On an approximation of the total aggregate risk distribution in a modified collective risk model

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Abstract We consider a portfolio of *n* risks $X_1,\cdots,X_n,\ n\in\mathbb{N},$ which are assumed to be independent and identically distributed with a finite expectation $\mu=E(X_k)$ and finite variance $\sigma^2=Var(X_k),\ k=1,\cdots,n.$ Moreover, we assume that only a certain random portion *p* of the contracts will be affected during the insurance period, and that multiple claims are not possible. In order to model this aspect we assume that $J_1,\cdots,J_n,\ n\in\mathbb{N},$ are additional conditionally independent binomially distributed random variables with a random success parameter $p=P(J_k=1)=1-P(J_k=0),\ k=1,\cdots,n,$ being Beta-distributed with shape parameters $\alpha>0$ and $\beta>0$ which are also independent of the risks under consideration.

1. Introduction We investigate a portfolio of *n* risks $X_1,\cdots,X_n,\ n\in\mathbb{N},$ being independent and identically distributed as X with a finite expectation $\mu=E(X)$ and finite variance $\sigma^2=Var(X).$ Additionally, we assume that only a certain random portion *p* of the contracts will be affected during the insurance period, and that multiple claims are not possible. In order to model this aspect we assume that $J_1,\cdots,J_n,\ n\in\mathbb{N},$ are additional conditionally independent binomially distributed random variables with a random success parameter $p=P(J_k=1)=1-P(J_k=0),\ k=1,\cdots,n,$ being Beta-distributed with shape parameters $\alpha>0$ and $\beta>0$ which are also independent of the risks under consideration. This model has been investigated recently in Pfeifer [5] (2022). We have, as is well-known,

$$E(p)=\frac{\alpha}{\alpha+\beta}\ \text{and}\ Var(p)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(cf. Johnson et al. [1], Chapter 3, p. 217). Then the total aggregate risk S_n is given by $S_n=\sum_{k=1}^{n}J_k\cdot X_k.$ Note that the distribution of S_n is stochastically equivalent to the distribution of $\tilde{S}_N=\sum_{k=1}^{N}X_k$ where $N_n=\sum_{k=1}^{n}J_k$ follows a Beta-Binomial distribution with parameters

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$$E(N_n) = \frac{n\alpha}{\alpha+\beta} \text{ and } Var(N_n) = \frac{n\alpha\beta \cdot (\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)} = n \cdot Var(p) \cdot (\alpha+\beta+n)$$

(cf. Johnson et al. [2], Chapter 6.2.2, p. 253). Note that for all $n \in \mathbb{N}$, N_n and the $\{X_k\}_{k\in\mathbb{N}}$ are independent. It follows that we have

$$E(S_n) = E(N) \cdot \mu = n \cdot E(p) \cdot \mu \quad \text{and}$$

$$Var(S_n) = E(N) \cdot Var(X) + Var(N) \cdot \{E(X)\}^2 = n \cdot E(p) \cdot \sigma^2 + n \cdot Var(p) \cdot (\alpha+\beta+n) \cdot \mu^2$$

Clearly, these moment relations follow from Wald's well-known formula and the Blackwell-Girshick-formula in collective risk theory (cf. Klugman [3], relation (9.9), p.143, or Rotar [6], Chapter 4, Propositions 1 and 2, p.200).

2.Approximations If $E(N_n)$ is large enough and $Var(N_n)$ is small enough it seems reasonable to approximate the distribution of S_n by a normal law (cf. Rotar [6], Chapter 4.1.1, p. 228). Note, however, that condition (4.1.6) or (4.1.7) of Theorem 12 in Rotar [6], p. 231 is not satisfied if α and β are constant since in this case,

$$\lim_{n\to\infty} \sqrt{\frac{Var(N_n)}{E(N_n)}} = \lim_{n\to\infty} \sqrt{\frac{\beta \cdot (\alpha+\beta+n)}{(\alpha+\beta)\cdot(\alpha+\beta+1)}} = \infty.$$

Alternatively, we may assume α and β to be dependent on *n*, say $\alpha = \gamma n$ and $\beta = \delta n$ with fixed $\gamma, \delta > 0$. in this case,

$$\lim_{n\to\infty} E(N_n) = \infty \quad \text{and}$$

$$\lim_{n\to\infty} \sqrt{\frac{Var(N_n)}{E(N_n)}} = \lim_{n\to\infty} \sqrt{\frac{\beta \cdot (\alpha+\beta+n)}{(\alpha+\beta)\cdot(\alpha+\beta+1)}} = \frac{\sqrt{\delta \cdot (\gamma+\delta+1)}}{\gamma+\delta}.$$

We can observe that for large values of *n* the corresponding Beta-Binomial distribution is close to the binomial distribution with success parameter $p = \frac{\gamma}{\gamma+\delta}$. Likewise, the corresponding Beta-Binomial distribution can be approximated by a normal law itself. We present some graphs for a visualization. The red line in Fig. 1 to Fig. 4 represents the counting density of the Beta-Binomial distribution, the blue line represents the counting density of the Binomial distribution. Fig. 5 to Fig. 8 show Quantile-Quantile-Plots for 100 simulated Beta-Binomial distributions each, cf. Pfeifer [4] (2019). *T* denotes the corresponding correlation based test statistic.

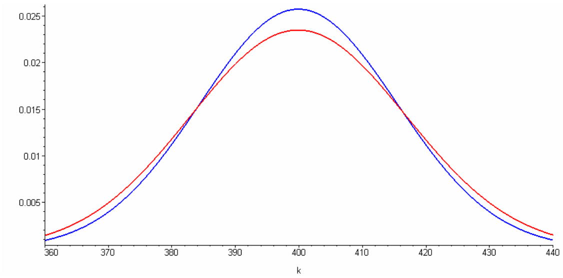

Fig. 1

$n = 1000,\ \alpha = 2n,\ \beta = 3n$

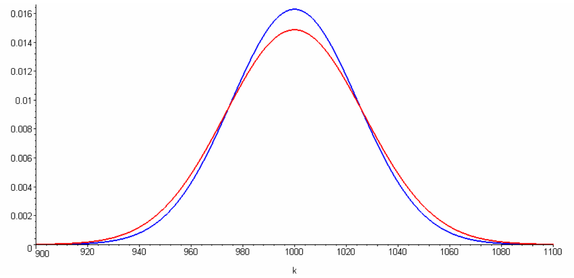

Fig. 2

$n = 2500,\ \alpha = 2n,\ \beta = 3n$

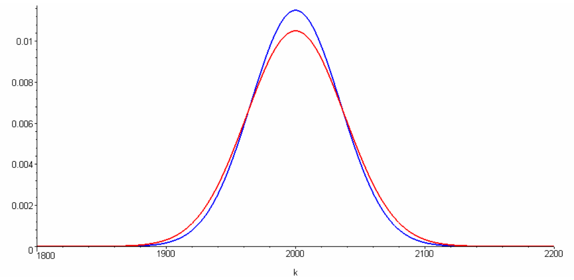

Fig. 3

$n = 5000,\ \alpha = 2n,\ \beta = 3n$

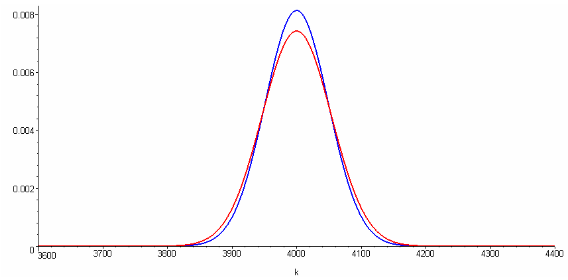

Fig. 4

$n = 10000,\ \alpha = 2n,\ \beta = 3n$

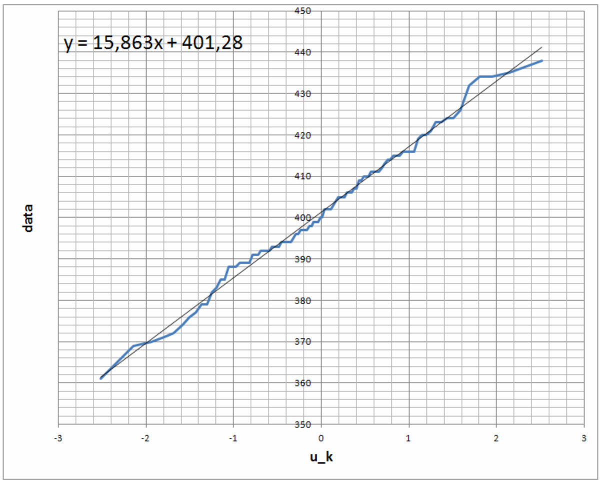

Fig. 5

$n = 1000,\ \alpha = 2n,\ \beta = 3n$

$T = 5,6477,\ p = 87,07\%$

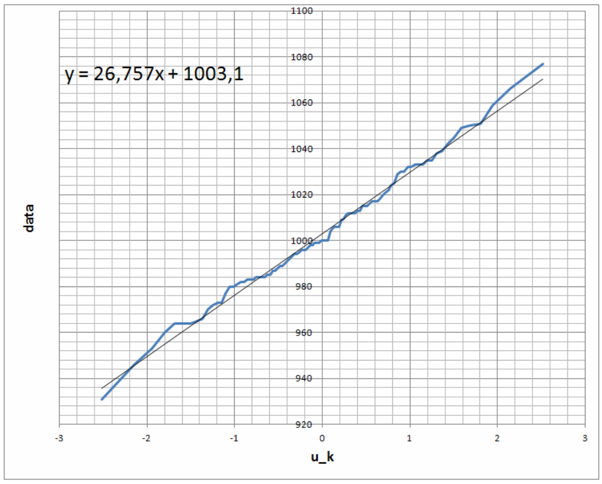

Fig. 6

$n = 2500,\ \alpha = 2n,\ \beta = 3n$

$T = 5,5583,\ p = 82,89\%$

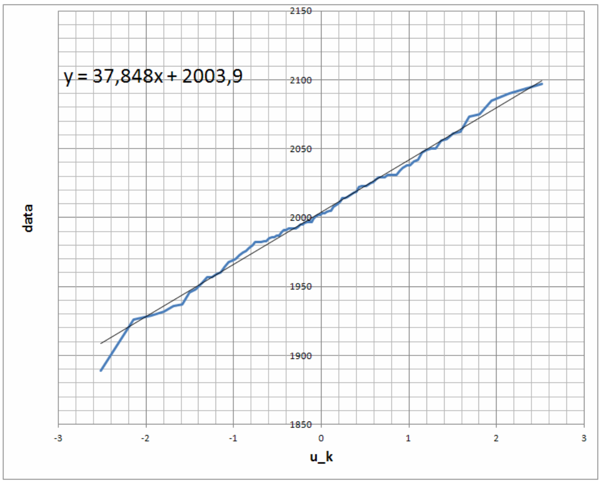

Fig. 7

$n = 5000,\ \alpha = 2n,\ \beta = 3n$

$T = 5{,}4882,\ p = 79{,}07\%$

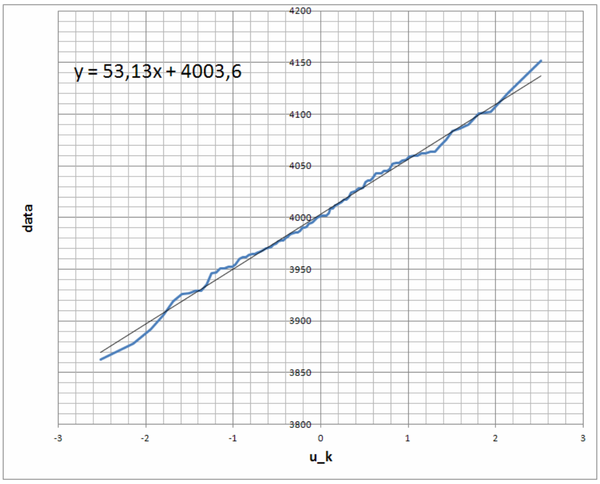

Fig. 8

$n = 10000,\ \alpha = 2n,\ \beta = 3n$

$T = 5{,}8477,\ p = 93{,}73\%$

Since the Binomial distribution fulfils the conditions of Theorem 12 in Rotar [6], p. 231, and the approximation of the Beta-Binomial distribution by a normal law under the conditions discussed above seems acceptable it seems reasonable to approximate the distribution of the total aggregate risk S_n itself by a normal law.

3 A Case Study In this section, we consider a portfolio with $n = 50.000$ insurance contracts and assume for simplicity that the sums insured follow a lognormal distribution with expectation $\mu_{SI} > 0$ and standard deviation $\sigma_{SI} > 0$. further, we assume that the individual loss realized is a beta distributed multiple of the individual sum insured, independent of the sums insured. The corresponding beta parameters are α_{SI} and β_{SI}. The expectation of the loss factor then is $\mu_{factor} = E(p) = \dfrac{\alpha_{SI}}{\alpha_{SI} + \beta_{SI}}$ and the standard deviation is $\sigma_{factor} = \sqrt{\dfrac{\alpha_{SI} \cdot \beta_{SI}}{(\alpha_{SI} + \beta_{SI})^2 (\alpha_{SI} + \beta_{SI} + 1)}}$ while the expectation of the realized loss is $\mu = \mu_{factor} \cdot \mu_{SI}$ and its standard deviation $\sigma = \sqrt{(\sigma_{SI}^2 + \mu_{SI}^2) \cdot (\sigma_{factor}^2 + \mu_{factor}^2) - \mu^2}$. Fig. 9 to Fig. 13 show Quantile-Quantile-Plots for 100 simulated aggregated losses, cf. Pfeifer [4] (2019).

Parameters	α	β	$E(N_n)$	α_{SI}	β_{SI}	μ_{factor}	σ_{factor}	μ_{SI}	σ_{SI}	μ	σ
Fig. 9	70	2.730	1.250	2	398	0,5%	0,35%	100.000	10.000	500	357,50
Fig. 10	90	4.410	1.000	2	198	1,0%	0,70%	100.000	10.000	1.000	712,36
Fig. 11	70	2.730	1.250	2	398	0,5%	0,35%	100.000	20.000	500	372,86
Fig. 12	70	$2.263,\bar{3}$	1.500	2	198	1,0%	0,70%	100.000	20.000	1.000	743,13
Fig. 13	70	$2.263,\bar{3}$	1.500	2	198	1,0%	0,70%	100.000	50.000	1.000	930,41

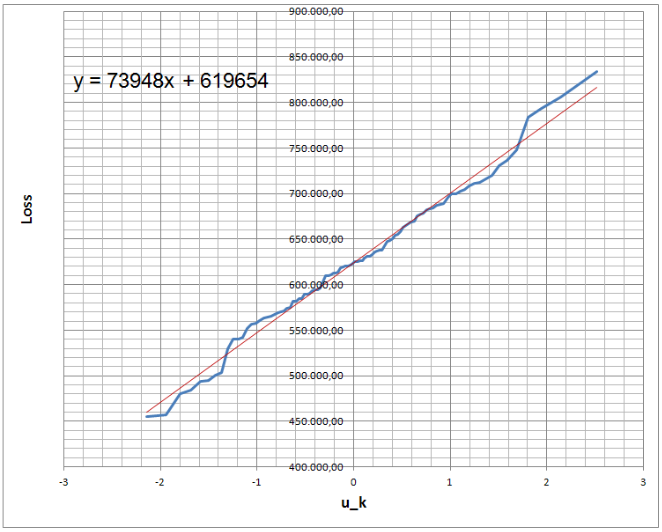

Fig. 9

$T = 5,2584, \quad p = 63,54\%$

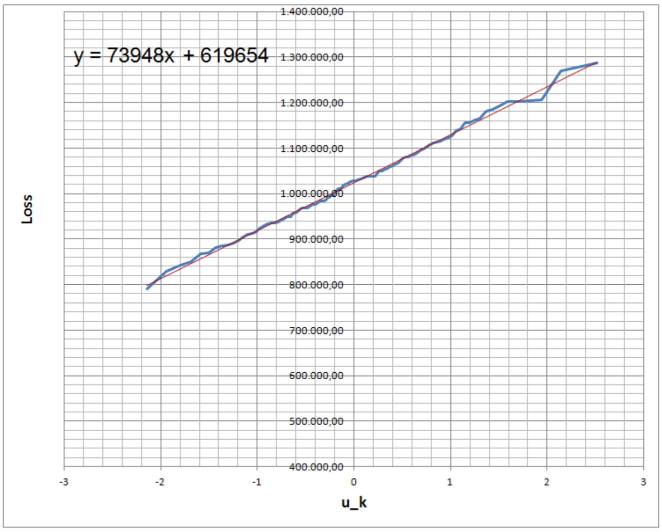

Fig. 10

$T = 6,4951, \quad p = 99,77\%$

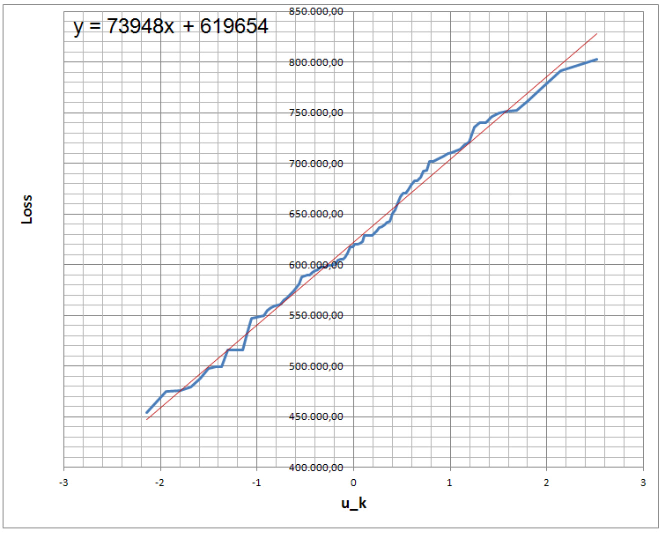

Fig. 11

$T = 5,5421, \quad p = 82,05\%$

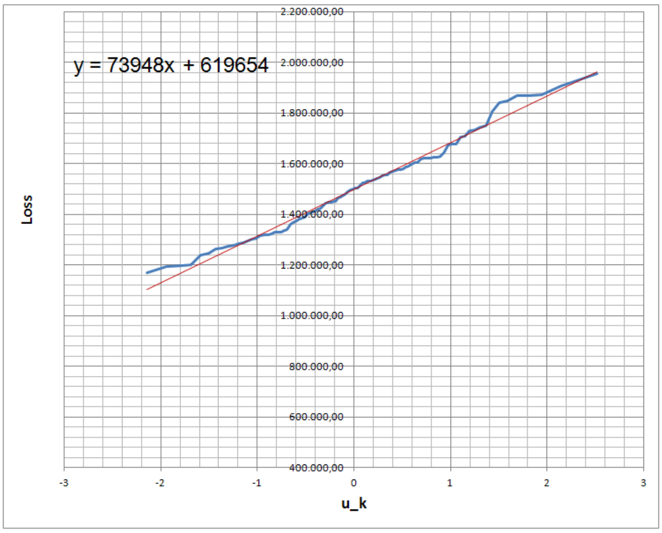

Fig. 12

$T = 5,0173, \quad p = 44,47\%$

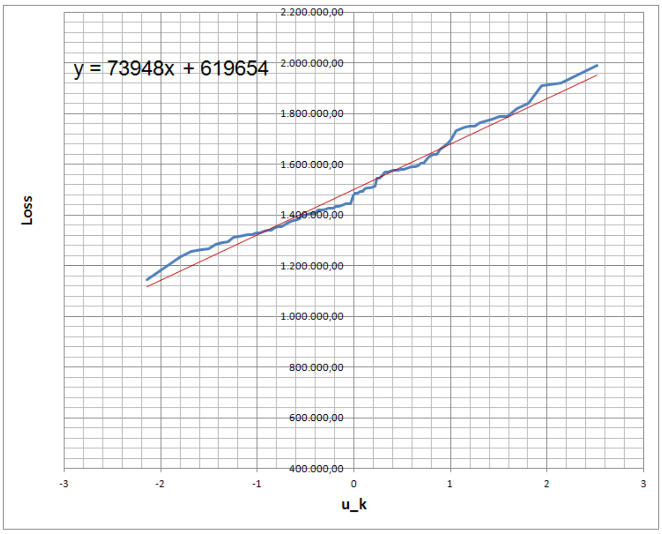

Fig. 13

$T = 4,5438, \quad p = 13,74\%$

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