*Adv. Appl. Prob.* **17**, 470–471 (1985)
*Printed in N. Ireland*


# ON A RELATIONSHIP BETWEEN RECORD VALUES AND ROSS'S MODEL OF ALGORITHM EFFICIENCY

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Recently Ross ((1981), (1983), Chapter 4.6) has developed a simple Markov chain model for an average-case analysis of the simplex algorithm in linear programming. Characteristically, this algorithm moves through the extreme points of the feasible region in such a way that only those points are successively considered which improve the actual value of the gain function (see e.g. Hadley (1962)). If we assume the $N$ (say) extreme points to be arranged in such a way that the first point gives the largest and the $N$th point the smallest value of the gain function, then the steps of the algorithm can appropriately be described by a finite Markov chain $S_1, \cdots, S_N$ with state space $\{1, \cdots, N\}$ such that

$$(1) \quad P(S_1 = k) = \frac{1}{N}, \qquad 1 \leqq k \leqq N \quad \text{and} \quad P(S_{n+1} = k \mid S_n = i) = \frac{1}{i-1}, \qquad 1 \leqq k < i \leqq N$$

with 1 being an absorbing state. For this model Ross (1981), (1983) has shown that if $T_N$ denotes the number of steps required to reach state 1 for the first time then $T_N$ is approximately (for large $N$) Poisson distributed over $\mathbb{N}$ with mean $\log N$. Here we shall demonstrate that this result can also be obtained by record value theory. In fact, if $\{X_n; n \in \mathbb{N}\}$ is an i.i.d. sequence of random variables following a uniform distribution over $\{1, \cdots, N\}$, then $\{S_n; 1 \leqq n \leqq N\}$ is identically distributed with the lower record value sequence $\{X_{U_n}; 1 \leqq n \leqq N\}$ where

$$(2) \qquad U_1 = 1, \qquad U_{n+1} = \begin{cases} \min\{k; X_k < X_{U_n}\} & \text{if} \quad X_{U_n} > 1, \\ U_n, & \text{otherwise.} \end{cases}$$

This follows readily by arguments as in Shorrock (1972). Especially, $T_N$ is identically distributed with $T = \min\{n; X_{U_n} = 1\}$.

Unfortunately, distribution theory for records from discrete distributions is rather cumbersome; however, to obtain the asymptotic results as indicated, we can use a continuous approximation in the following way. Obviously, nothing is seriously changed if we assume the random variables $\{X_n; n \in \mathbb{N}\}$ to be uniformly distributed over $\{1/N, \cdots, (N-1)/N, 1\}$ except that now $T = \min\{n; X_{U_n} = 1/N\} = \min\{n; X_{U_n} < 2/N\}$. But for large $N$, we may approximately assume the $X_n$'s to be uniformly distributed over the unit interval; then $T$ is close to the stopping time $T^* = \min\{n; X_{U_n} < 2/N\}$ where now $\{U_n; n \in \mathbb{N}\}$ is the associated record time sequence. But as is known from record value theory (see Shorrock (1972)), $\{-\log X_{U_n};\ n \in \mathbb{N}\}$ forms the arrival time sequence of a unit-rate Poisson process implying that $T^*$ follows exactly a Poisson distribution with mean $\log N + 1 - \log 2 \approx \log N$. This gives the desired result. Moreover, the above arguments suggest that for the original Markov chain $\{S_1, \cdots, S_N\}$ and large

Received 12 October 1984; revision received 30 January 1985.
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$N\{-\log S_n/N;\ 1 \leqq n \leqq N\}$ behaves approximately as the first $N$ arrival times $Z_1, \cdots, Z_N$ of a unit rate Poisson process, or equivalently,

$$S_n \approx \text{int}\,(N \exp(-Z_n)) + 1,\ 1 \leqq n \leqq N. \tag{3}$$

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