_{n=1}^{\infty}\left(1-F_n\left(R_{n-1}\right)\right) < \infty$ *a.s.*

which leads to the conditional version of (2.1). The unconditional version of (2.1) now us obtained by taking expectations on both sides. □

For the remainder of this section, we will assume that the underlying c.d.f.'s are of the form (1.3). From Theorem 2.1, the following limit law can be derived.

COROLLARY 2.2. *If* $\sum_{k=0}^{\infty} \frac{1}{\lambda_k^2} = \infty$, *then* $\log \Delta_n$ *is asymptotically normally distributed with*

$$(2.8) \qquad \frac{\frac{1}{\lambda_n}\log \Delta_n - \sum_{k=0}^{n-1}\frac{1}{\lambda_k}}{\sqrt{\sum_{k=0}^{n-1}\frac{1}{\lambda_k^2}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

If $\sigma^2 = \sum_{k=0}^{\infty}\frac{1}{\lambda_k^2} < \infty$, *then there exists a random variable X with zero mean and variance* σ^2 *such that*

$$(2.9) \qquad \frac{1}{\lambda_n}\log \Delta_n - \sum_{k=0}^{n-1}\frac{1}{\lambda_k} \to X \quad a.s.$$

PROOF. Since by continuity, the distribution of Δ_n only depends on $\lambda_1, \cdots, \lambda_n$, F may assumed to be the c.d.f. of an exponentially distributed r.v. with unit mean (that is, F_n is the c.d.f. of an exponentially distributed r.v. with mean $\frac{1}{\lambda_n}$). But in this case, $\left\{R_n;\, n \ge 0\right\}$ possesses independent exponentially distributed increments $\left\{Z_n;\, n \ge 0\right\}$ with $E\left(Z_n\right) = \frac{1}{\lambda_n}$ by [3], hence

$$(2.10) \qquad \frac{R_{n-1} - \sum_{k=0}^{n-1}\frac{1}{\lambda_k}}{\sqrt{\sum_{k=0}^{n-1}\frac{1}{\lambda_k^2}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ if } \sum_{k=0}^{\infty}\frac{1}{\lambda_k^2} = \infty, \text{ and}$$

$$(2.11) \qquad R_{n-1} - \sum_{k=0}^{n-1}\frac{1}{\lambda_k} \text{ converges } a.s. \text{ if } \sum_{k=0}^{\infty}\frac{1}{\lambda_k^2} < \infty.$$

While (2.11) is a simple consequence of Kolmogorov's theorem, (2.10) is a consequence of the Ljapunov type condition

$$(2.12)\quad \frac{\sum_{k=0}^{n-1} E\left(Z_k - \frac{1}{\lambda_k}\right)^4}{\left(\sqrt{\sum_{k=0}^{n-1} \frac{1}{\lambda_k^2}}\right)^2} = 9\frac{\sum_{k=0}^{n-1} \frac{1}{\lambda_k^4}}{\left(\sqrt{\sum_{k=0}^{n-1} \frac{1}{\lambda_k^2}}\right)^2} \to 0 \text{ for } n \to \infty.$$

Since in the case of exponential distributions, $G_n(x) = -\log\left(1 - F_n(x)\right) = \lambda_n x,\ x \geq 0,$ we have

$$(2.13)\quad \left|\frac{1}{\lambda_n}\log\Delta_n - R_{n-1}\right| = \frac{\log n}{\lambda_n} \cdot \left|\frac{\log\Delta_n - G_n\left(R_{n-1}\right)}{\log n}\right|,$$

hence (2.8) and (2.9) follow from (2.10) and (2.11), resp. applying Theorem 2.1.

It should be noted that in either case

$$(2.14)\quad E\left(\log\Delta_n\right) = \lambda_n \sum_{k=0}^{n-1} \frac{1}{\lambda_k} - C + \mathcal{O}\left(\frac{n}{2^n}\right) \text{ and}$$

$$(2.15)\quad Var\left(\log\Delta_n\right) = \lambda_n^2 \sum_{k=0}^{n-1} \frac{1}{\lambda_k^2} + \frac{\pi^2}{6} + \mathcal{O}\left(\frac{n^2}{2^n}\right),$$

where C denotes Euler's constant. This is obvious from the proof of the corresponding relations in the homogeneous case ([2]) and the fact that in the general model, Δ_n is stochastically not smaller than in the homogeneous case. □

Relation (2.8) essentially says that the asymptotic behaviour of inter-record times is robust against small alterations in the underlying distributions (cf. [1], [5]) whereas (2.9) indicated that major alterations can lead to a completely different limiting distribution, which can also be seen by the following example.

COROLLARY 2.3. *Let* $\lambda_n = n + k$ *with* $k \geq 1$ *being fixed. Then*

$$(2.16)\quad \frac{1}{n}\log\Delta_n - \log n \to X_k \quad a.s. \text{ for } n \to \infty$$

Where X_k *is a r.v. following a Smirnov-type extreme value distribution given by*

$$(2.17)\quad P\left(X_k \leq t\right) = \int_{e^{-t}}^{\infty} \frac{s^{k-1}}{(k-1)!} e^{-s}\, ds,\ t \in \mathbb{R}.$$

PROOF. Let again F be the c.d.f. of an exponentially distributed r.v. with unit mean. Then $R_{n-1} \stackrel{\mathcal{D}}{=} W_{(n)}$, the n-th order statistic of $n + k - 1$ i.i.d r.v.'s $W_1, \cdots, W_{n+k-1}$ with c.d.f. F hence

$W_{(n)} - \log n \xrightarrow{\mathcal{D}} X_k$. But by (2.11), $R_{n-1} - \log n \sim R_{n-1} - \sum_{j=0}^{n-2} \frac{1}{k+j} + C - \sum_{j=0}^{k-1} \frac{1}{j}$ converges *a.s.*, hence $R_{n-1} - \log n \to X_k$ *a.s.* Now by (2.13) and Theorem (2.1), $\frac{1}{n} \log \Delta_n - \log n \sim \frac{1}{n+k} \log \Delta_n - \log n \to X_k$ *a.s.* □

From Corollary 2.3, we also have $\frac{1}{n \log n} \log \Delta_n \to 1$ *a.s.*, while in the homogeneous case, $\frac{1}{n} \log \Delta_n \to 1$ *a.s.* ([5]).

Note that if $k = 1$ in Corollary 2.3 and F is the c.d.f. of an exponentially distributed r.v., the corresponding record counting process $N(t) = \#\{n; R_n \leq t\}$, $t \geq 0$ is a Furry-Yule process (cf. [4]); in this case, the limiting distribution for $\log \Delta_n$ simply is doubly exponential.

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Published in:

Journal of Organizational Behavior & Statistics, Vol. 1, No. 1, pp. 69 – 74.