On a Probabilistic Representation Theorem of Operator Semigroups with Bounded Generator

D. Pfeifer

(Technical University Aachen)

In this paper we derive an extension of a general probabilistic representation formula for strongly continuous operator semigroups $\{T(t);\, t \geq 0\}$ presented in [3], valid for semigroups with bounded infinitesimal generator *A*. Semigroups of this kind play an important role in probability theory, especially in the field of homogeneous Markov jump processes (see for example [4], [2]). For simplicity, our notation will closely follow [3]. (Note that in [3], the semigroup $\{T(t);\, t \geq 0\}$ was erroneously identified with a Banach algebra $\mathcal{T}$. Clearly, $\mathcal{T}$ should denote the Banach algebra generated by the semigroup.)

THEOREM. Let *N* be a non-negative integer-valued random variable with unit mean such that its characteristic function is analytic in some neighbourhood of the origin. Then if ψ_N denotes the moment generating function of *N*, we have for all $\xi > 0$

(1) $$T(\xi) = \lim_{n\to\infty} \left\{ \psi_N \left(I + \frac{\xi}{n} A \right) \right\}^n$$

where *I* denotes the identity operator.

PROOF. Heuristically, relation (1) is obtained from the following relation proved in [3]:

(2) $$T(\xi) = \lim_{n\to\infty} \left\{ \psi_N \left(T\left(\frac{\xi}{n} \right) \right) \right\}^n,$$

replacing $T\left(\frac{\xi}{n}\right)$ by the first two terms of the corresponding Taylor expansion ([1], Proposition 1.1.6).

For the proof, let $N_1, N_2, \cdots$ be independent copies of *N* and let $S_n = \sum_{k=1}^{n} N_k,\ n \in \mathbb{N}$. Then by [3],

(3) $$\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n = E\left(T\left(\frac{\xi}{n}S_n\right)\right) = E\left(T\left(\frac{\xi}{n}\right)^{S_n}\right),$$

and similarly,

(4) $$\left\{\psi_N\left(I+\frac{\xi}{n}A\right)\right\}^n = \left\{E\left(I+\frac{\xi}{n}A\right)^N\right\}^n = E\left(\left(I+\frac{\xi}{n}A\right)^{S_n}\right).$$

Again by [1], Proposition 1.1.6 and the fact that in our case, $T(t)=e^{At},\ t>0,$

$$\begin{aligned}
&\left\|\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n - \left\{\psi_N\left(I+\frac{\xi}{n}A\right)\right\}^n\right\| \le E\left(\left\|T\left(\frac{\xi}{n}\right)^{S_n} - \left(I+\frac{\xi}{n}A\right)^{S_n}\right\|\right)\\
&= E\left(\left\|\sum_{k=0}^{S_n-1} T\left(\frac{\xi}{n}\right)^{S_n-k}\left(I+\frac{\xi}{n}A\right)^k - T\left(\frac{\xi}{n}\right)^{S_n-k-1}\left(I+\frac{\xi}{n}A\right)^{k+1}\right\|\right)\\
&\le E\left(\sum_{k=0}^{S_n-1}\left\|T\left(\frac{\xi}{n}\right)^{S_n-k-1}\left\{T\left(\frac{\xi}{n}\right)-\left(I+\frac{\xi}{n}A\right)\right\}\left(I+\frac{\xi}{n}A\right)^k\right\|\right)\\
&\le \left\|T\left(\frac{\xi}{n}\right)-\left(I+\frac{\xi}{n}A\right)\right\| E\left(e^{\frac{\xi}{n}\|A\|S_n}\frac{\left(1+\frac{\xi}{n}\|A\|\right)^{S_n}-1}{\frac{\xi}{n}\|A\|}\right)\\
&\le \frac{n}{\xi}\|A\|\int_0^{\frac{\xi}{n}}\left(\frac{\xi}{n}-s\right)\|T(s)\|\,ds\ E\left(e^{2\frac{\xi}{n}\|A\|S_n}\right)\\
&\le \frac{n}{\xi}\|A\|e^{\frac{\xi}{n}\|A\|}\int_0^{\frac{\xi}{n}}\left(\frac{\xi}{n}-s\right)ds\ E\left(e^{2\frac{\xi}{n}\|A\|S_n}\right)\\
&\le \frac{\xi}{2n}\|A\|e^{\frac{\xi}{n}\|A\|}\ E\left(e^{2\frac{\xi}{n}\|A\|S_n}\right).
\end{aligned}$$

Following [3], $E\left(e^{2\frac{\xi}{n}\|A\|S_n}\right)\to e^{2\xi\|A\|}$ for $n\to\infty,$ hence for sufficiently large *n*,

(5) $$\left\|\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n - \left\{\psi_N\left(I+\frac{\xi}{n}A\right)\right\}^n\right\| \le \frac{\xi}{2n}\|A\|e^{3\xi\|A\|}.$$

Applying (2) now gives the desired result. ■

Note that by a slight modification of the foregoing proof, one could also obtain the following relation:

(6) $$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(I+\frac{A}{n}\right)\right\}^n,$$

where now N is a random variable with mean $\xi>0$.

For special choices of N, the following well-known representation formulae are reobtained:

A) $N\equiv 1$, i.e. $\psi_N(t)=t$:

$$T(\xi)=\lim_{n\to\infty}\left(I+\frac{\xi}{n}A\right)^n.$$

B) N being geometrically distributed with unit mean, i.e. $\psi_N(t)=\dfrac{1}{2-t}$:

$$T(\xi)=\lim_{n\to\infty}\left(2I-\left(I+\frac{\xi}{n}A\right)\right)^{-n}=\lim_{n\to\infty}\left(I-\frac{\xi}{n}A\right)^{-n}.$$

C) N being binomially distributed with mean $\xi\in(0,1)$, i.e. $\psi_N(t)=1-\xi+\xi t$:

$$T(\xi)=\lim_{n\to\infty}\left((1-\xi)I+\xi\left(I+\frac{A}{n}\right)\right)^n=\lim_{n\to\infty}\left(I+\frac{\xi}{n}A\right)^n.$$

D) N being Poisson-distributed with mean $\xi>0$, i.e. $\psi_N(t)=e^{-\xi}e^{t\xi}$:

$$T(\xi)=\lim_{n\to\infty}\left(e^{-\xi I+\xi\left(I+\frac{A}{n}\right)}\right)^n=e^{A\xi}.$$

Of course, a lot of further representation rtheorems can be obtained from the general formula. For instance, let N have a uniform distribution over the points $\{0,2\}$. Then $\psi_N(t)=\frac{1}{2}+\frac{1}{2}t^2$, hence

E) $$T(\xi) = \lim_{n\to\infty}\left(\frac{1}{2}I + \frac{1}{2}\left(I + \frac{\xi}{n}A\right)^2\right)^n = \lim_{n\to\infty}\left(I + \frac{\xi}{n}A + \frac{\xi^2}{2n^2}A^2\right)^n.$$

Note that in this case, $T\left(\frac{\xi}{n}\right)$ is replaced by the first three terms of the Taylor expansion in (2).

Or, if N has a uniform distribution over the points $\{0, 2\xi\}$, then $\psi_N(t) = \frac{1}{2} + \frac{1}{2}t^{2\xi}$, hence

F) $$T(\xi) = \lim_{n\to\infty}\left(\frac{1}{2}I + \frac{1}{2}\left(I + \frac{A}{n}\right)^{2\xi}\right)^n \text{ etc.}$$

REFERENCES

[1] Butzer, P.L. and Berens, M.: *Semi-Groups of Operators and Approximation*. Grundlehren der Mathematischen Wissenschaften 145. Berlin-Heidelberg-New York: Springer 1967.

[2] Freedman, D.: *Markov Chains*. San Francisco, Holden-Day 1971.

[3] Pfeifer, D.: On a general probabilistic representation formula for semigroups of operators. *J. of Mathematical Research and Exposition* 2, No. 4, 93 - 98 (1982).

[4] Pfeifer, D.: A semigroup-theoretic proof of Poisson's limit law. To appear in: *Semigroup Forum* (1983).