A note on a class of certain mixtures of product –gamma distributions

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July 2026

Key words: gamma distributions, mixtures of distributions,copulas
2020 Mathematics Subject Classification: 62E15,62H05

In this note, we investigate multivariate distributions with densities of the form

$h(x_1,\ldots,x_n)=\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)\cdot\exp\left(-\sum_{k=1}^{n}x_k\right),\ x_1,\ldots,x_n\geq 0$ with $n\in\mathbb{N}$. Note that this density can be rewritten as

$h(x_1,\ldots,x_n)=\frac{1}{n}\sum_{k=1}^{n}x_k\cdot\exp\left(-x_k\right)\cdot\prod_{\substack{i=1\\ i\neq k}}^{n}\exp(-x_i),\ x_1,\ldots,x_n\geq 0,$ i.e. it is the density of a mixture of n product-gamma distributions where exactly one component is representing a $\Gamma(2,1)$-distribution and all others represent independent $\mathcal{E}(1)$-Exponential distributions. Note that an $\mathcal{E}(1)$-Exponential distribution is identical to a $\Gamma(1,1)$-distribution (see [1], p. 337, (17.2)). This implies that we can write the corresponding c.d.f. H as follows:

$H(x_1,\ldots,x_n)=\frac{1}{n}\sum_{(i_1,\ldots,i_n)\in\mathcal{J}}\prod_{j=1}^{n}F_{i_j}\left(x_j\right)$ with $\mathcal{J}=\left\{(i_1,\ldots,i_n)\in\{0,1\}^n\,|\sum_{j=1}^{n}i_j=1\right\}$, i.e. $\mathcal{J}$ contains all $\{0,1\}$-valued n-tuples with all entries equal to zero except for exactly one entry equal to 1, and $F_0(x)=1-\exp(-x)$ and $F_1(x)=1-(1+x)\exp(-x),\ x\geq 0.$ The one-dimensional marginal c.d.f. H_1 is thus given by $H_1(x)=\frac{F_1(x)+(n-1)F_0(x)}{n}=1-\frac{n+x}{n}\exp(-x)$ with density $h_1(x)=\frac{f_1(x)+(n-1)f_0(x)}{n}=\left(1+\frac{x-1}{n}\right)\exp(-x),\ x\geq 0$ where $f_0(x)=\exp(-x)$ and $f_1(x)=x\exp(-x),\ x\geq 0$ while the two-dimensional marginal c.d.f. H_2 is correspondingly given by

$H_2(x,y)=\frac{F_0(x)F_1(y)+F_1(x)F_0(y)+(n-2)F_0(x)F_0(y)}{n}$ with density

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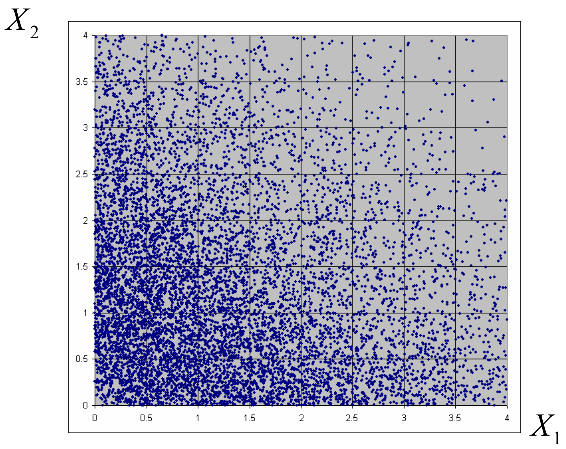
$$h_2(x,y)=\frac{f_0(x)f_1(y)+f_1(x)f_0(y)+(n-2)f_0(x)f_0(y)}{n},\ x,y\geq 0$$

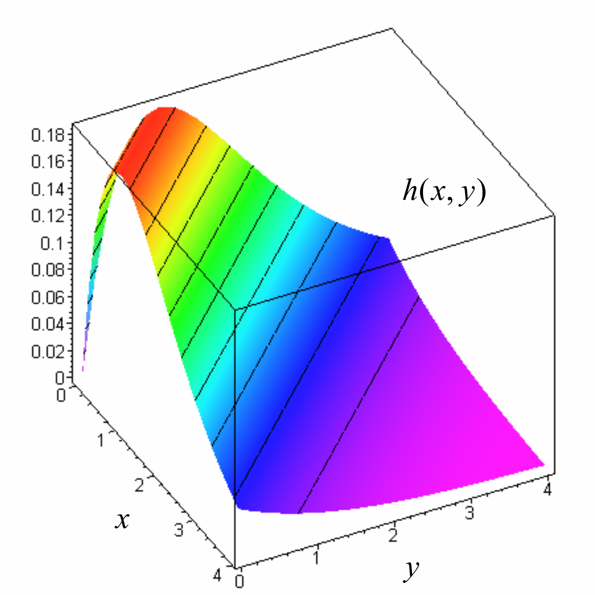
which implies that we have the following expressions for moments of random variables $\mathbf{X}=\left(X_1,\ldots,X_n\right)$ with density h:

$$E(X_i)=\frac{n+1}{n},\ E\left(X_i^2\right)=\frac{2n+4}{n},\ Var(X_i)=\frac{n^2+2n-1}{n^2},\ \mathrm{Kov}\left(X_i,X_j\right)=-\frac{1}{n^2},\ 1\leq i<j\leq n$$

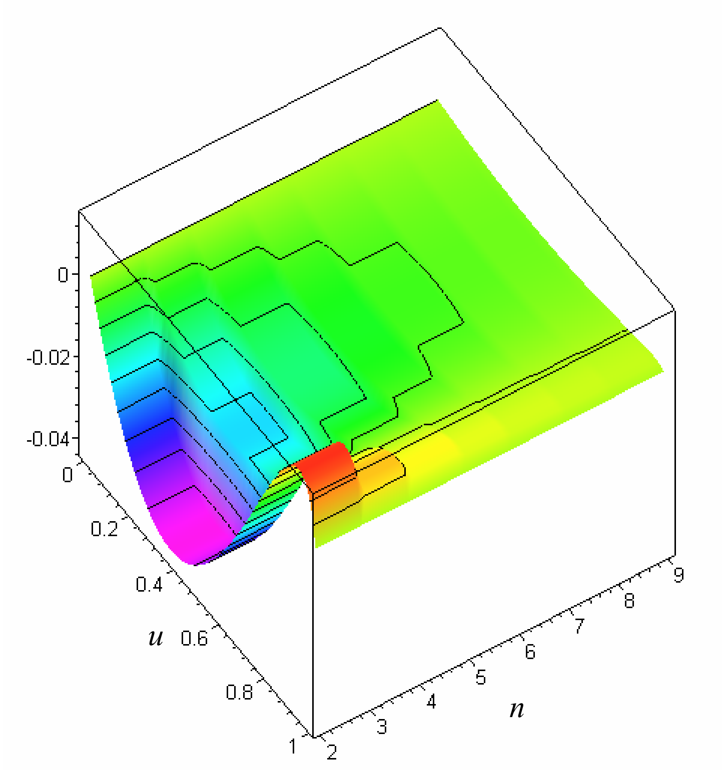
and correlation $\rho\left(X_i,X_j\right)=-\frac{1}{n^2+2n-1},\ 1\leq i<j\leq n$. The following graph shows a scatterplot of 10,000 simulated pairs $\left(X_1,X_2\right)$ for case $n=2$:



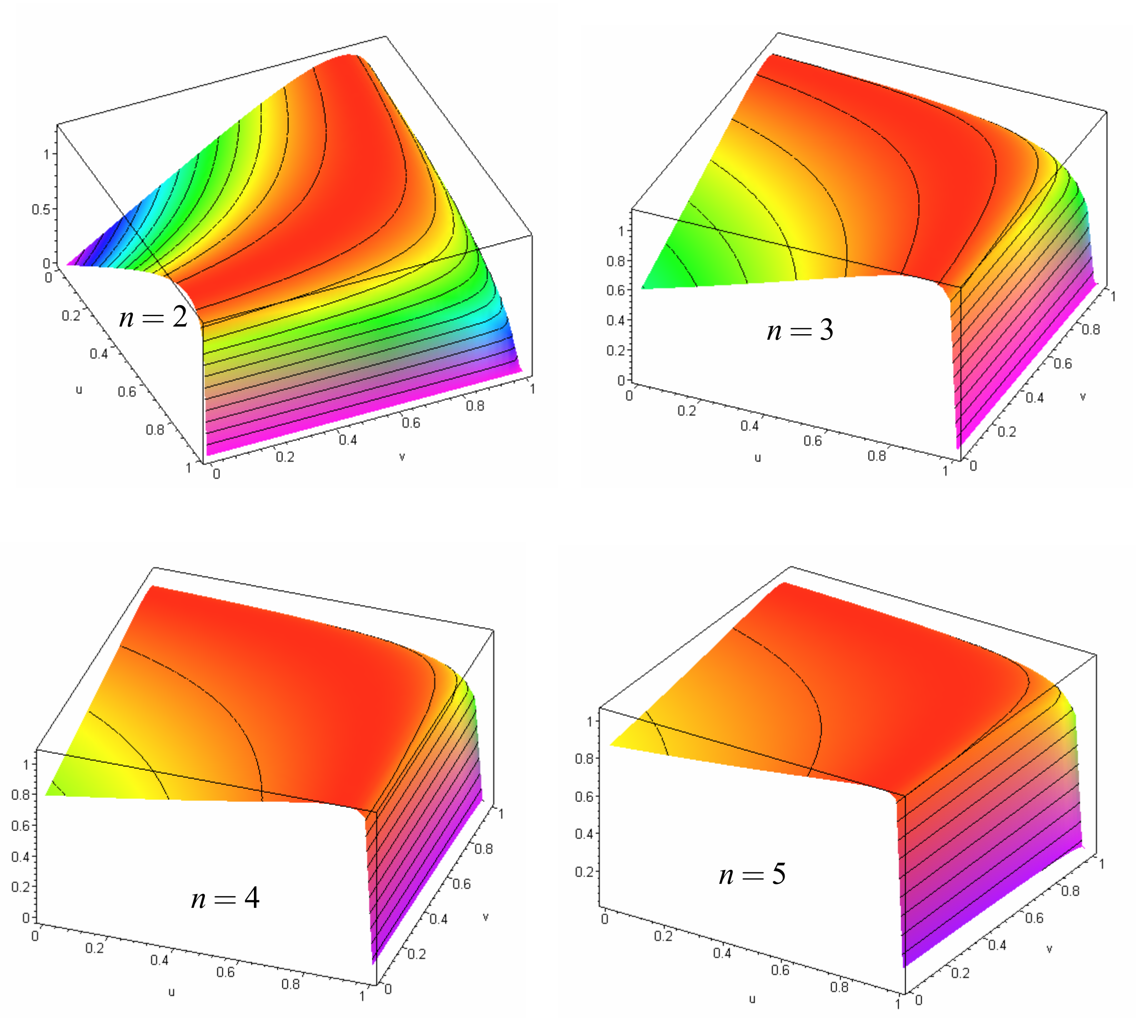
The following graph shows the corresponding density for $n=2$:



Unfortunately, the two-dimensional copulas C_2 derived from H_2 cannot be explicitly stated since the quantile function Q_1 pertaining to H_1 cannot be expressed in simple terms. But we can approximate H_1 as follows: $H_1(x) = 1 - \left(1 + \frac{x}{n}\right)\exp(-x) \approx 1 - \exp\left(-\frac{n+1}{n}x\right),\ x \geq 0$ from which we get an approximation Q_1^* to the true quantile function Q_1 as $Q_1^*(u,n) := -\frac{n}{n+1}\ln(1-u),\ u > 0,\ n \geq 2.$ The following 3d-graph shows the functions $H_1\left(Q_1^*(u,n)\right) - u$ for $0 < u \leq 1$ and $2 \leq n \leq 9$:



As to be expected, the approximation error gets smaller for increasing values of n. We thus get the following approximation $C^*(u,v,n) := H_2\left(Q_1^*(u,n), Q_1^*(u,n)\right)$ to the true copula $C(u,v) = \mathrm{H}_2\left(Q_1(u,v),\ Q_1(u,v)\right), 0 \leq u, v \leq 1,$ according to Sklar's theorem (cf. [2], p. 47, Corollary 2.10.10). The following graphs show the densities $c^*(u,v,n) = \frac{\partial^2}{\partial v \partial u} C^*(u,v,n)$ of $C^*(u,v,n)$ for various values of n:



Seemingly, with increasing n, $C^*(\bullet,\bullet,n)$ tends to the independence copula which is clear since $H_2(x,y)=\dfrac{F_0(x)F_1(y)+F_1(x)F_0(y)+(n-2)F_0(x)F_0(y)}{n}$ tends to $F_0(x)F_0(y)$ for increasing n, $x,y\geq 0$.

References

[1] N.L. Johnson, S. Kotz, N. Balakrishnan (1994): Continuous Univariate Distributions, Volume 1, 2nd ed.,Wiley, N.Y.

[2] R.B. Nelsen (2006): An Introduction to Copulas. 2nd ed., Springer, N.Y.