Tail-dependence properties of some new types of copula models (part II)

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Abstract We continue the investigation of the tail-dependence behaviour of some new types of copula models, published recently in [5] and [6].

1. Introduction. For the sake of simplicity, we concentrate our investigations to the two-dimensional case. Let U_1, U_2 be standard random variables, i.e. they follow a uniform distribution over the interval $[0,1]$ each. Let further T_1, T_2 be real continuous functions over $\mathbb{R}^2$ and $W_1 = T_1(U_1, U_2),\ W_2 = T_2(U_1, U_2)$. If W_1, W_2 already follow a continuous uniform distribution over $[0,1]$ each, then (W_1, W_2) is a representative of a two-dimensional copula. Otherwise, $(V_{1,} V_2) := (F_1(W_1),\ F_2(W_2))$ is a representative of a two-dimensional copula if F_i denotes the continuous c.d.f. of $W_i,\ \mathrm{i} = 1, 2.$

Of particular interest especially for financial markets or risk management is the tail dependence of copulas which was explicitly treated for dependence-of-unity copulas in [2], [3] and [4], and for the new approach in [6], which we shall continue here. The simplest definition of the coefficient λ_U of upper and λ_L of lower tail dependence is

$$\lambda_U = \lim_{t \uparrow 1} \frac{P\big(W_1 > F_1^{-1}(t), W_2 > F_2^{-1}(t)\big)}{1-t},\ \lambda_L = \lim_{t \downarrow 0} \frac{P\big(W_1 \le F^{-1}(t), W_2 \le G^{-1}(t)\big)}{t},$$ see e.g. [1], Def. 7.36, p.247.

In case that $F_1 = F_2$, i.e. W_1 and W_2 have same distribution, we also have

$$\lambda_U = \lim_{s \uparrow \infty} \frac{P\big(W_1 > s,\ W_2 > s\big)}{1-F(s)},\ \lambda_L = \lim_{s \downarrow -\infty} \frac{P\big(W_1 \le s,\ W_2 \le s\big)}{F(s)}.$$

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2. Particular Cases.

Case 1. Here we consider the choice

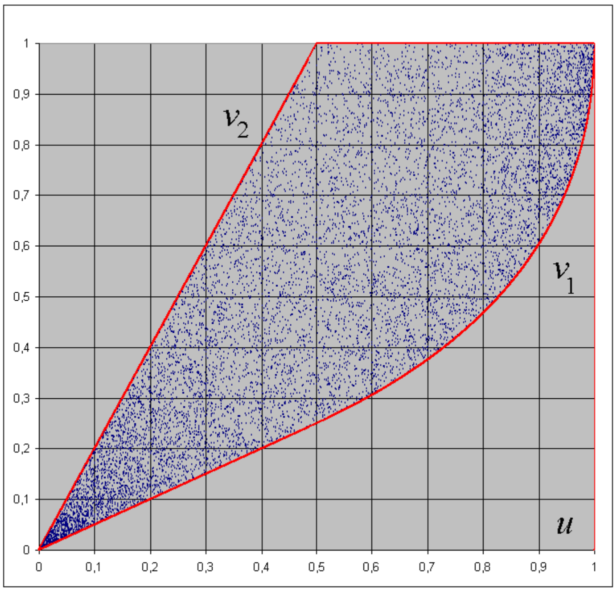
$$W_1 = T_1(U_1,\ U_2) = U_1 + U_2,\ W_2 = T_2(U_1,\ U_2) = \max\left(U_1, U_2\right).$$

It is easy to see that the corresponding c.d.f.'s are given by

$$F_1(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x \le 1 \\ 1 - 2\left(1 - \dfrac{x}{2}\right)^2, & 1 \le x \le 2. \end{cases} \quad \text{and}$$

$$F_2(x) = x^2,\ 0 \le x \le 1.$$

The following graph shows 10,000 simulations of $(V_1,\ V_2) = \left(F_1\left(W_1\right),\ F_2\left(W_2\right)\right)$.



The red lines (u,v) represent the (sharp) lower and upper envelopes of the copula, which are given by

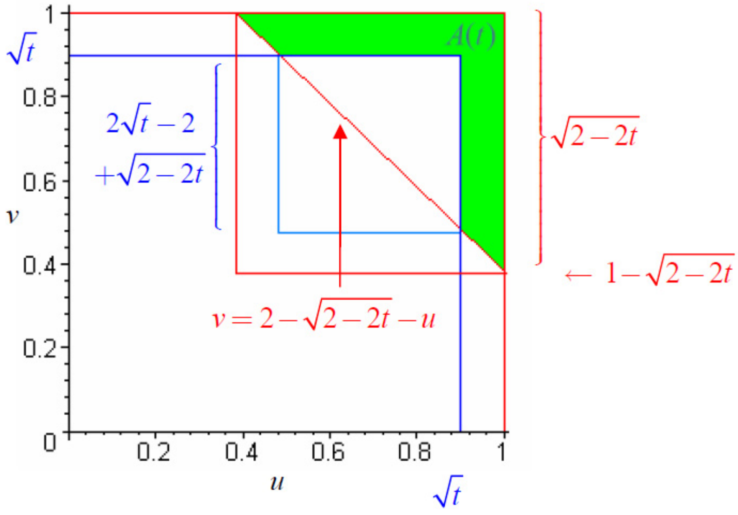
$$v_1 = v_{lower} = \begin{cases} \dfrac{u}{2}, & \text{if } u \le \dfrac{1}{2} \\ \left(1 - \sqrt{\dfrac{1-u}{2}}\right)^2, & \text{otherwise} \end{cases}$$

and $v_2 = v_{upper} = \begin{cases} 2u, \text{ if } u \le \dfrac{1}{2} \\ 1, \ \text{ if } u > \dfrac{1}{2} \end{cases}$,

The lower bound is reached if V_1 and V_2 are close to each other, while the upper bound is reached if one of V_1 or V_2 is close to zero.

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 0$.

We start with some preliminary inequalitites.



We have, for $t > \dfrac{1}{2}$, with μ denoting Lebesgue measure,

$$P\left(W_1 > F_1^{-1}(t), W_2 > F_2^{-1}(t)\right) = P\left(\max(U_1, U_2) > \sqrt{t},\, U_2 > 2 - \sqrt{2-t} - U_1\right)$$

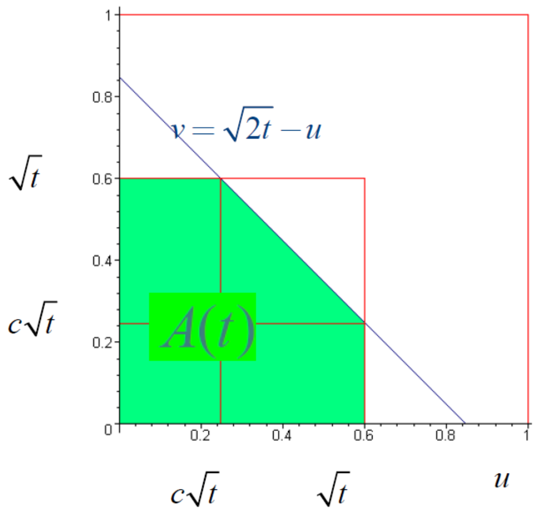
$$= \mu\left(A(t)\right) = \frac{1}{2}\left(\sqrt{2-2t}^{\,2} - \left(2\sqrt{t} - 2 + \sqrt{2-2t}\right)^2\right)$$

or, by a Taylor expansion around the point $t = 1$,

$$\mu\left(A(t)\right) = \sqrt{2}(1-t)^{3/2} + \mathcal{O}\left((1-t)^2\right), \text{ hence } \lambda_U = \lim_{t\to 1} \frac{\mu\left(A(t)\right)}{1-t} = 0, \text{ as stated.}$$

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 2\left(\sqrt{2}-1\right) = 0{,}828427\ldots$.

We start again with some preliminary inequalitites.



We have, for $t > \frac{1}{2}$, with $c = \sqrt{2} - 1$,

$$P\left(\max(U_1,U_2) \le F_2^{-1}(t),\, U_1 + U_2 \le F_1^{-1}(t)\right) = P\left(\max(U_1,U_2) \le \sqrt{t},\, U_2 \le \sqrt{2t} - U_1\right)$$

$$= \mu\left(A(t)\right) = t - \frac{\left\{(1-c)\sqrt{t}\right\}^2}{2} = t\left(1 - \frac{(1-c)^2}{2}\right) = 2ct$$

and hence $\lambda_L = \lim_{t \downarrow 0} \frac{\mu\left(A(t)\right)}{t} = 2c = 2\left(\sqrt{2} - 1\right) = 0{,}828427...$, as stated.

Case 2. Here we consider the choice

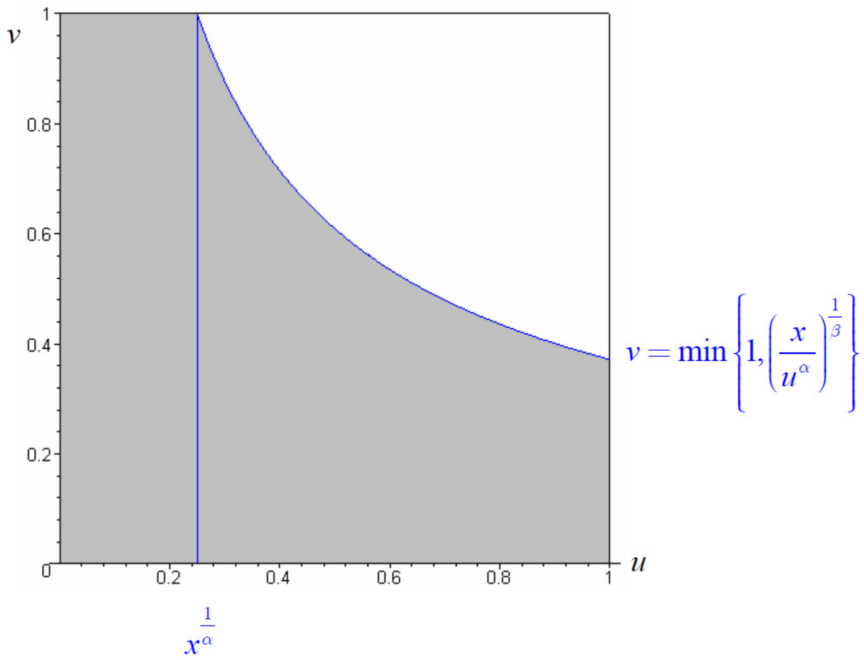
$W_1 = T_1(U_1,\ U_2) = U_1^{\alpha} U_2^{\beta},\ W_2 = T_2(U_1,\ U_2) = U_1^{\beta} U_2^{\alpha}$ with real $\alpha, \beta > 0$.

For simplicity, let $U := U_1,\ V := U_2$.

The common c.d.f. of U and V is given by

$$F(x) = \frac{\alpha}{\alpha - \beta} x^{\frac{1}{\alpha}} - \frac{\beta}{\alpha - \beta} x^{\frac{1}{\beta}},\ 0 < x < 1$$

which can be seen as follows.



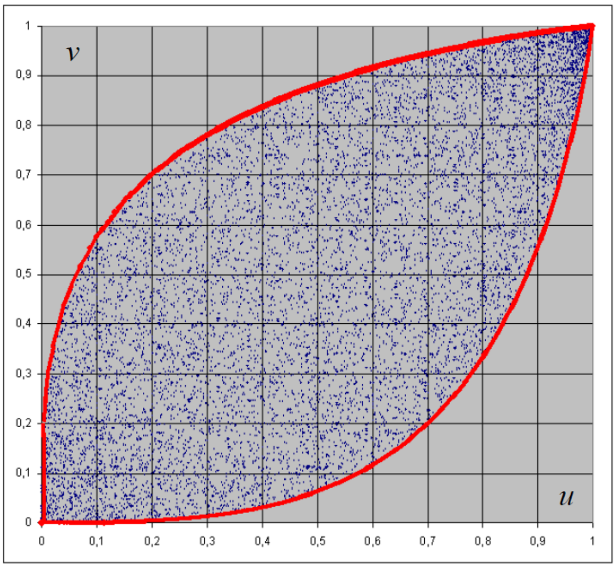
For $0 < x < 1$, there holds

$$F(x) = P\left(U^{\alpha} V^{\beta} \le x\right) = P\left(V \le \left(\frac{x}{U^{\alpha}}\right)^{\frac{1}{\beta}}\right) = \int_0^1 P\left(V \le \left(\frac{x}{u^{\alpha}}\right)^{\frac{1}{\beta}}\right) du = \int_0^1 \min\left\{1, \left(\frac{x}{u^{\alpha}}\right)^{\frac{1}{\beta}}\right\} du$$

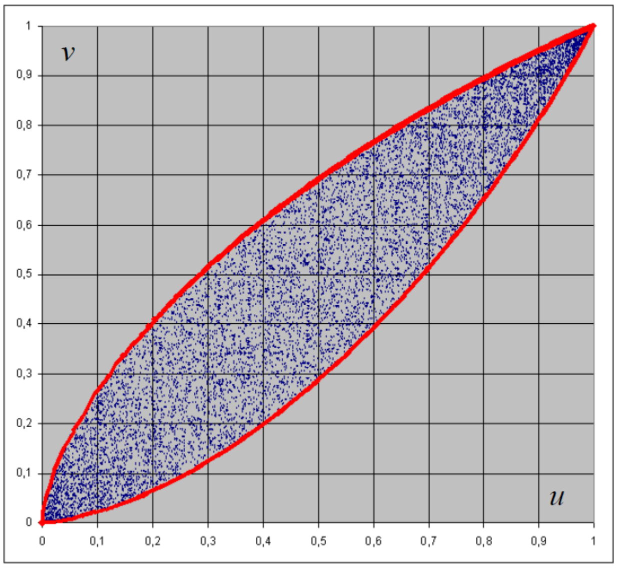
$$= x^{\frac{1}{\alpha}} + x^{\frac{1}{\beta}} \int_{x^{\frac{1}{\alpha}}}^{1} \frac{1}{u^{\frac{\alpha}{\beta}}} du = x^{\frac{1}{\alpha}} + \frac{\beta}{\beta - \alpha} x^{\frac{1}{\beta}} \left[1 - x^{\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)}\right] = \frac{\alpha}{\alpha - \beta} x^{\frac{1}{\alpha}} - \frac{\beta}{\alpha - \beta} x^{\frac{1}{\beta}} = P\left(U^{\beta} V^{\alpha} \le x\right)$$

by symmetry reasons.

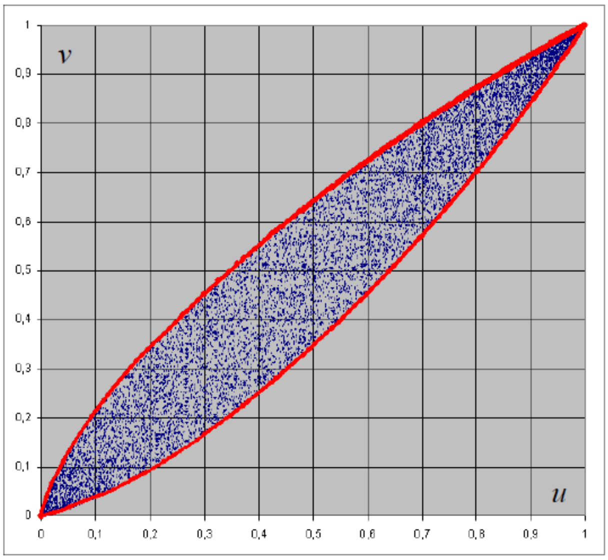
The following graphs show 10,000 simulations each of the copula given by $\left(F\left(W_1\right), F(W_2)\right)$, for different values of α and β.



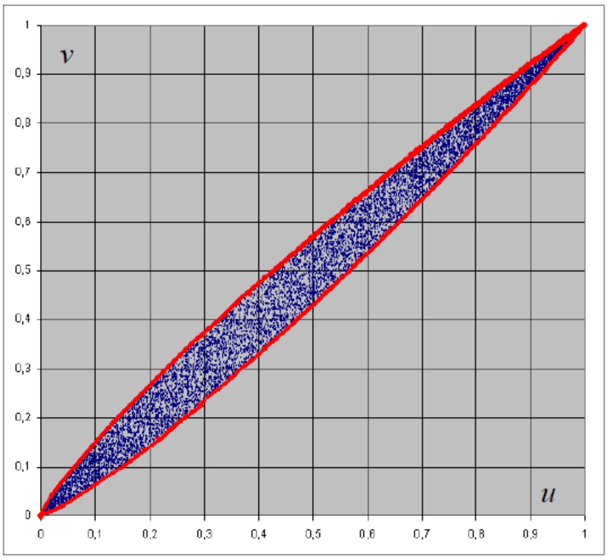
$\alpha = 3,\ \beta = 1$



$\alpha = 3,\ \beta = 2$



$\alpha = 4,\ \beta = 3$



$\alpha = 4,\ \beta = 3.5$

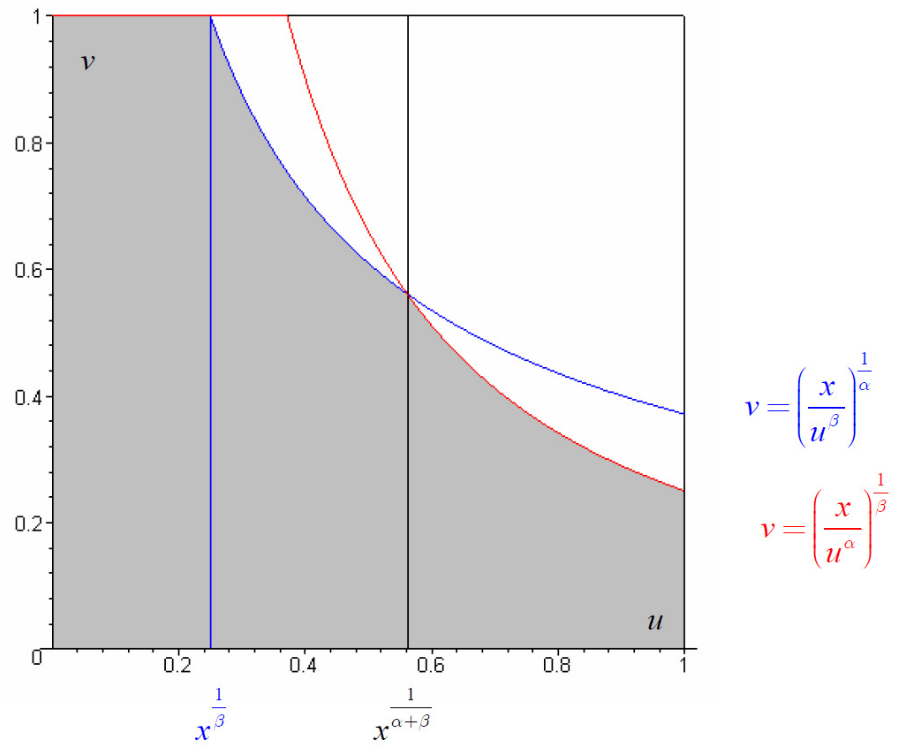
The red lines (u,v) represent the (sharp) lower and upper envelopes of the copula, which are given by

$$v_{lower} = F\left(\left(F^{-1}(u)\right)^{\frac{\beta}{\alpha}}\right) \text{ and } v_{upper} = F\left(\left(F^{-1}(u)\right)^{\frac{\alpha}{\beta}}\right),\ 0 < \mathrm{u} < 1.$$

Alternatively, the lower envelope can be described by the points $\left[F(u), F\left(u^{\frac{\beta}{\alpha}}\right)\right]$ and the upper envelope by the points $\left[F(u), F\left(u^{\frac{\alpha}{\beta}}\right)\right]$, $0 < \mathrm{u} < 1$.

Note that for $\alpha \gg \beta$, the copula tends to the independence copula, and for $\alpha \approx \beta$, we obtain the upper Fréchet bound. Note also that the copula is symmetric in α, β.

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 0$. First notice that for $0 < x < 1$, the intersection point of $\left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}}$ and $\left(\frac{x}{u^\beta}\right)^{\frac{1}{\alpha}}$ is given by $u_x = x^{\frac{1}{\alpha+\beta}}$ with value u_x.



Next we have

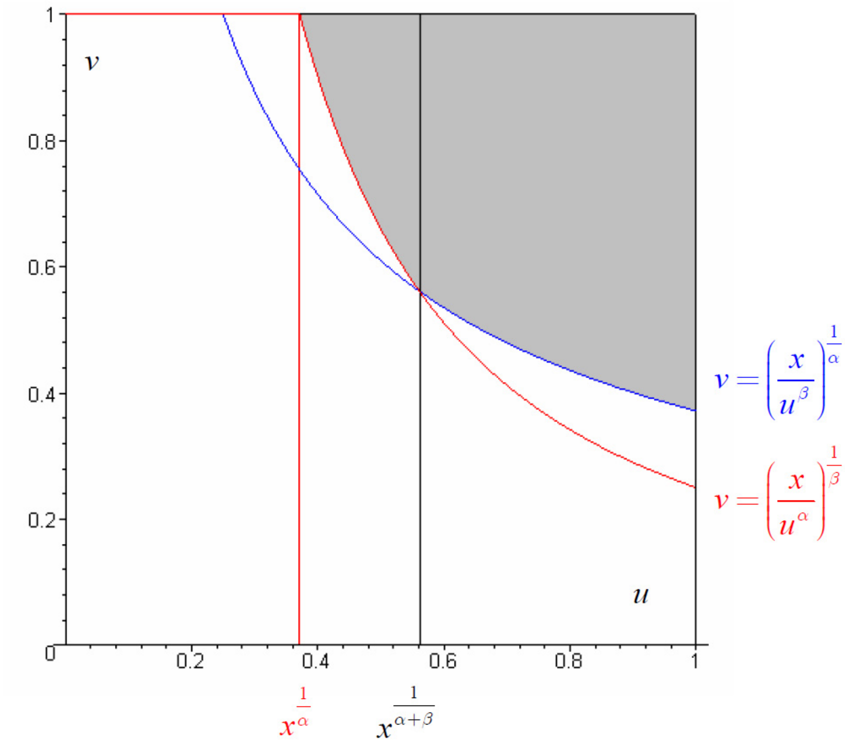
$$P\left(U^\alpha V^\beta \le x,\ U^\beta V^\alpha \le x\right) = x^{\frac{1}{\beta}} + \int_{x^{\frac{1}{\beta}}}^{x^{\frac{1}{\alpha+\beta}}} \left(\frac{x}{u^\beta}\right)^{\frac{1}{\alpha}} du + \int_{x^{\frac{1}{\alpha+\beta}}}^{1} \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}} du = x^{\frac{2}{\alpha+\beta}}$$

with

$$\frac{F(x)}{P\left(U^\alpha V^\beta \le x,\ U^\beta V^\alpha \le x\right)} = \frac{\alpha}{\alpha-\beta} x^{\frac{\beta-\alpha}{\alpha\cdot(\alpha+\beta)}} - \frac{\beta}{\alpha-\beta} x^{\frac{\alpha-\beta}{\alpha\cdot(\alpha+\beta)}} \text{ and hence}$$

$$\lim_{x\to 0} \frac{F(x)}{P\left(U^\alpha V^\beta \le x,\ U^\beta V^\alpha \le x\right)} = \infty, \text{ i.e. } \lim_{x\to 0} \frac{P\left(U^\alpha V^\beta \le x,\ U^\beta V^\alpha \le x\right)}{F(x)} = 0 = \lambda_L.$$

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = \frac{2\beta}{\alpha+\beta} > 0$ if $\alpha > \beta$.



If $\alpha>\beta$,

$$P\left(U^{\alpha}V^{\beta}>x,\ U^{\beta}V^{\alpha}>x\right)=1-x^{\frac{1}{\alpha}}-\int_{x^{\frac{1}{\alpha}}}^{x^{\frac{1}{\alpha+\beta}}}\left(\frac{x}{u^{\alpha}}\right)^{\frac{1}{\beta}}du-\int_{x^{\frac{1}{\alpha+\beta}}}^{1}\left(\frac{x}{u^{\beta}}\right)^{\frac{1}{\alpha}}du=1-\frac{2\alpha}{\alpha-\beta}x^{\frac{1}{\alpha}}+\frac{\beta}{\alpha-\beta}x^{\frac{2}{\alpha+\beta}}$$

with, by a Taylor expansion around $x=1$,

$$\frac{P\left(U^{\alpha}V^{\beta}>x,\ U^{\beta}V^{\alpha}>x\right)}{1-F(x)}=\frac{2\beta}{\alpha+\beta}-\frac{2(\alpha-\beta)}{3(\alpha+\beta)^{2}}(x-1)+\mathcal{O}\left((x-1)^{2}\right),\text{ hence }\lambda_{U}=\frac{2\beta}{\alpha+\beta}>0.$$

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