

RESEARCH ARTICLE

PROBABILISTIC REPRESENTATIONS OF OPERATOR  
SEMIGROUPS - A UNIFYING APPROACH

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In the recent years probabilistic representations of strongly continuous operator semigroups have re-raised a great deal of attention ([2], [6], [7], [8], [10]); a possible explanation for this might be the fact that a probabilistic approach to representation theory seems to be the most natural one (cf. [8]). Besides individual representation theorems also quite general probabilistic representation formulas have been given in the literature ([2], [3], [6], [7]), the most interesting ones being derived from the famous law of large numbers in probability theory. As will be shown in the present paper, all of these can be subsumed under a single probabilistic representation theorem based on a weak law of large numbers for a random number of summands which now also includes the continuous versions of Hille's and Phillips' exponential formulas (compare [6], Corollary 2). Moreover, the general theorem gives rise to certain product representations with unequal factors which to our knowledge have not yet been considered in the literature before. As a main tool, we use a rigorous approach to integration theory of semigroup operator-valued random variables by means of an extended form of Pettis' integral.

1. PRELIMINARIES

For a Banach space  $\mathcal{X}$  with norm  $\|\cdot\|$ , let  $\mathcal{E}(\mathcal{X})$  denote the Banach algebra of bounded endomorphisms on  $\mathcal{X}$ , and  $\mathcal{B}(\mathcal{X})$  denote the Borel  $\sigma$ -field generated by the norm-topology over  $\mathcal{X}$ . We consider a strongly continuous operator semigroup  $\{T(t) \mid t \geq 0\} \subseteq \mathcal{E}(\mathcal{X})$ , i.e.

$$T(t+s) = T(t) \circ T(s), \quad s, t \geq 0 \quad (1)$$

$$T(0) = I \quad (\text{the identity operator}) \quad (2)$$

$$\lim_{t \rightarrow 0} \|T(t)f - f\| = 0, \quad f \in \mathcal{X}. \quad (3)$$

It can be shown (cf. [1]) that (3) implies strong continuity, and that there exist constants  $M \geq 1$  and  $\omega \geq 0$  such that

$$\|T(t)\| \leq M e^{\omega t}, \quad t \geq 0. \quad (4)$$

As usual, let  $A$  denote the infinitesimal generator of the semigroup, and  $R(\lambda) = \int_0^{\infty} e^{-\lambda t} T(t)(\cdot) dt$ ,  $\lambda > \omega$  denote its resolvent. For a non-negative real-valued random variable  $X$  defined on some probability space  $(\Omega, \mathcal{A}, P)$  let  $\psi_X(t) = E(t^X)$ ,  $t \geq 0$ , denote the probability generating function of  $X$ , and  $\psi_X^*(t) = E(e^{tX})$ ,  $t \in \mathbb{R}$ , denote the moment-generating function of  $X$ , where  $E(\cdot)$  means expectation. If  $N \geq 0$  is an integer-valued random variable, then also

$$\psi_N(t) = \sum_{k=0}^{\infty} P(N=k) t^k, \quad t \geq 0, \quad (5)$$

and if  $\{Y_k \mid k \in \mathbb{N}\}$  is a sequence of independent, identically (as  $Y \geq 0$ , say) distributed random variables, independent of  $N$ , then for the random sum  $X = \sum_{k=1}^N Y_k$  (the empty sum being zero),

$$\psi_X^*(t) = \psi_N(\psi_Y^*(t)), \quad t \in \mathbb{R}. \quad (6)$$

(For a detailed probabilistic background, cf. e.g. Feller's monograph [4].)

## 2. INTEGRATION THEORY

The necessity for a separate integration theory for semigroup operator-valued random variables of the form  $T(X)$  where  $X \geq 0$  is a suitable real-valued random variable is due to the fact that in the general strongly continuous case the mapping  $t \rightarrow T(t)$  usually is neither Borel-measurable w.r.t.  $\mathcal{B}(\mathcal{E}(X))$  nor separably valued, which means that possibly  $E[T(X)]$  does not exist as a Bochner- or Pettis integral in  $\mathcal{E}(X)$  unless  $X$  is countably valued or the semigroup is uniformly continuous (i.e.  $A$  is bounded). More precisely, the following statement holds.

THEOREM 1. If  $\liminf_{t \rightarrow t_0} \|T(t) - T(t_0)\| > 0$  for some  $t_0 > 0$ , then the mapping  $t \rightarrow T(t)$  is neither Borel-measurable (and hence not strongly measurable) nor separably valued.

PROOF. It is easy to see that for any strongly continuous semigroup  $\{T(t) \mid t \geq 0\}$  with  $T(t) \neq I$  there is some neighbourhood  $\mathcal{U} \subseteq \mathbb{R}^+$  of the origin such that on  $\mathcal{U}$ ,  $T(\cdot)$  is injective. Further, by the assumptions of the theorem, there exist  $\delta, c > 0$  such that

$$\inf_{t_0 < t \leq t_0 + \delta} \|T(t) - T(t_0)\| \geq c \quad (7)$$

which implies that for arbitrary  $0 \leq s \leq t_0$

$$\inf_{s < t \leq s + \delta} \|T(t) - T(s)\| \geq \frac{c}{M} e^{-\omega t_0} = c^* > 0. \quad (8)$$

Without loss of generality, we may assume that  $\delta \leq t_0$  and  $T(\cdot)$  is injective in  $\mathcal{U} \subseteq [0, \delta]$ , hence by (8),

$$\|T(t) - T(s)\| \geq c^* \text{ for } s, t \in \mathcal{U}, s \neq t. \quad (9)$$

Now choose a non-measurable set  $\mathcal{N} \subset \mathcal{U}$  (which always exists); then  $T(\mathcal{N})$  consists of uncountably many separated points in  $\mathcal{E}(\mathcal{X})$ , i.e.  $T(\mathcal{N})$  is closed (and hence a measurable set in  $\mathcal{B}(\mathcal{E}(\mathcal{X}))$ ) and non-separable. But  $\mathcal{N} = T^{-1}(T(\mathcal{N})) \cap \mathcal{U}$  which is not measurable by assumption hence  $T(\cdot)$  is neither Borel-measurable nor separably valued. ■

A simple example of a semigroup fulfilling the conditions of Theorem 1 is the semigroup of left translations on the space  $\mathcal{X} = \text{UCB}(\mathbb{R})$  of all uniformly continuous and bounded functions on  $\mathbb{R}$  (cf. [1]); here  $\|T(t) - T(s)\| = 2$  for  $s \neq t$ .

In order to be able to define expectations also for the general strongly continuous case, we now introduce an extension of Pettis' integral for  $\mathcal{E}(\mathcal{X})$  - valued mappings. The idea behind this extension is the fact that for the Banach space  $\mathcal{E}(\mathcal{X})$  less bounded linear functionals are necessary to guarantee uniqueness of the (extended) integral.

DEFINITION. Let  $(\mathcal{M}, \mathcal{F}, \mu)$  be a measure space and  $S : \mathcal{M} \rightarrow \mathcal{E}(\mathcal{X})$  a mapping such that  $f^*(S(\cdot)f)$  is Borel-measurable for all  $f \in \mathcal{X}$  and  $f^* \in \mathcal{X}^*$ , the dual space of  $\mathcal{X}$ .  $S$  is called  $\mu$ -integrable if there exists an element  $J \in \mathcal{E}(\mathcal{X})$  such that

$$f^*(J(f)) = \int_{\mathcal{M}} f^*(S(\cdot)f) d\mu \text{ for all } f \in \mathcal{X}, f^* \in \mathcal{X}^*. \quad (10)$$

$J$  is then called the  $\mu$ -integral of  $S : J = \int_{\mathcal{M}} S d\mu$ . If  $\mu$  is a probability measure, then  $J$  is also called expectation of  $S : J = E(S)$ .

By a corollary of the Hahn-Banach theorem,  $J$  is uniquely determined in case of existence since for  $J_1, J_2 \in \mathcal{E}(\mathcal{X})$  with  $f^*(J_1(f)) = f^*(J_2(f))$ ,  $f \in \mathcal{X}$ ,  $f^* \in \mathcal{X}^*$ , we have  $J_1(f) = J_2(f)$ ,  $f \in \mathcal{X}$  and hence  $J_1 = J_2$ . Further,  $F^*(J) = f^*(J(f))$  for fixed  $f \in \mathcal{X}$ ,  $f^* \in \mathcal{X}^*$  and arbitrary  $J \in \mathcal{E}(\mathcal{X})$  defines a bounded linear functional  $F^* \in \mathcal{E}(\mathcal{X})^*$

with  $\|F^*\| \leq \|f^*\| \|f\|$ , i.e. if  $S$  is Pettis-integrable in the ordinary sense, then  $S$  is also extended Pettis-integrable, and both integrals coincide.

Note that if additionally  $S(\cdot)f$  is Borel-measurable w.r.t.  $\mathcal{B}(\mathcal{X})$  for some  $f \in \mathcal{X}$ , then also  $\int_{\mathcal{M}} S(\cdot)f d\mu$  exists as an ordinary Pettis-integral in  $\mathcal{X}$ ; in this case, also

$$\left(\int_{\mathcal{M}} S d\mu\right)f = \int_{\mathcal{M}} S(\cdot)f d\mu. \quad (11)$$

We shall now give a simple sufficient condition for the existence of an extended Pettis-integral.

LEMMA 1. Suppose that  $S(\cdot)f$  is Borel-measurable and separably valued for every  $f \in \mathcal{X}$  and that  $\|S\|$  is dominated by some integrable function  $g \geq 0$ . Then  $S$  is extended Pettis-integrable, and

$$\left\|\int_{\mathcal{M}} S d\mu\right\| \leq \int_{\mathcal{M}} g d\mu.$$

PROOF. Since  $S(\cdot)f$  is Borel- and hence weakly measurable and separably valued, by  $\|S(\cdot)f\| \leq g\|f\|$ ,  $S(\cdot)f$  is Bochner-integrable, hence Pettis-integrable with

$$\left\|\int_{\mathcal{M}} S(\cdot)f d\mu\right\| \leq \|f\| \int_{\mathcal{M}} g d\mu. \quad (12)$$

But then

$$J(f) = \int_{\mathcal{M}} S(\cdot)f d\mu, \quad f \in \mathcal{X} \quad (13)$$

defines a bounded endomorphism  $J \in \mathcal{E}(\mathcal{X})$  with

$$\|J\| \leq \int_{\mathcal{M}} g d\mu \quad (14)$$

by (12), and

$$f^*(J(f)) = f^*\left(\int_{\mathcal{M}} S(\cdot)f d\mu\right) = \int_{\mathcal{M}} f^*(S(\cdot)f) d\mu, \quad f \in \mathcal{X}, f^* \in \mathcal{X}^* \quad (15)$$

by the Pettis-integrability of  $S(\cdot)f$  which says that  $S$  is extended Pettis-integrable with  $J = \int_{\mathcal{M}} S d\mu$ . Hence by (14),

$$\left\| \int_{\mathcal{M}} S d\mu \right\| \leq \int_{\mathcal{M}} g d\mu. \quad \blacksquare \quad (16)$$

Note that  $\|S\|$  need not be a measurable function, hence the inequality

$$\left\| \int_{\mathcal{M}} S d\mu \right\| \leq \int_{\mathcal{M}} \|S\| d\mu \quad (17)$$

may be meaningless unless  $\|S\|$  is measurable.

COROLLARY 1. Let  $X \geq 0$  be a real-valued random variable such that  $\psi_X^*(\omega) < \infty$  where  $\omega$  is as in (4). Then  $T(X)$  is extended Pettis-integrable with

$$\|E[T(X)]\| \leq M \psi_X^*(\omega),$$

and

$$E[T(X)]f = E[T(X)f], \quad f \in \mathcal{X}$$

where the integral on the right hand side is an ordinary Pettis-integral.

PROOF. Obvious from (4), (11) and Lemma 1 since  $T(\cdot)f$  is continuous for every  $f \in \mathcal{X}$ , hence  $T(X)f$  is measurable and separably valued.  $\blacksquare$

COROLLARY 2. Let  $X, Y \geq 0$  be independent real-valued random variables such that  $\psi_X^*(\omega)$  and  $\psi_Y^*(\omega) < \infty$ . Then  $T(X)$ ,  $T(Y)$  and  $T(X) \circ T(Y)$  are extended Pettis-integrable, and

$$E[T(X) \circ T(Y)] = E[T(X)] \circ E[T(Y)].$$

PROOF. By Corollary 1,  $T(X)$ ,  $T(Y)$  and  $T(X) \circ T(Y) = T(X+Y)$  are extended Pettis-integrable (the latter since  $\psi_{X+Y}^*(\omega) = \psi_X^*(\omega)\psi_Y^*(\omega)$ ). To prove Corollary 2, it suffices to show

$$f^*((E[T(X)] \circ E[T(Y)])f) = f^*(E[T(X+Y)]f) \quad (18)$$

for all  $f \in \mathcal{X}$ ,  $f^* \in \mathcal{X}^*$ . But

$$h^*(h) = f^*(E[T(X) | h]) = f^*(E[T(X)h]), \quad h \in \mathcal{X} \quad (19)$$

defines a bounded linear functional  $h^* \in \mathcal{X}^*$  with

$$\|h^*\| \leq \|f^*\| \|E[T(X)]\| \leq \|f^*\| M \psi_X^*(\omega). \quad (20)$$

Now by definition and some well-known integration theorems,

$$\begin{aligned} f^*((E[T(X)] \circ E[T(Y) | \cdot])f) &= h^*(E[T(Y)]f) = E[h^*(T(Y)f)] \\ &= \int h^*(T(Y)f) P^Y(dy) = \int f^*(E[T(X) \circ T(Y)f]) P^Y(dy) \\ &= \int E[f^*(T(X) \circ T(Y)f)] P^Y(dy) = \iint f^*(T(x) \circ T(y)f) P^X(dx) P^Y(dy) \\ &= \iint f^*(T(x+y)f) P^{(X,Y)}(dx, dy) = \int f^*(T(X+Y)f) dP \\ &= f^*E[T(X+Y)]f, \end{aligned}$$

where, for a random variable  $Z$ ,  $P^Z$  denotes the distribution of  $Z$ . ■

Note that Corollary 2 provides a rigorous proof of a similar relation in [3], p. 157 which was stated only heuristically.

The following result is a generalization of Corollary 2 to a random number of summands.

COROLLARY 3. Let  $N \geq 0$  be an integer-valued random variable and  $Y \geq 0$  a real-valued random variable such that  $\psi_Y^*(\omega) < \infty$  and  $\psi_N(\psi_Y^*(\omega)) < \infty$ . Let further  $\{Y_k | k \in \mathbb{N}\}$  be a sequence of independent identically (as  $Y$ ) distributed random variables, independent of  $N$ , and  $X = \sum_{k=1}^N Y_k$ . Then  $T(X)$  is extended Pettis-integrable, and

$$E[T(X)] = \psi_N(E[T(Y)]) = \sum_{m=0}^{\infty} P(N=m) \{E[T(Y)]\}^m,$$

where  $\{E[T(Y)]\}^0 = I$ .

PROOF. By Corollary 2, for  $m \geq 1$ ,

$$\begin{aligned} \|\{E[T(Y)]\}^m\| &= \|E[T(\sum_{k=1}^m Y_k)]\| \leq M \psi_{\sum_{k=1}^m Y_k}^* (\omega) \\ &= M\{\psi_Y^*(\omega)\}^m \end{aligned} \quad (21)$$

which by  $M \geq 1$  also holds for  $m = 0$ . But then

$$\begin{aligned} \sum_{m=0}^{\infty} P(N=m) \|\{E[T(Y)]\}^m\| &\leq M \sum_{m=0}^{\infty} P(N=m) \{\psi_Y^*(\omega)\}^m \\ &= M \psi_N(\psi_Y^*(\omega)) < \infty \end{aligned} \quad (22)$$

by assumption, i.e.  $\sum_{m=0}^{\infty} P(N=m) \{E[T(Y)]\}^m$  is absolutely convergent, hence convergent in  $\mathcal{E}(\mathcal{X})$  to  $\psi_N(E[T(Y)])$ . Now with  $f \in \mathcal{X}$ ,  $f^* \in \mathcal{X}^*$ ,

$$\begin{aligned} f^*(E[T(X)]f) &= \sum_{m=0}^{\infty} P(N=m) E[f^*(T(\sum_{k=1}^m Y_k)f)] \\ &= \sum_{m=0}^{\infty} P(N=m) f^*(\{E[T(Y)]\}^m f) \\ &= f^*(\sum_{m=0}^{\infty} P(N=m) \{E[T(Y)]\}^m f), \end{aligned}$$

hence

$$E[T(X)] = \sum_{m=0}^{\infty} P(N=m) \{E[T(Y)]\}^m. \quad \blacksquare \quad (23)$$

### 3. THE REPRESENTATION THEOREM

Before stating the main theorem we shall prove a weak law of large numbers for random sums in the setting of Corollary 3 which turns out to be basic for the general representation formula.



LEMMA 2. Let  $\{N(\tau) \mid \tau > 0\}$  be a family of non-negative integer-valued random variables such that  $\frac{1}{\tau} N(\tau)$  converges in probability to some constant  $\zeta > 0$  for  $\tau \rightarrow \infty$ . Let further  $\{Y_k \mid k \in \mathbb{IN}\}$  be a sequence of independent identically (as  $Y \geq 0$ ) distributed random variables, independent of  $\{N(\tau) \mid \tau > 0\}$ , with  $E(Y) = \gamma$ . Then the random variables  $X(\tau) = \sum_{k=1}^{N(\tau)} Y_k$ ,  $\tau > 0$  obey a law of large numbers, i.e.

$$\frac{1}{\tau} X(\tau) = \frac{1}{\tau} \sum_{k=1}^{N(\tau)} Y_k \rightarrow \zeta \gamma$$

in probability for  $\tau \rightarrow \infty$ .

PROOF. For any event  $C \in \mathcal{A}$ , let  $1_C$  denote the indicator random variable, i.e.  $1_C(x) = 1$  iff  $x \in C$ , and 0 otherwise. To prove the theorem, we split up the random sum into three parts:

$$\begin{aligned} X(\tau) = & \sum_{1 \leq k \leq \zeta \tau} Y_k + 1_{\{\zeta \tau < N(\tau)\}} \sum_{\zeta \tau < k \leq N(\tau)} Y_k \\ & - 1_{\{\zeta \tau > N(\tau)\}} \sum_{N(\tau) < k \leq \zeta \tau} Y_k. \end{aligned} \quad (24)$$

By the classical law of large numbers,

$$\frac{1}{\tau} \sum_{1 \leq k \leq \zeta \tau} Y_k = \zeta \frac{1}{\zeta \tau} \sum_{1 \leq k \leq \zeta \tau} Y_k \rightarrow \zeta \gamma \quad (25)$$

in probability for  $\tau \rightarrow \infty$ . For the remainder terms in (24), choose  $\varepsilon > 0$  and  $0 < \eta \leq \frac{\varepsilon}{1+\gamma}$ . Then again by the law of large numbers,

$$\frac{1}{\tau} \sum_{\zeta \tau < k \leq (\zeta + \eta) \tau} Y_k = \eta \frac{1}{\eta \tau} \sum_{\zeta \tau < k \leq \zeta \tau + \eta \tau} Y_k \rightarrow \eta \gamma < \varepsilon \quad (26)$$

in probability for  $\tau \rightarrow \infty$ , hence

$$\begin{aligned}
 & P(1_{\{\zeta\tau < N(\tau)\}} \frac{1}{\tau} \sum_{\zeta\tau < k \leq N(\tau)} Y_k \geq \epsilon) \\
 & \leq P(\{ \zeta\tau < N(\tau) \} \cap \{ \frac{1}{\tau} \sum_{\zeta\tau < k \leq N(\tau)} Y_k \geq \epsilon \}) + P(N(\tau) > (\zeta + \eta)\tau) \\
 & \leq P(\frac{1}{\tau} \sum_{\zeta\tau < k \leq (\zeta + \eta)\tau} Y_k \geq \epsilon) + P(\frac{1}{\tau} N(\tau) > \zeta + \eta) \rightarrow 0 \quad (27)
 \end{aligned}$$

by (26) and the assumptions of the lemma, hence

$$\frac{1}{\tau} 1_{\{\zeta\tau < N(\tau)\}} \sum_{\zeta\tau < k \leq N(\tau)} Y_k \rightarrow 0 \quad (28)$$

in probability for  $\tau \rightarrow \infty$ ; similarly for the third summand in (24). Hence by (25),

$$\frac{1}{\tau} X(\tau) = \frac{1}{\tau} \sum_{k=1}^{N(\tau)} Y_k \rightarrow \zeta\gamma \quad (29)$$

in probability for  $\tau \rightarrow \infty$ . ■

THEOREM 2. Let  $\{T(t) \mid t \geq 0\}$  be a strongly continuous operator semigroup, and let  $\{N(\tau) \mid \tau > 0\}$  and  $Y$  be as in Lemma 2 such that for the probability generating functions  $\psi_{N(\tau)}$ ,  $\psi_{N(\tau)}(\delta_1) < \infty$  for some  $\delta_1 > 1$  and all  $\tau > 0$ , and that  $\psi_Y^*(\delta_2) < \infty$  for some  $\delta_2 > 0$ . Then if

$$\limsup_{\tau \rightarrow \infty} \psi_{N(\tau)}(\psi_Y^*(\frac{r\omega}{\tau})) < \infty \quad (30)$$

for some  $r > 1$ , we have  $\psi_{N(\tau)}(E[T(\frac{Y}{\tau})]) \in \mathcal{E}(\mathcal{X})$  for sufficiently large  $\tau$  with

$$\|\psi_{N(\tau)}(E[T(\frac{Y}{\tau})])\| \leq M \psi_{N(\tau)}(\psi_Y^*(\frac{\omega}{\tau})), \quad (31)$$

and the semigroup representation

$$T(\xi) = \lim_{\tau \rightarrow \infty} \psi_{N(\tau)}(E[T(\frac{Y}{\tau})])$$

holds in the strong sense where  $\xi = \zeta\gamma$ .

PROOF. Choose  $\{Y_k \mid k \in \mathbb{N}\}$  as in Lemma 2. Then by Corollary 3, for sufficiently large  $\tau$ ,  $T(\frac{1}{\tau} X(\tau))$  is extended Pettis-integrable with

$$E|T(\frac{1}{\tau} X(\tau))| = \psi_{N(\tau)}(E|T(\frac{Y}{\tau})|) \in \mathcal{E}(\mathcal{X}) \quad (32)$$

and

$$\|\psi_{N(\tau)}(E|T(\frac{Y}{\tau})|)\| \leq M \psi_{N(\tau)}(\psi_Y^*(\frac{\omega}{\tau})) \quad (33)$$

by (22). Further, for sufficiently large  $\tau$ ,

$$\begin{aligned} E(\|T(\frac{1}{\tau} X(\tau))f\|^r) &\leq M^r \|f\|^r \psi_{X(\tau)}^*(\frac{r\omega}{\tau}) \\ &= M^r \|f\|^r \psi_{N(\tau)}(\psi_Y^*(\frac{r\omega}{\tau})) \end{aligned} \quad (34)$$

by (4), hence by (11), (32), Lemma 2, the assumptions and Lemma 1 in [3], the theorem follows. ■

Note that (30) is always fulfilled for any such random variable  $Y$  if

$$\limsup_{\tau \rightarrow \infty} \psi_{N(\tau)}^*(\frac{\delta}{\tau}) < \infty \quad (35)$$

for some  $\delta > \omega\gamma$  since by Taylor expansion,

$$\psi_Y^*(\frac{r\omega}{\tau}) = 1 + \frac{r\omega\gamma}{\tau} + \mathcal{O}(\frac{1}{\tau^2}) \quad (\tau \rightarrow \infty), \quad (36)$$

hence for sufficiently large  $\tau$  and  $1 < r < \frac{\delta}{\omega\gamma}$ ,

$$\begin{aligned} \psi_{N(\tau)}(\psi_Y^*(\frac{r\omega}{\tau})) &= E|(1 + \frac{r\omega\gamma}{\tau} + \mathcal{O}(\frac{1}{\tau^2}))^{N(\tau)}| \\ &\leq E[\exp\{(\frac{r\omega\gamma}{\tau} + \mathcal{O}(\frac{1}{\tau^2}))N(\tau)\}] \\ &= \psi_{N(\tau)}^*(\frac{r\omega\gamma}{\tau} + \mathcal{O}(\frac{1}{\tau^2})) \leq \psi_{N(\tau)}^*(\frac{\delta}{\tau}). \end{aligned} \quad (37)$$

The condition  $\delta > \omega\gamma$  is not very restrictive since for equi-bounded semigroups (i.e.  $\omega = 0$ ) it reduces to  $\delta > 0$ ;

in the general case, instead of  $T(t)$ ,  $t \geq 0$  the equibounded semigroup  $S(t) = e^{-\omega t} T(t)$ ,  $t \geq 0$  might be considered.

In what follows we shall show how the various representation theorems given in the literature follow from Theorem 2 by specialisation.

COROLLARY 4. Let  $N$  be a non-negative integer-valued random variable with  $E(N) = \zeta$  and  $Y \geq 0$  be a real-valued random variable with  $E(Y) = \gamma$  such that  $\psi_N(\delta_1) < \infty$  for some  $\delta_1 > 1$  and  $\psi_Y^*(\delta_2) < \infty$  for some  $\delta_2 > 0$ . Then for sufficiently large  $n \in \mathbb{N}$ ,  $\psi_N(E[T(\frac{Y}{n})]) \in \mathcal{E}(\mathcal{X})$  with

$$\|\psi_N(E[T(\frac{Y}{n})])\| \leq M \psi_N(\psi_Y^*(\frac{\omega}{n})), \quad (38)$$

and

$$T(\xi) = \lim_{n \rightarrow \infty} \{\psi_N(E[T(\frac{Y}{n})])\}^n$$

in the strong sense where  $\xi = \zeta\gamma$ .

PROOF. Let  $\{N_k \mid k \in \mathbb{N}\}$  be a sequence of independent, identically (as  $N$ ) distributed random variables, and let

$$N(\tau) = \sum_{1 \leq k \leq \tau} N_k, \quad \tau > 0. \quad (39)$$

Then by the law of large numbers,

$$\frac{1}{\tau} N(\tau) \rightarrow \zeta \quad (40)$$

in probability for  $\tau \rightarrow \infty$ . Further, since

$$\psi_{N(\tau)} = \psi_N^{\llbracket \tau \rrbracket}, \quad \tau > 0 \quad (41)$$

(where  $\llbracket \tau \rrbracket$  denotes the greatest integer not exceeding  $\tau$ ), we have

$$\psi_{N(\tau)}^*(\frac{\delta}{\tau}) = \{\psi_N^*(\frac{\delta}{\tau})\}^{\llbracket \tau \rrbracket} = (1 + \frac{\delta\zeta}{\tau} + \mathcal{O}(\frac{1}{\tau^2}))^{\llbracket \tau \rrbracket} \rightarrow e^{\delta\zeta} \quad (42)$$

for  $\tau \rightarrow \infty$  and every  $\delta > 0$ , hence the corollary follows by

(35) and Theorem 2 for the integer subsequence. ■

Corollary 4 is the main theorem in [6] from which by specifying the distributions of  $N$  and  $Y$  as binomial, geometric, exponential etc. the representation theorems of Butzer-Hahn [2], Chung [3], Kendall [5], Shaw [10] and others immediately follow (see [6]).

COROLLARY 5. Let  $\{N(\tau) \mid \tau > 0\}$  be as in Lemma 2 such that  $\psi_{N(\tau)}(\delta_1) < \infty$  for some  $\delta_1 > 1$  and all  $\tau > 0$ , and that for some  $\delta_2 > 2\omega \frac{\xi}{\zeta}$

$$\limsup_{\tau \rightarrow \infty} \psi_{N(\tau)}^* \left( \frac{\delta_2}{\tau} \right) < \infty.$$

Then all of the relations

$$T(\xi) = \lim_{\tau \rightarrow \infty} \psi_{N(\tau)} \left( T\left(\frac{\xi}{\zeta\tau}\right) \right) \quad (43)$$

$$T(\xi) = \lim_{\tau \rightarrow \infty} \psi_{N(\tau)} \left( \frac{\zeta\tau}{\xi} R\left(\frac{\zeta\tau}{\xi}\right) \right) \quad (44)$$

$$T(\xi) = \lim_{\tau \rightarrow \infty} \psi_{N(\tau)} \left( \sum_{k=0}^m \frac{1}{k!} \left(\frac{\xi}{\zeta\tau}\right)^k A^k \right), \quad m \in \mathbb{N} \quad (45)$$

hold in the strong sense (the latter only in case  $A$  is bounded).

PROOF. Obvious from (35) and Theorem 2 by choosing  $Y \equiv \frac{\xi}{\zeta}$  or  $Y$  being exponentially distributed with mean  $\frac{\xi}{\zeta}$ . Relation (45) follows from (43) by Taylor expansion of the semi-group using methods sketched in [7]. ■

Choosing  $\{N(\tau) \mid \tau > 0\}$  as a Poisson process with parameter  $\xi$ , Hille's and Phillips' exponential formulas are reobtained from Corollary 5.

COROLLARY 6. For all  $\xi > 0$ ,

$$T(\xi) = \lim_{\tau \rightarrow \infty} \exp \left\{ \xi\tau \left[ T\left(\frac{1}{\tau}\right) - I \right] \right\} \quad (46)$$

$$T(\xi) = \lim_{\tau \rightarrow \infty} \exp \{ \xi \tau^2 R(\tau) - \xi \tau I \} \quad (47)$$

$$T(\xi) = \exp \{ \xi A \} \quad (48)$$

in the strong sense (the latter only in case A is bounded).

PROOF. For a Poisson process with parameter  $\xi$ ,  $\frac{1}{\tau} N(\tau) \rightarrow \xi$  in probability for  $\tau \rightarrow \infty$ , and  $\psi_{N(\tau)}(t) = \exp(\xi \tau(t-1))$ , hence  $\psi_{N(\tau)}(\delta_1) < \infty$  for all  $\delta_1 > 1$  and  $\tau > 0$ , and

$$\psi_{N(\tau)}^* \left( \frac{\delta_2}{\tau} \right) = \exp(\xi \tau (e^{\delta_2/\tau} - 1)) \rightarrow e^{\xi \delta_2} \quad (49)$$

as  $\tau \rightarrow \infty$  for all  $\delta_2 > 0$ . Relations (46) to (48) now follow from (43) to (45) (with  $m = 1$ ). ■

COROLLARY 7. Let A be bounded and N be a non-negative integer-valued random variable with  $E(N) = \zeta$  such that  $\psi_N(\delta) < \infty$  for some  $\delta > 1$ . Then for all  $m \in \mathbb{N}$ ,

$$T(\xi) = \lim_{n \rightarrow \infty} \psi_N^n \left( \sum_{k=0}^m \frac{1}{k!} \left( \frac{\xi}{\zeta n} \right)^k A^k \right)$$

in the uniform sense.

PROOF. Obvious from (45) and the proof of Corollary 4; note that any  $\delta_2 \geq 2 \|A\| \frac{\xi}{\zeta}$  can be chosen (cf. also (42)). The uniform convergence can be deduced from uniform estimations in the proof of Theorem 2. ■

Corollary 7 is a slight generalization of the representation theorems given in [7].

COROLLARY 8. Let  $\{N_k \mid k \in \mathbb{N}\}$  be a sequence of non-negative integer-valued random variables with  $E(N_k) = \zeta_k$  such that  $\psi_{N_k}(\delta_1) < \infty$  for some  $\delta_1 > 1$  and all  $k$ , and let  $Y \geq 0$  be a real-valued random variable with  $E(Y) = \gamma$  such that  $\psi_Y^*(\delta_2) < \infty$  for some  $\delta_2 > 0$ . Then if

$$\limsup_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k} \left( \psi_Y^* \left( \frac{\gamma \omega}{n} \right) \right) < \infty$$

for some  $r > 1$  (or alternatively,

$$\limsup_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k}^* \left( \frac{\delta}{n} \right) < \infty \text{ for some } \delta > \omega\gamma$$

and if

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \text{Var}(N_k) = 0$$

(where Var means variance) and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \zeta_k = \zeta \text{ exists,}$$

the semigroup representation

$$T(\xi) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k} \left( E \left[ T \left( \frac{Y}{n} \right) \right] \right)$$

holds in the strong sense where  $\xi = \zeta\gamma$ .

PROOF. Without loss of generality we may assume all random variables to be independent. As in the proof of Corollary 4, let  $N(\tau) = \sum_{1 < k < \tau} N_k$ . Then by our assumptions and the law of large numbers, again  $\frac{1}{\tau} N(\tau) \rightarrow \zeta$  in probability for  $\tau \rightarrow \infty$ . Since  $\psi_{N(\tau)} = \prod_{1 < k < \tau} \psi_{N_k}$ , the corollary now follows from Theorem 2 and (35). ■

Corollary 8 is an extension of Corollary 4 to a product representation formula with not necessarily equal factors. The following result is the corresponding analogue of Corollary 5.

COROLLARY 9. Let  $\{N_k \mid k \in \mathbb{N}\}$  be as in Corollary 8. If

$$\limsup_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k}^* \left( \frac{\delta}{n} \right) < \infty \text{ for some } \delta > 2\omega \frac{\xi}{\zeta}$$

and if

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \text{Var}(N_k) = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \zeta_k = \zeta \text{ exists,}$$

then

$$T(\xi) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k} \left( T\left(\frac{\xi}{\zeta n}\right) \right) \quad (50)$$

$$T(\xi) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k} \left( \frac{\zeta n}{\xi} R\left(\frac{\zeta n}{\xi}\right) \right) \quad (51)$$

$$T(\xi) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \psi_{N_k} \left( \sum_{j=0}^m \left(\frac{\xi}{\zeta n}\right)^j A^j \right), \quad m \in \mathbb{N} \quad (52)$$

in the strong sense (the latter only in case A is bounded).

COROLLARY 10. If

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \zeta_k = \zeta \text{ exists,}$$

then

$$T(\xi) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \exp \{ \zeta_k [T\left(\frac{\xi}{\zeta n}\right) - I] \} \quad (53)$$

$$T(\xi) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \exp \{ \zeta_k \left[ \frac{\zeta n}{\xi} R\left(\frac{\zeta n}{\xi}\right) - I \right] \} \quad (54)$$

in the strong sense.

PROOF. Let  $N_k$  be Poisson distributed with mean  $\zeta_k$ . Then for every  $\delta > 0$ ,

$$\begin{aligned} \prod_{k=1}^n \psi_{N_k}^* \left( \frac{\delta}{n} \right) &= \prod_{k=1}^n \exp \{ \zeta_k (e^{\delta/n} - 1) \} \\ &\leq \exp \left\{ \frac{\delta}{n} \sum_{k=1}^n \zeta_k + \mathcal{O}\left(\frac{1}{n^2}\right) \right\} \rightarrow e^{\delta \zeta} \end{aligned} \quad (55)$$

for  $n \rightarrow \infty$ . Also, since  $\text{Var}(N_k) = \zeta_k$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n \text{Var}(N_k) = 0$ , hence the statement follows from Corollary 9. ■

Note that Corollary 10 is a natural generalization of Hille's and Phillips' exponential formulas to product representations of operator semigroups.



It should be pointed out that the general probabilistic approach to representation theory as chosen in this paper also permits statements on rates of convergence in the general as well as in the individual representation formulas (cf. also [2]); these approximation theoretic aspects of representation theory are dealt with in more detail in a forthcoming paper [9].

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