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ON AN ASYMPTOTIC RELATIONSHIP BETWEEN SYMMETRIC BETA AND NORMAL DISTRIBUTIONS

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In this paper, which is a follow-up publication to [4], we show that normal distributions can be considered as weak limits of appropriately rescaled symmetric beta distributions.

Let X be a random variable with a symmetric standard beta distribution, i.e., with a p.d.f. f given by

$$f = (x;\ \alpha) = \frac{x^{\alpha-1} \cdot (1-x)^{\alpha-1}}{B(\alpha,\ \alpha)}, \quad 0 < x < 1; \quad \alpha > 0,$$

where $B(\alpha,\ \beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}$ denotes Euler's Beta function (cf. [2], (25.2). The first moments are given by

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$E(X) = \frac{1}{2}$, $Var(X) = \frac{1}{4 \cdot (2\alpha + 1)}$ (cf. [2], relations (25.15a) and (25.15b)).

Now let $Y := c \cdot (2X - 1)$ with a real $c > 0$. Then Y follows a generalized symmetric beta distribution with support $[-c, c]$ and p.d.f.

$$g(y;\ \alpha,\ c) = \frac{1}{2c} f\left[\frac{1}{2} + \frac{y}{2c},\ \alpha\right], \quad -c < y < c \text{ (cf. [2], (25.3) and (25.5)).}$$

For the first moments, we thus obtain

$$E(Y) = 0, \quad Var(Y) = \frac{c^2}{(2\alpha + 1)}.$$

For the sequel, we need the following Lemma.

Lemma. *There holds* $k(\alpha) < B(\alpha, \alpha) < k(\alpha) \cdot e^{1/(6\alpha)}$ *with* $k(\alpha) = 2\sqrt{\frac{\pi}{\alpha}} \cdot 4^{-\alpha}$, $\alpha > 0$ *and hence* $0 < B(\alpha, \alpha) - k(\alpha) < \frac{k(\alpha)}{6\alpha} \cdot \exp\left(\frac{1}{6\alpha}\right)$ *with* $\lim_{\alpha \to \infty} \frac{B(\alpha, \alpha)}{k(\alpha)} = 1$.

Proof. By the known analytical estimate for Stirling's formula (cf. [1], (6.1.38))

$$\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x < \Gamma(x) < \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x \cdot \exp\left(\frac{1}{12x}\right), \quad x > 0$$

there follows

$$\frac{2\pi}{x}\left(\frac{x}{e}\right)^{2x} < \Gamma^2(x) < \frac{2\pi}{x}\left(\frac{x}{e}\right)^{2x} \cdot \exp\left(\frac{1}{6x}\right)$$

and $$\sqrt{\frac{\pi}{x}}\left(\frac{2x}{e}\right)^{2x} < \Gamma(2x) < \sqrt{\frac{\pi}{x}}\left(\frac{2x}{e}\right)^{2x} \cdot \exp\left(\frac{1}{24x}\right),$$

hence

$$0 < \exp\left(-\frac{1}{24\alpha}\right) < \frac{B(\alpha, \alpha)}{k(\alpha)} = \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha) \cdot k(\alpha)} < \exp\left(\frac{1}{6\alpha}\right), \quad \alpha > 0$$

from which the Lemma follows immediately. Note that $\exp(x) - 1 < x \cdot \exp(x),\ x > 0.$

The following figure shows the function $h(\alpha) = \frac{B(\alpha, \alpha) - k(\alpha)}{k(\alpha)} \cdot 6(\alpha),$ $0 < \alpha < 100$:

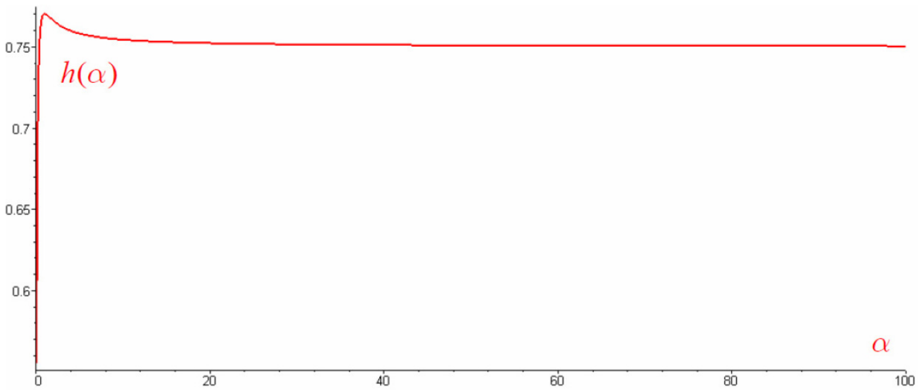

Figure 1. Plot of $g(\alpha)$.

This leads to the following

Theorem. *Let* $\sigma > 0$ *be fixed and* $c := \sqrt{2\alpha} \cdot \sigma$ *for* $\alpha > 0$. *Then* $\lim_{\alpha \to \infty} g(y;\ \alpha,\ c) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2\sigma^2}\right)$, *i.e., the rescaled symmetric beta distribution converges weakly to the normal distribution with mean zero and variance* σ^2.

Proof. We have

$$g(y;\ \alpha,\ c) = \frac{1}{2c} f\left(\frac{1}{2} + \frac{y}{2c},\ \alpha\right) = \frac{\left(\frac{1}{2} + \frac{y}{2c}\right)^{\alpha - 1} \cdot \left(\frac{1}{2} - \frac{y}{2c}\right)^{\alpha - 1}}{B(\alpha, \alpha) \cdot 2c}$$

$$= \frac{\left(1 - \frac{y^2}{c^2}\right)^{\alpha-1}}{4^{\alpha-1} \cdot k(\alpha) \cdot 2c} \cdot \frac{k(\alpha)}{B(\alpha,\ \alpha)} = \ldots$$

$$\ldots = \frac{\left(1 - \frac{y^2}{c^2}\right)^{\alpha-1}}{\sqrt{2\pi}\sigma} \cdot \frac{k(\alpha)}{B(\alpha,\ \alpha)}, \quad -c < y < c.$$

Now since $\lim_{\alpha\to\infty}\left(1 - \frac{y^2}{c^2}\right)^{\alpha-1} = \lim_{\alpha\to\infty}\left(1 - \frac{y^2}{2\alpha\sigma^2}\right)^{\alpha-1} = \exp\left(-\frac{y^2}{2\sigma^2}\right)$, the Theorem is proved.

The following figures show some comparisons between $g(y;\ \alpha,\ c)$ (plotted in red) and $n(y;\ \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)$ (plotted in blue), for several choices of α and σ.

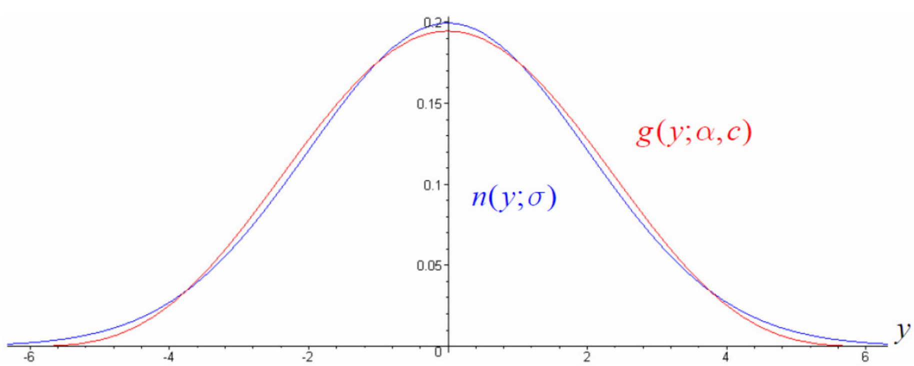

Figure 2. $\alpha = 10$, $\sigma = 2$, $c = 8.944\ldots$.

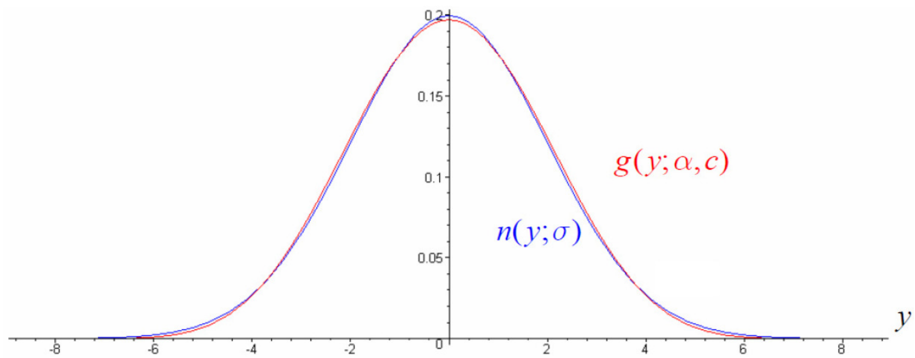

Figure 3. $\alpha = 5$, $\sigma = 2$, $c = 6.324...$.

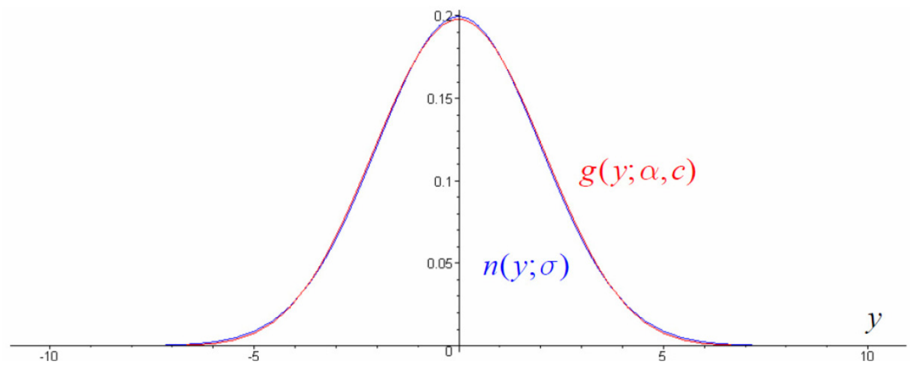

Figure 4. $\alpha = 15$, $\sigma = 2$, $c = 10.954...$.

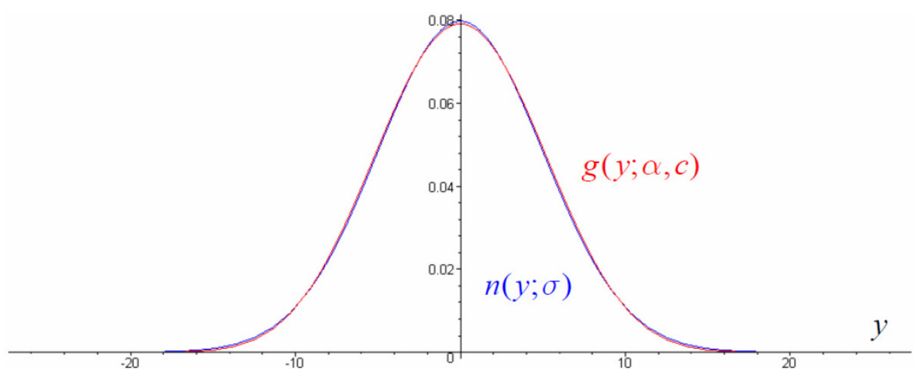

Figure 5. $\alpha = 15$, $\sigma = 5$, $c = 27.386...$.

Note that for the variance of Y, we have $Var(Y) = \frac{c^2}{(2\alpha+1)}$ $= \frac{2\alpha\sigma^2}{(2\alpha+1)} \approx \sigma^2$ for large values of α.

Further relationships between beta and other distributions are discussed in [3].

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