The zero utility principle for scale families of risk distributions

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1. Introduction

A premium calculation principle is a general rule that assigns a premium H to a given risk X. Intuitively, H is what the insurance carrier charges (apart from an expense allowance) for taking over the risk X. As a policy of premium calculation, the insurance carrier might base his decision on a utility function u, which is assumed to be a monotonically increasing, differentiable and concave function on $\mathbb{R}$. The principle requires that the utility u (0) before assuming responsibility for the claims be equal to the expected utility $E[u(H - X)]$ after taking over such responsibility in exchange for the premium H. Mathematically X is a random variable, and H depends on X through its distribution function and the utility function, as well. In the case $u(0) = 0$ we obtain the zero utility principle given as

$$E[u(H - X)] = 0.$$

So far this principle has been mainly of theoretical interest. If we take the utility function as exponential, then the zero utility principle equals the exponential principle and has an explicit solution. For twice differentiable non-exponential utility functions we have, moreover, the variance principle as a first approximation, see Heilmann (1987). Since this is the only approach known so far to deal with general utility functions, many authors *define* a utility function as being twice differentiable, see Goovaerts and de Vylder (1979), Gerber (1985), Reich (1986), Kremer (1986) or Heilmann and Schröter (1987).

In this paper, we show that for certain classes of not necessarily twice differentiable utility functions – which will be characterized as being *scale* invariant – and for certain classes of risk distributions where the expectation plays the role of the "natural" parameter we find explicit representations for the zero utility premium. Further, comparisons with other premium calculation principles are given.

2. Scale invariant utility functions

In this paper, we consider throughout continuous utility functions $u: \mathbb{R} \to \mathbb{R}$, differentiable at zero, with the following characteristic properties:

$$u(0) = 0, \quad u'(0) = 1 \qquad \text{[normalization]} \quad (1)$$

$$x \leq y \Rightarrow u(x) \leq u(y), \quad x, y \in \mathbb{R} \qquad \text{[monotonicity]} \quad (2)$$

$$u(tx + (1-t)y) \geq t\,u(x) + (1-t)\,u(y), \quad x, y \in \mathbb{R},\ 0 < t < 1 \qquad \text{[concavity]} \quad (3)$$

Note that we do not assume that u should be twice differentiable.

Clearly, the normalization condition says that a zero quantity x (e.g., an empty financial budget) has no utility, and that the absolute utility of "small" quantities x is almost identical with that quantity (i.e., $u(x) \approx x$). The concavity condition says that the relative increase of the utility becomes smaller when the size x of the quantity grows.

Suppose that $\mathscr{U} = \{u_a \mid a > 0\}$ is a given family of utility functions fulfilling (1), (2) and (3) above. We say that $\mathscr{U}$ is *scale invariant* if there exists some function $h: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the property

$$u_a(\mu x) = \mu\, u_{h(a,\mu)}(x) \quad \text{for all } a, \mu > 0,\ x \in \mathbb{R}, \qquad (4)$$

where $\mathbb{R}^+$ denotes the set of all *positive* real numbers. This notion of scale invariance reflects the idea that structural properties of utility functions should not essentially depend on the scale in which the variable x is measured [e.g., if x represents a monetary quantity, there should not be basic differences whether the currency unit is DM or US-$ or £].
The following Theorem given a characterization for such classes of utility functions.

Theorem 1. Let u be a utility function in the sense of (1), (2) and (3), and $g: \mathbb{R}^+ \to \mathbb{R}^+$ be surjective. Then

$$u_a(x) := \frac{u(g(a)x)}{g(a)}, \quad a > 0, x \in \mathbb{R} \tag{5}$$

defines a scale invariant class $\mathscr{U}$ of utility functions. Conversely, if $\mathscr{U} = \{u_a \mid a > 0\}$ is a scale invariant class of utility functions and $h(a, \cdot)$ is surjective for at least one $a > 0$, then there exists some utility function u and a surjective function g such that (5) holds for all $a > 0$.

Proof. Given (5), we see that for all $a, \mu > 0$,

$$u_a(\mu x) = \frac{u(g(a)\mu x)}{g(a)} = \mu \frac{u(g(a')x)}{g(a')} = \mu u_{a'}(x), \quad x \in \mathbb{R}$$

where a' is any solution of the equation $g(a') = g(a)\mu$ [which exists by the surjectivity of g]. Since a' depends in general on both a and μ, it is clear that $h(a, \mu) = a'$ defines a function h fulfilling (4), hence $\mathscr{U}$ is scale invariant.
Conversely, suppose that $\mathscr{U}$ is a scale invariant class of utility functions. Fix $a > 0$ such that $h(a, \cdot)$ is surjective. Then for all $c > 0$,

$$u_a(cx) = c\, u_{h(a,c)}(x), \quad x \in \mathbb{R},$$

hence

$$u_{h(a,c)}(x) = \frac{u_a(cx)}{c}, \quad x \in \mathbb{R}.$$

By the surjectivity of $h(a, \cdot)$, there exists at least one $c > 0$ with $h(a, c) = c'$, for every $c' > 0$. Choosing c appropriately we see that there exists a function $g: \mathbb{R}^+ \to \mathbb{R}^+$ such that $c = g(c')$, i.e. $h(a, g(c')) = c'$ and hence

$$u_{c'}(x) = \frac{u_a(g(c')x)}{g(c')}, \quad x \in \mathbb{R},$$

for all $c' \in \mathbb{R}^+$, which is (5). In particular, for the fixed a, we have

$$u_a(x) = \frac{u_a(g(a)x)}{g(a)}, \quad x \in \mathbb{R}.$$

It remains to show that g is surjective. But for every $\mu > 0$, we have, with a still being fixed,

$$u_a(\mu x) = \mu \frac{u_a(g(a)\mu x)}{g(a)\mu} = \mu u_{h(a,\mu)}(x) = \mu u_{h(a,g(c''))}(x) = \mu \frac{u_a(g(c'')x)}{g(c'')}, \quad x \in \mathbb{R},$$

where $c'' = h(a, \mu)$. It follows that

$$\frac{u_a(g(a)\mu x)}{g(a)\mu} = \frac{u_a(g(c'')x)}{g(c'')}, \quad x \in \mathbb{R},$$

hence, denoting $r = \dfrac{\mu g(a)}{g(c'')}$, we obtain, putting $y = g(c'')x$,

$$u_a(ry) = r u_a(y), \quad y \in \mathbb{R}.$$

It follows that $u_a(r^2 y) = r u_a(ry) = r^2 u_a(y)$, $u_a(r^3 y) = r u_a(r^2 y) = r^3 u_a(y)$, and hence by induction,

$$u_a(r^n y) = r^n u_a(y), \quad n \in \mathbb{N}, y \in \mathbb{R}.$$

Similarly,

$$u_a(y) = u_a(r^n r^{-n} y) = r^n u_a(r^{-n} y), \quad n \in \mathbb{N}, y \in \mathbb{R},$$

or, in general,

$$u_a(r^n y) = r^n u_a(y), \quad n \in \mathbb{Z}, y \in \mathbb{R}.$$

Suppose first that $r \neq 1$. Let $s_n = r^n$ if $r < 1$ and $s_n = r^{-n}$ if $r > 1$, for $n \in \mathbb{N}$. Then $s_n \to 0$ for $n \to \infty$, hence

$$\frac{u_a(y)}{y} = \lim_{n \to \infty} \frac{u_a(s_n y)}{s_n y} = u_a'(0) = 1, \quad y \neq 0,$$

i.e.

$$u_a(y) = y, \quad y \in \mathbb{R}.$$

In this case,

$$u_{h(a,\mu)}(y) = \frac{u_a(\mu y)}{\mu} = y, \quad y \in \mathbb{R},$$

for all $\mu > 0$, independent of μ, hence we are free to modify h in such a way that $h(a, \mu) = \mu$, i.e. $g(\mu) = \mu$ for all $\mu > 0$, which is surjective.
In case that $r = 1$, we have $g(c'') = g(a)\mu$, and since $g(a)\mu$ ranges all over R^+ when μ does so, we see that g is surjective, as requested.
This proves Theorem 1. □

By relation (5), we see that in the case of surjectivity of g, we could likewise put $u_a = v_{g(a)}$, so that the re-parameterized family $\mathscr{V} = \{v_c | c > 0\}$ with

$$v_c(x) = \frac{u(cx)}{c}, \quad x \in \mathbb{R}$$

would be an equivalent representation of the class $\mathscr{U}$, with the "natural" scale parameter $c > 0$.

Example 1. The following classes $\mathscr{U} = \{u_a | a > 0\}$ of utility functions are scale invariant:

$$u_a(x) := \min\{x, a\} = \frac{u(x/a)}{1/a} \quad \text{[truncated linear utility]} \tag{6}$$

with $u(x) = \min\{x, 1\} = u_1(x)$, $x \in \mathbb{R}$, $g(a) = 1/a$

$$u_a(x) := \frac{1}{a}[1 - e^{-ax}] = \frac{u(ax)}{a} \quad \text{[exponential utility]} \tag{7}$$

with $u(x) = 1 - e^{-x} = u_1(x)$, $x \in \mathbb{R}$, $g(a) = a$

$$u_a(x) := \begin{cases} x - \frac{x^2}{2a}, & x \le a \\ \frac{a}{2}, & x > a \end{cases} = \frac{u(x/a)}{1/a} \quad \text{[quadratic utility]} \tag{8}$$

with $u(x) = u_1(x)$, $x \in \mathbb{R}$, $g(a) = 1/a$

$$u_a(x) := \begin{cases} x, & x \le 0 \\ v(x), & x > 0 \end{cases} \quad \text{[left linearized utility]} \tag{9}$$

with $v(x)$ being an arbitrary utility function.

Note that the utility functions of type (9) are differentiable, but not twice differentiable at zero in general.
If we assume that $u_a \in \mathscr{U}$ [where $\mathscr{U}$ is scale invariant] is twice differentiable at zero and if we use the first three terms in the Taylor expansion of the generating utility function u in (5), we obtain

$$u(x) \approx x + \frac{\varrho}{2} x^2$$

for values of x near zero, where $\varrho = u''(0) \le 0$ by the concavity of u. If further the random variable X denotes a risk with existing mean μ and variance σ^2, the zero utility premium H can approximately be calculated according to the solution of the equation [cf. Heilmann (1988), p. 154 f].

$$0 = E[u(H - X)] \approx E[(H - X)] + \frac{\varrho}{2} E[(H - X)^2] = H - \mu + \frac{\varrho}{2}[(H - \mu)^2 + \sigma^2],$$

giving

$$H \approx \mu - \frac{\varrho \sigma^2}{1 + \sqrt{1 - \varrho^2 \sigma^2}} \approx \mu - \frac{\varrho}{2} \sigma^2$$

provided that $\varrho \sigma < 1$. The zero utility premium H_a for the given class $\mathscr{U}$ can hence also be approximated by a variance premium, using the corresponding expansion

$$u_a(x) = \frac{u(g(a)x)}{g(a)} \approx x + \frac{g(a)\varrho}{2} x^2$$

for values of x near zero, giving

$$H_a \approx \mu - \frac{g(a)\varrho}{2} \sigma^2 \approx \mu + \frac{g(a)}{g(1)}(H_1 - \mu), \quad a > 0.$$

3. Scale families of risk distributions

In what follows we shall assume that the random variable X denotes a non-negative risk with existing means $\mu > 0$ and finite variance σ^2. For a given distribution Q, let Q_μ denote the scaled distribution defined by

$$Q_\mu(A) := Q\left(\frac{1}{\mu} A\right) \tag{10}$$

for all $\mu > 0$ and Borel sets $A \subseteq \mathbb{R}$, where $\frac{1}{\mu} A = \{a/\mu \mid a \in A\}$. We say that the risk distribution P^X belongs to a *scale family* $\mathscr{P}$ of distributions if there exists a fixed distribution Q such that $\mathscr{P} = \{Q_\mu \mid \mu > 0\}$. If, in particular, $P^X = Q_\mu$ and simultaneously $E(X) = \mu$, $\mu > 0$, then $\mathscr{P}$ is also called a *natural scale family* of distributions since in such a class, the expectation of the risk plays the role of the "natural" parameter. Note that if X is an arbitrary risk with distribution Q and the expectation $E(X) = 1$, then the scaled risks μX, $\mu > 0$ belong to the natural scale family $\mathscr{P} = \{Q_\mu \mid \mu > 0\}$, and the variances grow quadratically with μ: $\operatorname{Var}(\mu X) = \mu^2 \operatorname{Var}(X) = \mu^2 \sigma^2$, $\mu > 0$.

Example 2. The following classes of risk distributions form natural scale families [in terms of densities f_μ, $\mu > 0$]:

$$f_\mu(x) = \frac{1}{\mu} e^{-x/\mu}, \qquad x > 0 \qquad \text{[exponential distributions]} \tag{11}$$

$$f_\mu(x) = \left(\frac{\alpha}{\mu}\right)^\alpha \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha x/\mu}, \qquad x > 0\ (\alpha > 0) \qquad \text{[gamma distributions]} \tag{12}$$

$$f_\mu(x) = \frac{\alpha - 1}{\alpha \mu} \frac{1}{\left(1 + \frac{x}{\alpha \mu}\right)^\alpha}, \qquad x > 0\ (\alpha > 1) \qquad \text{[Pareto distributions]} \tag{13}$$

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{1}{2}\ln^2(x/\mu)}, \qquad x > 0 \qquad \text{[log-normal distributions]} \tag{14}$$

Note that in general, if f is the density of a risk X with expectation $E(X) = 1$, then $\frac{1}{\mu} f\left(\frac{x}{\mu}\right)$, $x > 0$ is the density of the scaled risk μX, with expectation μ.

4. Zero utility premiums in scale families

In this section we shall show that premium calculations based on the zero utility principle can be essentially simplified if scale invariance is given, and the risk distribution is a member of a natural scale family of distributions. In this case, the zero utility premium can be expressed in terms of the net risk premium.

Theorem 2. Suppose that $\mathscr{U} = \{u_a \mid a > 0\}$ is a scale-invariant class of utility functions according to (4), and that the distribution P^X of the risk X belongs to a natural scale family $\mathscr{P}$. Suppose further that for all $a > 0$, the premium $H_{a,1}$ is the unique solution of the equation

$$E[u_a(H_{a,1} - X)] = 0$$

for a risk X with $E(X) = 1$. Then the zero utility premium $H_{a,\mu}$ for the scaled risk $Y = \mu X$ is in general also uniquely determined, and is given by

$$H_{a,\mu} = \mu H_{h(a,\mu),1}, \quad a, \mu > 0 .$$

Proof. Let X be a risk with $E(X) = 1$ and $Y = \mu X$, $\mu > 0$. Then the premium $H := H_{h(a,\mu),1}$ is, by assumption, uniquely determined by the equation

$$E[u_{h(a,\mu)}(H - X)] = 0 ,$$

hence

$$0 = \mu \cdot 0 = \mu E[u_{h(a,\mu)}(H - X)] = E[\mu u_{h(a,\mu)}(H - X)]$$
$$= E[u_a(\mu \cdot (H - X))] = E[u_a(\mu H - Y)],$$

which implies that $\mu H = \mu H_{h(a,\mu),1}$ is a zero utility premium for Y. Suppose now that $\mu H'$ is another zero utility premium for Y. Then a correspond backwards calculation shows that

$$E[u_{h(a,\mu)}(H' - X)] = 0, \quad a > 0,$$

and hence $H' = H$ by the uniqueness of H.
This proves Theorem 2. □

Theorem 2 says that for the general premium calculation, it suffices to calculate the premiums for risks with unit expectation solely, from which the general solution is obtained essentially via the mapping h which is a kind of inverse for the structural function g of the class of utility functions.
The following result is concerned with the "converse" question, i.e. whether the general premium can be calculated from the general risk, with a particular choice of the utility shape parameter a.

Theorem 3. Suppose that $\mathscr{U} = \{u_a \mid a > 0\}$ is a scale-invariant class of utility functions according to (4), and that the distribution P^X of the risk X belongs to a natural scale family $\mathscr{P}$. Suppose further that the map $h(a, \cdot)$ is surjective for at least one $a = a_0 > 0$, say, and the zero utility premium $H_{a_0,\mu}$ is uniquely determined by the equation

$$E[u_{a_0}(H_{a_0,\mu} - X)] = 0$$

and all risks X with $E(X) = \mu > 0$. Then the premium $H_{a',1}$ is in general also uniquely determined, and is given by

$$H_{a',1} = \frac{g(a_0)}{g(a')} H_{a_0 g(a')/g(a_0)}, \quad a' > 0,$$

where g is as in (5).

Proof. By Theorem 1, we can assume that the utility functions u_a possess a representation according to relation (5). Fix $a_0 > 0$ such that $H_{a_0,\mu}$ is uniquely determined by the relation $E[u_{a_0}(H_{a_0,\mu} - X)] = 0$ for all risks X with $E(X) = \mu > 0$. If we put, in particular, $\mu := g(a')/g(a_0)$ for arbitrary $a' > 0$ and $Y := X/\mu$, then Y is a risk whose distribution belongs to the same scale family $\mathscr{P}$, with $E(Y) = 1$, and

$$E\left[u\left(g(a')\left\{\frac{g(a_0)}{g(a')} H_{a_0,\mu} - Y\right\}\right)\right] = E[u(g(a_0)(H_{a_0,\mu} - X))] = 0$$

which implies that

$$H_{a',1} = \frac{g(a_0)}{g(a')} H_{a_0,\mu} = \frac{g(a_0)}{g(a')} H_{a_0, g(a')/g(a_0)},$$

as requested. The uniqueness of $H_{a',1}$ now follows from the uniqueness of $H_{a_0,\mu}$. □

If we apply Theorems 2 and 3 in the special case of $g(a) = 1/a$ [examples (6) and (8)] or $g(a) = a$, $a > 0$, respectively [example (7)], we obtain

$$H_{a',\mu} = \frac{a'}{a} H_{a,a\mu/a'}, \quad a, a', \mu > 0 \quad \text{if } g(a) = 1/a,\ a > 0, \tag{15}$$

$$H_{a',\mu} = \frac{a}{a'} H_{a,a'\mu/a}, \quad a, a', \mu > 0 \quad \text{if } g(a) = a,\ a > 0. \tag{16}$$

In particular, zero utility premiums can in these cases easily be obtained from the premiums $H_{1,c}$ for $c > 0$, which might be an alternative to the procedure in Theorem 2:

$$H_{a,\mu} = a\, H_{1,\mu/a}, \quad a, \mu > 0 \quad \text{if } g(a) = 1/a, a > 0, \tag{17}$$

$$H_{a,\mu} = \frac{1}{a} H_{1,a\mu}, \quad a, \mu > 0 \quad \text{if } g(a) = a, a > 0, \tag{18}$$

which means that in these cases, the graphs of $H_{a,\bullet}$ differ only in a simultaneous proportional rescaling of the two plot axes, for different values of $a > 0$.

5. Numerical Examples

In this section, we shall present some numerical evaluations of zero utility premiums for the utility classes given by (6), (8) and (9), for risk distributions of exponential- and Pareto type [relations (11) and (13)].

Example 3. [Linear truncated utility]

Recall that the utility functions are of the form $u_a(x) = \min\{x, a\}$ here; $x \in \mathbb{R}$, $a > 0$.

i) Suppose the risk distribution is exponential with unit expectation (cf. (11)). Then we have, for $H \in \mathbb{R}$,

$$E[u_a(H - X)] = \begin{cases} a - e^{-(H-a)}, & H \geq a \\ H - 1, & H \leq a, \end{cases} \tag{19}$$

from which it follows that

$$H_{a,1} = \begin{cases} a - \ln a, & a \leq 1 \\ 1, & a \geq 1 \end{cases} \tag{20}$$

and hence, by Theorem 2 or relation (15),

$$H_{a,\mu} = \mu \cdot H_{a/\mu,1} = \begin{cases} \mu, & 0 \leq \mu \leq a \\ a + \mu \ln\left(\dfrac{\mu}{a}\right), & \mu \geq a. \end{cases} \tag{21}$$

Note that here the zero utility premium is linear for small values of μ, i.e. the zero utility premium coincides with the net risk premium here, and that $u''(0) = 0$ so that a formal application of the variance principle approximation would likewise yield $H_{a,\mu} \approx \mu$.

ii) Suppose the risk distribution is of Pareto type with unit expectation (cf. (13)), with shape parameter $\alpha > 1$. Then we have, for $H \in \mathbb{R}$,

$$E[u_a(H - X)] = \begin{cases} a - \left(1 + \dfrac{H-a}{\alpha}\right)^{-\alpha}, & H \geq a \\ H - 1, & H \leq a, \end{cases} \tag{22}$$

from which it follows that

$$H_{a,1} = \begin{cases} a - \alpha + \dfrac{\alpha}{\sqrt[\alpha]{a}}, & a \leq 1 \\ 1, & a \geq 1 \end{cases} \tag{23}$$

and hence, by Theorem 2 or relation (15),

$$H_{a,\mu} = \mu \cdot H_{a/\mu,1} = \begin{cases} \mu, & 0 \leq \mu \leq a \\ a - \alpha\mu + \alpha\mu \sqrt[\alpha]{\dfrac{\mu}{a}}, & \mu \geq a. \end{cases} \tag{24}$$

Note that in this case, for $\alpha \to \infty$, the Pareto distribution converges weakly to the exponential distribution; likewise converges the zero utility premium of the Pareto distribution to that of the exponential distribution.

Example 4. [Quadratic utility]

Recall that the utility functions are of the form $u_a(x) = \begin{cases} x - \dfrac{x^2}{2a}, & x \leq a \\ \dfrac{a}{2}, & x \geq a \end{cases}$ here; $x \in \mathbb{R}$, $a > 0$.

i) Suppose again that the risk distribution is exponential with unit expectation (cf. (11)). Then we have, for $H \in \mathbb{R}$, after some elaborate calculus,

$$E[u_a(H - X)] = \begin{cases} \dfrac{a^2 - 2e^{-(H-a)}}{2a} & H \geq a \\ \dfrac{(a^2 - 1) - (H - (a+1))^2}{2a}, & H \leq a, \end{cases} \tag{25}$$

from which it follows that

$$H_{a,1} = \begin{cases} a - \ln\left(\dfrac{a^2}{2}\right), & a \leq \sqrt{2} \\ a + 1 - \sqrt{a^2 - 1}, & a \geq 1 \end{cases} \tag{26}$$

and hence, by Theorem 2 or relation (15),

$$H_{a,\mu} = \mu \cdot H_{a/\mu,1} = \begin{cases} a + \mu - \sqrt{a^2 - \mu^2}, & 0 \leq \mu \leq a/\sqrt{2} \\ a + \mu \ln\left(\dfrac{2\mu^2}{a^2}\right), & \mu \geq a/\sqrt{2}. \end{cases} \tag{27}$$

Note that the variance principle approximation would yield $H_{a,\mu} \approx \mu + \dfrac{\mu^2}{2a}$ for small values of μ here since $\varrho = u''(0) = -\dfrac{1}{a}$.

(ii) Suppose again that the risk distribution is of Pareto type with unit expectation (cf. (13)), with shape parameter $\alpha > 1$. Then we have, for $H \in \mathbb{R}$,

$$E[u_a(H-X)] = \begin{cases} \dfrac{a^2 - \dfrac{2\alpha}{\alpha-1}\left(1+\dfrac{H-a}{\alpha}\right)^{-(\alpha-1)}}{2a}, & H \geq a \\ \dfrac{\left(a^2 - \dfrac{\alpha+1}{\alpha-1}\right) - (H-(a+1))^2}{2a}, & H \leq a, \end{cases} \tag{28}$$

from which it follows that

$$H_{a,1} = \begin{cases} a - \alpha + \dfrac{\alpha}{\sqrt[\alpha-1]{\dfrac{\alpha-1}{2\alpha}a^2}}, & a \leq \sqrt{\dfrac{2\alpha}{\alpha-1}} \\ a + 1 - \sqrt{a^2 - \dfrac{\alpha+1}{\alpha-1}}, & a \geq \sqrt{\dfrac{2\alpha}{\alpha-1}} \end{cases} \tag{29}$$

and hence, by Theorem 2 or relation (15),

$$H_{a,\mu} = \mu \cdot H_{a/\mu,1} = \begin{cases} a + \mu - \sqrt{a^2 - \dfrac{\alpha+1}{\alpha-1}\mu^2}, & 0 \leq \mu \leq a\sqrt{\dfrac{\alpha-1}{2\alpha}} \\ a - \alpha\mu + \alpha\mu \sqrt[\alpha-1]{\dfrac{2\alpha\mu^2}{(\alpha-1)a^2}}, & \mu \geq a\sqrt{\dfrac{\alpha-1}{2\alpha}}. \end{cases} \tag{30}$$

Again we see that for $\alpha \to \infty$, the zero utility premium from the Pareto distribution converges to the zero utility premium for the exponential distribution.

Example 5. [Left linearized quadratic utility]

Recall that the utility functions are of the form $u_a(x) = \begin{cases} x, & x \leq 0 \\ x - \dfrac{x^2}{2a}, & 0 \leq x \leq a \\ \dfrac{a}{2}, & x \geq a \end{cases}$ here;

$x \in \mathbb{R}$, $a > 0$. In particular, u_a is not twice differentiable at the origin.
Suppose again that the risk distribution is exponential with unit expectation (cf. (11)). Then we have, for $H \in \mathbb{R}$, similarly as before,

$$E[u_a(H-X)] = \begin{cases} \dfrac{a^2 + 2e^{-H} - 2e^{-(H-a)}}{a}, & H \geq a \\ \dfrac{2e^{-H} + (H-1)(2a-(H-1)) - 1}{2a}, & H \leq a. \end{cases} \tag{31}$$

This equation can explicitly be solved only for $H \geq a$, giving

$$H_{a,1} = \ln\left(\frac{2(e^a-1)}{a^2}\right), \quad a \leq a_0 = 1.1760\ldots \tag{32}$$

and hence, by Theorem 2 or relation (15),

$$H_{a,\mu} = \mu \cdot H_{a/\mu,1} = \mu \ln\left(\frac{2\mu^2(e^{a/\mu}-1)}{a^2}\right), \quad \mu \geq \frac{a}{a_0}. \tag{33}$$

The left branch of the general zero utility premium is the solution of the equation

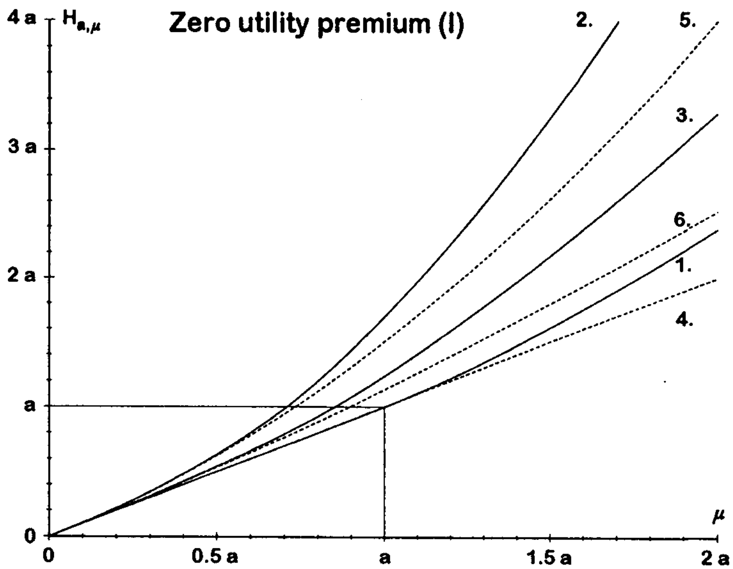
$$2\mu^2 e^{-H/\mu} + (H-\mu)(2a-(H-\mu)) - \mu^2, \quad \text{for } \mu \leq \frac{a}{a_0}. \tag{34}$$

Note that no variance principle approximation in the usual sense is possible here since $u''(0)$ does not exist. However, $u''_+(0) = -\frac{1}{a}$ [right second derivative], $u''_-(0) = 0$ [left second derivative], which implies that $H_{a,\mu}$ should lie between μ and $\mu + \frac{\mu^2}{2a}$ for small values of μ *) (see (37)). However, if we formally replace H by $\mu + c\mu^2$ in (34) for $\mu > 0$ and divide by μ^2, we obtain the equation

$$\frac{2}{e}e^{-c\mu} + 2ac - c^2\mu^2 - 1 = 0. \tag{35}$$

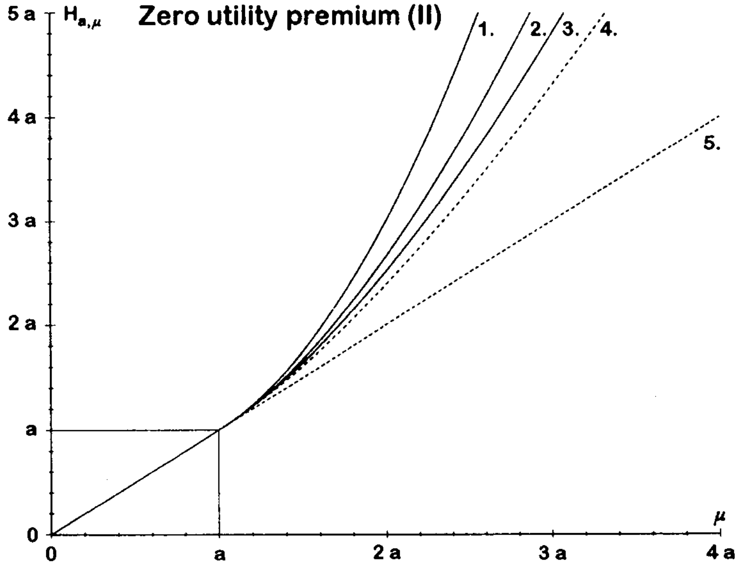
Considering the limit for $\mu \to 0$ in (35), we see that c must take the value

$$c = \frac{e-2}{2ae}. \tag{36}$$



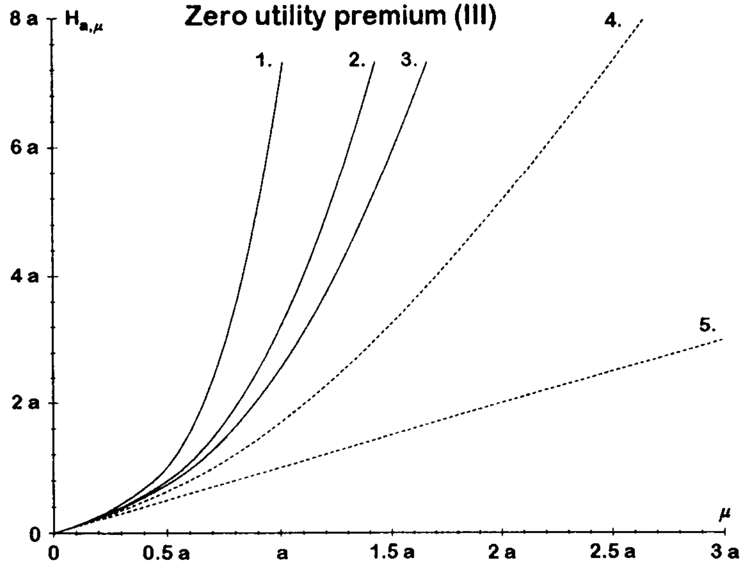
Exponential risk distribution

1.: truncated linear utility, 2.: quadratic utility, 3.: left linearized quadratic utility, 4.: net risk premium, 5.: approximative variance principle (quadratic utility), 6.: approximative variance principle (left linearized quadratic utility)



Pareto and exponential risk distribution; truncated linear utility

Pareto distributions: 1.: limiting case $\alpha \downarrow 1$, 2.: $\alpha = 2$, 3.: $\alpha = 3$, 4.: limiting exponential distribution, 5.: net risk premium



Pareto and exponential risk distribution; quadratic utility

Pareto distributions: 1.: $\alpha = 2$, 2.: $\alpha = 3$, 3.: $\alpha = 4$, 4.: limiting exponential distribution, 5.: net risk premium

The appropriate variance principle approximation for this case is hence given by

$$H_{a,\mu} \approx \mu + \frac{e-2}{2ae}\mu^2 = \mu + 0.264\frac{\mu^2}{2a}, \quad \mu \geq 0, \tag{37}$$

for small values of μ, which shows that the above suggestion *) is correct since the factor $\frac{e-2}{e} = 0.264\ldots$ is between 0 and 1.

The preceding graphs which have been produced by use of the computer algebra system MAPLE show plots of some of the zero utility premiums above. Note that for Example 5, the left branch of the premium was plotted using the implicit plot command of MAPLE, according to (34).

It is clearly seen from the plots above that the approximation with the variance principle is, in general, not very good for larger values of μ, in particular for "dangerous" distributions such as the Pareto risk distribution. Note that in this case, for small values of α, the quadratic zero utility premium increases extremely fast with μ.

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Zusammenfassung

Zum Nullnutzen-Prinzip für Skalenfamilien von Risikoverteilungen

Es wird gezeigt, daß in natürlichen Skalenfamilien von Risikoverteilungen vereinfachte Berechnungs- und Vergleichsmöglichkeiten für die Nullnutzen-Prämien existieren, wenn die Nutzenfunktionen eine Skaleninvarianzeigenschaft besitzen.

Summary

The zero utility principle for scale families of risk distributions

It is shown that in natural scale families of risk distributions simplified calculations and comparisons of zero utility premiums are possible, if the class of utility functions considered is scale invariant.