

Approximation-Theoretic Aspects of Probabilistic Representations for Operator Semigroups

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Communicated by P. L. Butzer

Received May 8, 1984

In the present paper explicit sharp estimations are provided for the rate of convergence of basically all known representation formulae for operator semigroups which in an earlier paper have been shown to arise from a single general probabilistic representation theorem based on a special version of the weak law of large numbers. As a main tool, sharp estimations for moments and moment-generating functions of suitable random variables are used. Some of the results are applied to exponential operators as well as to a class of Poisson approximation theorems in probability theory. © 1985 Academic Press, Inc.

1. INTRODUCTION

In a preceding paper [23] we have shown how basically all known representation formulae for (C_0) -semigroups of operators such as those of Hille and Phillips [12], Kendall [15], Widder and Chung [6], Shaw [27] and others can be subsumed under a single general probabilistic representation theorem derived from a weak law of large numbers for a random number of summands. Conversely, for a very large class of representation theorems, it can be shown that only such probabilistic representations are possible [22]. This emphasizes the importance of probabilistic methods also for an approximation-theoretic analysis in this field. In fact, early approaches to the estimation of rates of convergence for some of these theorems given by Hsu [13] and Ditzian [7-9] use hidden probabilistic arguments such as Markov-type inequalities and moment calculations, while in the recent paper [4] by Butzer and Hahn probability calculus is extensively applied. However, the latter approach deals only with the second modulus of continuity, and, due to its special setting, does not provide the best possible results. In the present paper, we shall therefore establish explicit sharp estimations for the rates of convergence, both generally and individually, involving different kinds of moduli of continuity

and various direct theorems. This is possible by thorough estimations of moments and moment-generating functions of the underlying random variables and a more general Taylor expansion of the semigroup. The probabilistic approach chosen in this paper also permits an improvement of the order of convergence in the spirit of Bernstein [1] and Voronovskaja [28]; their ideas carry over to semigroup theory almost literally. Finally, the results obtained can be applied to classical operators of approximation theory, among them Bernstein polynomials and the operators of Szász, Baskakov, Meyer and König and Zeller and others which are also referred to as exponential operators in May [17], and which in fact share many properties of operator semigroups due to their common probabilistic background (cf. also Ismail and May [14], Ramanujan [25] and Lindvall [16]). To demonstrate the broadness of possible applications, we shall conclude with investigations of the rates of convergence for a class of Poisson approximation theorems in probability theory, being initiated by Le Cam [5] and Serfling [26] (cf. also [20] for a corresponding semigroup setting).

Due to lack of space, proofs will not always be given in full detail. These, however, can easily be completed by Ref. [24], of which the present paper is an extended part.

2. PRELIMINARIES

Although our notation will closely follow [23] it will be necessary to give a short account of basic definitions and theorems from semigroup theory and probability. For further details, we refer to the monographs of Butzer and Berens [3] and Billingsley [2].

In what follows we are concerned with a Banach space \mathcal{X} with norm $\|\cdot\|$ (which will also denote the operator norm) and the Banach algebra $\mathcal{E}(\mathcal{X})$ of bounded endomorphisms on \mathcal{X} . $\mathcal{B}(\mathcal{X})$ denotes the Borel σ -field generated by the strong topology over \mathcal{X} . We consider a (C_0) -semigroup of operators $\{T(t) \mid t \geq 0\} \subseteq \mathcal{E}(\mathcal{X})$, for which finite constants $M \geq 1$ and $\omega \geq 0$ exist such that

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0. \quad (2.1)$$

For the infinitesimal generator A let A^r , $r \geq 1$, denote the r th power of A with domain $D(A^r)$, and $R(\lambda) = \int_0^\infty e^{-\lambda t} T(t) dt$, $\lambda > \omega$, denote the resolvent of the semigroup which is to be understood as an extended Pettis integral (see [23] for definitions and properties).

For a real-valued random variable X defined on some probability space (Ω, \mathcal{A}, P) let $E(X)$ denote its expectation. If $\xi = E(X)$ exists then $\sigma^2 =$

$\sigma^2(X) = E(X - \xi)^2$ is also called the variance of X . Let further $\psi_X(t) = E(t^X)$, $t \geq 0$, and $\psi_X^*(t) = E(e^{tX})$, $t \in \mathbb{R}$, denote the probability generating function and the moment-generating function of X , resp. If $N \geq 0$ is an integer-valued random variable, then also $\psi_N(t) = \sum_{k=0}^{\infty} P(N=k) t^k$, $t \geq 0$ (which explains the name), and if $\{Y_k; k \in \mathbb{N}\}$ is a sequence of independent, identically (as Y , say) distributed random variables, independent of N , then for the random sum $X = \sum_{k=1}^N Y_k$ we have

$$\psi_X^*(t) = \psi_N(\psi_Y^*(t)), \quad t \in \mathbb{R}. \tag{2.2}$$

For convenience, the symbol $E(\cdot)$ will also be used for the extended Pettis expectation introduced in [23]. The following important theorems of probability theory will be of essential interest in the sequel.

THEOREM 2.1 (Jensen's Inequality). *Let X be a real-valued random variable with values in some interval K and with finite expectation ξ . Then $\xi \in K$, and for any convex function, continuous on K , we have*

$$E(g(X)) \geq g(\xi) \tag{2.3}$$

provided $g(X)$ is integrable, i.e., $E(g(X))$ is finite.

THEOREM 2.2 (Markov's Inequality). *Let X be a real-valued random variable with $E(|X|^\alpha) < \infty$ for some $\alpha > 0$. Then for every $x > 0$,*

$$P(|X| > x) \leq x^{-\alpha} E(|X|^\alpha). \tag{2.4}$$

THEOREM 2.3 (Extended Central Limit Theorem). *Let X be a real-valued random variable with $E(X) = 0$ and $\sigma^2 = \sigma^2(X) > 0$. Then for any sequence $\{X_n; n \in \mathbb{N}\}$ of independent, identically (as X) distributed random variables,*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n X_k \xrightarrow{\mathcal{D}} Y \quad (n \rightarrow \infty) \tag{2.5}$$

where Y is a normally distributed random variable with mean 0 and variance σ^2 , and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. If further the moment-generating function ψ_X^ exists in some neighbourhood of the origin, then also*

$$E\left(\left|\frac{1}{\sqrt{n}} \sum_{k=1}^n X_k\right|^\alpha\right) \rightarrow E(|Y|^\alpha) \quad (n \rightarrow \infty). \tag{2.6}$$

While the first part of Theorem 2.3 is well known it should be mentioned that relation (2.6) is a consequence of the uniform integrability of the sequence $\{(1/\sqrt{n}) \sum_{k=1}^n X_k; n \in \mathbb{N}\}$, due to the existence of ψ_X^* (cf. also Billingsley [2]).

3. ESTIMATIONS OF MOMENTS AND GENERATING FUNCTIONS

The usefulness of estimations of the above kind is due to the fact that in probabilistic representation theorems the extended Pettis expectation $E[T(X)]$ plays a central role where X is a suitable non-negative real random variable; in fact, we have

$$\|E[T(X)]\| \leq M\psi_X^*(\omega) \quad (3.1)$$

where M and ω are the characteristic constants of the semigroup according to (2.1) (cf. [23]).

Further, if X is, in some probabilistic sense, close to its expectation ξ , then

$$E[T(X)]f - T(\xi)f \approx \frac{\sigma^2}{2} T(\xi)A^2f \quad (3.2)$$

for $f \in D(A^3)$ as will—among other results—be shown in Section 4. In particular, if $X = (1/n) \sum_{k=1}^n X_k$ is the arithmetic mean of non-trivial, independent and identically distributed (i.i.d.) random variables, relation (3.2) provides a large class of representation theorems due to the classical law of large numbers (see Butzer and Hahn [4] and [19, 21]). In this case, the rate of convergence is exactly $\mathcal{O}(1/n)$ for $n \rightarrow \infty$ since then $\sigma^2 = n^{-2} \sum_{k=1}^n \sigma^2(X_k) = n^{-1} \sigma^2(X_1)$, and the remainder terms are of order $\mathcal{O}(n^{-3/2})$; this is typical for approximation by operators of probabilistic type (see also Hahn [10]). Further, by the Extended Central Limit Theorem 2.3, $E(|(1/n) \sum_{k=1}^n X_k - \xi|^\alpha) = \mathcal{O}(n^{-\alpha/2})$ for $n \rightarrow \infty$ and $\alpha > 0$ which allows for explicit estimations of the remainder terms using moments and generating functions (cf. also Hall [11] and the references given therein for related problems of moment estimations). For a thorough treatise of the latter kind of approximations the following results will be needed.

LEMMA 3.1. *Let X be a non-negative real random variable with $\psi_X^*(\delta) < \infty$ for some $\delta > 0$. Then all positive moments $E(X^\alpha)$ exist, and*

$$E(X^\alpha) \leq \left(\frac{\alpha}{\delta}\right)^\alpha e^{-\alpha} \psi_X^*(\delta) \quad (\alpha > 0). \quad (3.3)$$

Further, for $\xi = E(X)$, we have

$$\psi_X^*(t) = 1 + \xi t + R^*(t), \quad t \leq \delta, \quad (3.4)$$

where

$$0 \leq R^*(t) \leq (2t^2/e^2(\eta - |t|)^2) \psi_X^*(\eta) \quad (3.5)$$

for $|t| < \eta$, and $0 < \eta \leq \delta$.

Proof. For $\alpha, \beta > 0, x \geq 0$ we have

$$x^\alpha \leq (\alpha/\beta)^\alpha e^{-\alpha} e^{\beta x} \tag{3.6}$$

with equality for $x = \alpha/\beta$, from which (3.3) follows by integration, letting $\beta = \delta$. To prove (3.4) and (3.5), note that for all real x ,

$$0 \leq e^x - 1 - x \leq (x^2/2) e^{|x|}, \tag{3.7}$$

hence $0 \leq R^*(t) = \psi_x^*(t) - 1 - \xi t \leq (t^2/2) E(X^2 e^{|t|X})$. The statement now follows from (3.6) by integration with $\alpha = 2, \beta = \eta - |t|$. ■

THEOREM 3.1. *Let X be a non-negative real random variable with expectation $E(X) = \xi$ and with $\psi_x(\delta) < \infty$ for some $\delta > 0$. Let further $\{X_n; n \in \mathbb{N}\}$ be an i.i.d. sequence of random variables, identically distributed with X . Then, if $\bar{X}_n = (1/n) \sum_{k=1}^n X_k$ denotes the arithmetic mean of X_1, \dots, X_n , we have*

$$e^{t\xi} \leq \psi_{\bar{X}_n}^*(t) \leq e^{t\xi} \exp \left\{ \frac{2t^2 \psi_x^*(\eta)}{e^2 n (\eta - |t|/n)^2} \right\} \tag{3.8}$$

for $|t| < n\eta$, and $0 < \eta \leq \delta$. Further,

$$E(|\bar{X}_n - \xi|^\alpha) \leq 2n^{-\alpha/2} (\alpha \psi_x^*(\eta) / e \eta^2)^{\alpha/2} \tag{3.9}$$

for $n \geq 15\alpha / \psi_x^*(\eta)$ ($\alpha > 0$).

Proof. Since X_1, \dots, X_n are i.i.d., we have

$$\begin{aligned} \psi_{\bar{X}_n}^*(t) &= \left\{ \psi_x^* \left(\frac{t}{n} \right) \right\}^n = \left\{ 1 + \frac{t}{n} \xi + R^* \left(\frac{t}{n} \right) \right\}^n \\ &\leq \exp \left\{ t\xi + nR^* \left(\frac{t}{n} \right) \right\}, \quad \left| \frac{t}{n} \right| < \eta, \end{aligned} \tag{3.10}$$

from which the r.h.s. of (3.8) follows by Lemma 3.1. The l.h.s. of (3.8) is a consequence of Jensen's inequality, Theorem 2.1. By the inequalities $e^{|x|} \leq e^x + e^{-x}, x \in \mathbb{R}$, and (3.6), we can deduce from (3.8)

$$E(|\bar{X}_n - \xi|^\alpha) \leq 2n^{-\alpha/2} (\alpha/\beta)^\alpha e^{-\alpha} \exp \left\{ \frac{2\beta^2 \psi_x^*(\eta)}{e^2 (\eta - \beta/\sqrt{n})^2} \right\} \tag{3.11}$$

for $\alpha, \beta > 0, n > (\beta/\eta)^2$. Choosing n at least as large as $15(\beta/\eta)^2$, (3.11) simplifies into

$$E(|\bar{X}_n - \xi|^\alpha) \leq 2n^{-\alpha/2} (\alpha/\beta)^\alpha e^{-\alpha} \exp \left\{ \frac{\beta^2 \psi_x^*(\eta)}{2\eta^2} \right\}. \tag{3.12}$$

But the function $g(\beta) = \beta^{-\alpha} e^{c\beta^2}$ with some positive constant c is minimal for $\beta = \sqrt{\alpha/2c}$. Letting $c = \psi_{X_n}^*(\eta)/2\eta^2$, this gives (3.9). ■

Note that the inequalities (3.8) and (3.9) are sharp in that they imply

$$\psi_{X_n}^*(t) = e^{t\xi} + \mathcal{O}(t^2) \quad (t \rightarrow 0, n \text{ fixed}) \quad (3.13)$$

and

$$\psi_{X_n}^*(t) = e^{t\xi} + \mathcal{O}\left(\frac{1}{n}\right) \quad (n \rightarrow \infty, t \text{ fixed}) \quad (3.14)$$

which are in fact the exact orders of convergence, and that by the Extended Central Limit Theorem, $\mathcal{O}(n^{-\alpha/2})$ is the exact order of convergence for the central mean of order α .

It should be pointed out that the estimations given by (3.8) and (3.9) are especially well suited for our purposes (compared with classical ones such as in Hall [11]) since they essentially use the moment-generating function of the underlying random variable X which in the most interesting cases (see [19, 21, 22]) is closely connected with the semigroup representation via (2.2). It will thus be possible to give estimations of the rate of convergence mainly by means of the different "representation functions" used in the representation theorems themselves, avoiding the probabilistic terminology which is necessary for the proof of the corresponding results.

4. SOME DIRECT PROBABILISTIC APPROXIMATION THEOREMS

In this section we will establish a collection of various direct theorems connected with relation (3.2) and others, both for several general cases as well as for the individual representation theorems mentioned in the Introduction. In contrast to the recent paper by Butzer and Hahn [4], one of the main tools here will be the general Taylor expansion of the semigroup as stated in the following result.

LEMMA 4.1. For $f \in D(A^r)$, $r \geq 1$, and arbitrary $s, t \geq 0$ we have

$$T(t)f - T(s)f = \sum_{k=1}^{r-1} \frac{(t-s)^k}{k!} T(s) A^k f + \int_s^t \frac{(t-u)^{r-1}}{(r-1)!} T(u) A^r f du \quad (4.1)$$

where for $s > t$

$$\int_s^t \frac{(t-u)^{r-1}}{(r-1)!} T(u) A^r f du = - \int_t^s \frac{(t-u)^{r-1}}{(r-1)!} T(u) A^r f du \quad (4.2)$$

by definition.

Proof. Apply linear functionals and proceed by induction, using partial integration. ■

THEOREM 4.1. *Let X be a non-negative real-valued random variable with $\psi_X^*(\delta) < \infty$ for some $\delta > 0$. Let further $\xi = E(X)$ denote the expectation of X and $\sigma^2 = \sigma^2(X)$ its variance. Then the following relations hold:*

$$\begin{aligned} & \|E[T(X)] f - T(\xi) f\| \\ & \leq M \|Af\| \{e^{\omega\xi} E(|X - \xi|) + \omega \{E(X - \xi)^4\}^{1/2} \{\psi_X^*(2\omega)\}^{1/2}\} \\ & \leq M \|Af\| \{\sigma e^{\omega\xi} + \omega \{E(X - \xi)^4\}^{1/2} \{\psi_X^*(2\omega)\}^{1/2}\} \\ & \text{for } f \in D(A), \text{ if } \delta \geq 2\omega; \end{aligned} \tag{4.3}$$

$$\begin{aligned} & \|E[T(X)] f - T(\xi) f\| \\ & \leq \frac{M}{2} \|A^2 f\| \{\sigma^2 e^{\omega\xi} + \omega \{E(X - \xi)^4\}^{3/4} \{\psi_X^*(4\omega)\}^{1/4}\} \\ & \text{for } f \in D(A^2), \text{ if } \delta \geq 4\omega; \end{aligned} \tag{4.4}$$

$$\begin{aligned} & \left\| E[T(X)] f - T(\xi) f - \frac{\sigma^2}{2} T(\xi) A^2 f \right\| \\ & \leq \frac{M}{6} \|A^3 f\| \{e^{\omega\xi} E(|X - \xi|^3) + \omega \{E(X - \xi)^6\}^{2/3} \{\psi_X^*(3\omega)\}^{1/3}\} \\ & \text{for } f \in D(A^3), \text{ if } \delta \geq 3\omega. \end{aligned} \tag{4.5}$$

Here M and ω denote the characteristic constants of the semigroup given by (2.1).

Proof. By Lemma 4.1, for $f \in D(A')$,

$$\int_s^t \frac{(t-u)^{r-1}}{(r-1)!} T(u) A' f \, du$$

is a strongly continuous function of the variables s and t , hence also $\mathcal{B}(\mathcal{X})$ -measurable, implying

$$E[T(X)] f - T(\xi) f = E \left[\int_\xi^X T(u) A f \, du \right], \quad f \in D(A). \tag{4.6}$$

Let 1_B denote the indicator function for the event $B \in \mathcal{A}$, i.e., $1_B(x) = 1$ iff $x \in B$, and 0 otherwise. Then

$$\begin{aligned}
& \left\| E \left[\int_{\xi}^X T(u) Af du \right] \right\| \\
&= \left\| E \left[1_{\{X > \xi\}} \int_{\xi}^X T(u) Af du - 1_{\{X \leq \xi\}} \int_X^{\xi} T(u) Af du \right] \right\| \\
&\leq E \left[1_{\{X > \xi\}} \left\| \int_{\xi}^X T(u) Af du \right\| + 1_{\{X \leq \xi\}} \left\| \int_X^{\xi} T(u) Af du \right\| \right] \\
&\leq M \|Af\| E \left[1_{\{X > \xi\}} \int_{\xi}^X e^{\omega u} du + 1_{\{X \leq \xi\}} \int_X^{\xi} e^{\omega u} du \right] \\
&\leq M \|Af\| E [1_{\{X > \xi\}} (X - \xi) e^{\omega X} + 1_{\{X \leq \xi\}} (\xi - X) e^{\omega \xi}] \\
&= M \|Af\| E [|X - \xi| e^{\omega \xi} + 1_{\{X > \xi\}} (X - \xi) (e^{\omega X} - e^{\omega \xi})] \\
&\leq M \|Af\| E [|X - \xi| e^{\omega \xi} + \omega 1_{\{X > \xi\}} (X - \xi)^2 e^{\omega X}] \\
&\leq M \|Af\| E [|X - \xi| e^{\omega \xi} + \omega (X - \xi)^2 e^{\omega X}] \\
&\leq M \|Af\| \{ e^{\omega \xi} E(|X - \xi|) + \omega \{ E(X - \xi)^4 \}^{1/2} \{ E(e^{2\omega X}) \}^{1/2} \} \quad (4.7)
\end{aligned}$$

by Hölder's inequality. This proves (4.3), since by Jensen's inequality, Theorem 2.1, $\{E(|X - \xi|)\}^2 \leq E(X - \xi)^2 = \sigma^2$. Relations (4.4) and (4.5) follow similarly, using Lemma 4.1 with $r=2$ and $r=3$, resp. ■

Of course, other conjugate indices for Hölder's inequality could also be used in the preceding proof; however, the choice made above turns out to be of special importance since the fourth and sixth central moments are more easy to calculate than moments of non-integer order.

It should be pointed out that for equi-bounded semigroups, i.e., $\omega=0$ (which cover also contraction semigroups), the existence of the moment-generating function ψ_X^* is not necessary; instead, only the existence of the second and third moment of X , resp., has to be imposed.

As a first important consequence of Theorem 4.1, we shall for simplicity reformulate the general probabilistic representation theorem for equi-bounded operator semigroups, endowed with estimations for the rate of convergence (cf. [23, Theorem 2]).

THEOREM 4.2. *Let $\{N(\tau) \mid \tau > 0\}$ be a family of non-negative integer-valued random variables with $E(N(\tau)) = \tau\zeta$ for some $\zeta > 0$ such that $\sigma^2(N(\tau)) = o(\tau^2)$ for $\tau \rightarrow \infty$. Then $(1/\tau)N(\tau)$ converges in probability to ζ for $\tau \rightarrow \infty$. Let further Y be a non-negative random variable with $E(Y) = \gamma$ such that $\sigma^2(Y)$ exists. Then for any equi-bounded operator semigroup, we have*

$$T(\xi) = \lim_{\tau \rightarrow \infty} \psi_{N(\tau)} \left(E \left[T \left(\frac{Y}{\tau} \right) \right] \right) \quad (4.8)$$

in the strong sense where $\xi = \zeta\gamma$. Moreover, for $f \in D(A)$, we have

$$\left\| T(\xi)f - \psi_{N(\tau)} \left(E \left[T \left(\frac{Y}{\tau} \right) \right] \right) f \right\| \leq M \|Af\| \sqrt{\frac{\xi}{\tau} \sigma^2(Y) + \frac{\gamma^2}{\tau^2} \sigma^2(N(\tau))} \quad (4.9)$$

while for $f \in D(A^2)$,

$$\begin{aligned} & \left\| T(\xi)f - \psi_{N(\tau)} \left(E \left[T \left(\frac{Y}{\tau} \right) \right] \right) f \right\| \\ & \leq \frac{M}{2} \|A^2f\| \left\{ \frac{\xi}{\tau} \sigma^2(Y) + \frac{\gamma^2}{\tau^2} \sigma^2(N(\tau)) \right\}, \quad \tau > 0. \end{aligned} \quad (4.10)$$

Proof. Obvious from a modified version of the general Theorem 2 in [23], Theorem 4.1 and Lemma 3 of Chung [6]. ■

The most important specialisations of (4.8) are given by $Y \equiv \gamma$, leading to so-called first main theorems, and Y being exponentially distributed with mean γ , leading to so-called second main theorems since then $E[T(Y/\tau)] = (\tau/\gamma) R(\tau/\gamma)$.

Choosing $\{N(\tau) \mid \tau > 0\}$ as a Poisson process in Theorem 4.2, Hille and Phillips' representation theorems are reobtained, with rates of convergence even for the general case of arbitrary operator semigroups (cf. Corollary 6 in [23]).

THEOREM 4.3. *Let $\{T(t) \mid t \geq 0\}$ be an arbitrary strongly continuous operator semigroup, and let $A_h = (1/h)(T(h) - I)$ for $h > 0$. Then*

$$\begin{aligned} & \|T(\xi)f - e^{\xi A_{1/\tau}}f\| \\ & \leq M e^{\omega\xi} \|Af\| \left\{ \sqrt{\frac{\xi}{\tau}} + 2\omega \frac{\xi}{\tau} \exp \left\{ \frac{\omega^2 \xi}{\tau} e^{2\omega/\tau} \right\} \right\} \\ & \text{for } f \in D(A) \text{ and } \tau \geq \frac{1}{\xi}; \end{aligned} \quad (4.11)$$

$$\begin{aligned} & \|T(\xi)f - e^{\xi A_{1/\tau}}f\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2f\| \left\{ \frac{\xi}{\tau} + 2\omega \sqrt{\frac{2\xi^3}{\tau^3}} \exp \left\{ \frac{2\omega^2 \xi}{\tau} e^{4\omega/\tau} \right\} \right\} \\ & \text{for } f \in D(A^2) \text{ and } \tau \geq \frac{1}{\xi}; \end{aligned} \quad (4.12)$$

$$\begin{aligned} & T(\xi)f - e^{\xi A_{1/\tau}}f = -\frac{\xi}{2\tau} T(\xi) A^2f + \mathcal{O} \left(\frac{1}{\tau \sqrt{\tau}} \right) \\ & \text{for } f \in D(A^3) \text{ and } \tau \rightarrow \infty, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} & \|T(\xi) f - \exp\{\xi\tau^2 R(\tau) - \xi\tau I\} f\| \\ & \leq M e^{\omega\xi} \|Af\| \left\{ \sqrt{\frac{2\xi}{\tau}} + 4\omega \frac{e^\xi}{\tau} \exp\left\{\frac{2\omega^2\xi}{\tau - 2\omega}\right\} \right\} \\ & \quad \text{for } f \in D(A) \text{ and } \tau > \max(4; 2\omega); \end{aligned} \quad (4.14)$$

$$\begin{aligned} & \|T(\xi) f - \exp\{\xi\tau^2 R(\tau) - \xi\tau I\} f\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left\{ \frac{2\xi}{\tau} + 6\omega \frac{e^\xi}{\tau\sqrt{\tau}} \exp\left\{\frac{4\omega^2\xi}{\tau - 4\omega}\right\} \right\} \\ & \quad \text{for } f \in D(A^2) \text{ and } \tau > \max(16; 4\omega); \end{aligned} \quad (4.15)$$

$$\begin{aligned} T(\xi) f - \exp\{\xi\tau^2 R(\tau) - \xi\tau I\} f &= -\frac{\xi}{\tau} T(\xi) A^2 f + \mathcal{O}\left(\frac{1}{\tau\sqrt{\tau}}\right) \\ & \quad \text{for } f \in D(A^3) \text{ and } \tau \rightarrow \infty. \end{aligned} \quad (4.16)$$

Moreover, a possible estimation of the remainder terms $R_1(f)$ and $R_2(f)$, say, in (4.13) and (4.16), resp. is given by

$$\begin{aligned} \|R_1(f)\| & \leq \frac{M}{6} e^{\omega\xi} \|A^3 f\| \left\{ \frac{3}{\tau\sqrt{\tau}} \exp\left(\frac{\xi}{2} e^{1/\sqrt{\tau}}\right) \right. \\ & \quad \left. + \frac{38\omega}{\tau^2} \exp\left(\frac{\xi}{3} e^{1/\sqrt{\tau}} + \frac{3\omega^2\xi}{2\tau} e^{3\omega/\tau}\right) \right\}, \quad \tau > 0, \end{aligned} \quad (4.17)$$

$$\begin{aligned} \|R_2(f)\| & \leq \frac{M}{6} e^{\omega\xi} \|A^3 f\| \left\{ \frac{3}{\tau\sqrt{\tau}} e^{2\xi/3} + \frac{38\omega}{\tau^2} e^\xi \exp\left(\frac{3\omega^2\xi}{\tau - 3\omega}\right) \right\} \\ & \quad \text{for } \tau > \max(9; 3\omega). \end{aligned} \quad (4.18)$$

Proof. Let $\{N(\tau) \mid \tau > 0\}$ be a Poisson process with parameter ξ , i.e., $E(N(1)) = \xi$, and let $\{Y_n \mid n \in \mathbb{N}\}$ be an i.i.d. sequence of random variables with unit mean, independent of the Poisson process, distributed as $Y \geq 0$, say. Define

$$X(\tau) = \frac{1}{\tau} \sum_{k=1}^{N(\tau)} Y_k, \quad \tau > 0. \quad (4.19)$$

In order to prove (4.11) to (4.13) and (4.17), choose $Y \equiv 1$. Then $E(X(\tau)) = \xi$, $\sigma^2(X(\tau)) = \xi/\tau$, and

$$E((X(\tau) - \xi)^4) = 3 \frac{\xi^2}{\tau^2} + \frac{\xi}{\tau^3} \leq 4 \frac{\xi^2}{\tau^2} \quad \text{for } \tau \geq \frac{1}{\xi}, \quad (4.20)$$

$$\psi_{X(\tau)}^*(t) = \exp\{\xi\tau(e^{t/\tau} - 1)\}, \quad t \geq 0. \quad (4.21)$$

Further, for $\alpha > 0$,

$$\begin{aligned} E(|X(\tau) - \xi|^\alpha) &\leq 2\tau^{-\alpha/2}(\alpha/e)^\alpha e^{-\sqrt{\tau}\xi} \psi_{X(\tau)}^*(\sqrt{\tau}) \\ &\leq 2\tau^{-\alpha/2}(\alpha/e)^\alpha \exp\left(\frac{\xi}{2} e^{1/\sqrt{\tau}}\right), \quad \tau > 0, \end{aligned} \tag{4.22}$$

which can be proven by methods parallel to those of the proof of Theorem 3.1. Combining (4.20) to (4.22) with Theorem 4.1, the first part of the theorem follows. For the remainder part, let Y be exponentially distributed. Then again $E(X(\tau)) = \xi$, $\sigma^2(X(\tau)) = 2\xi/\tau$, and for $\alpha > 0$,

$$E(|X(\tau) - \xi|^\alpha) \leq 2\tau^{-\alpha/2}(\alpha/e)^\alpha \exp\left(\frac{\xi\tau}{\tau - \sqrt{\tau}}\right), \quad \tau > 1. \tag{4.23}$$

Also,

$$\psi_{X(\tau)}^*(t) = \exp\left(\frac{\xi\tau t}{\tau - t}\right), \quad \tau > t \geq 0, \tag{4.24}$$

since $\psi_{X(\tau)}^*(t) = \psi_{N(\tau)}(\psi_Y^*(t/\tau))$ and

$$\psi_Y^*(t) = \frac{1}{1-t}, \quad 0 \leq t < 1. \tag{4.25}$$

Using again Theorem 4.1 and the monotonicity of the mapping $\tau \rightarrow \tau/(\tau - \sqrt{\tau})$, $\tau > 1$, the proof is completed. ■

It is no surprise that the rate of convergence in (4.13) and (4.16) is exactly $\mathcal{O}(1/\tau)$ with remainder terms of order $\mathcal{O}(\tau^{-3/2})$ since the Poisson process is an independent increments process, hence in a certain sense behaves like a sum of i.i.d. random variables (cf. also (3.2) and the subsequent remarks). A corresponding direct theorem for this important case is given in the following result (cf. also Corollary 4 in [23], and [21]).

THEOREM 4.4. *Let N be a non-negative integer-valued random variable with $E(N) = \zeta$, and $Y \geq 0$ be a real-valued random variable with $E(Y) = \gamma$ such that $\psi_N(\psi_Y^*(\delta)) < \infty$ for some $\delta > 0$. Then for an arbitrary strongly continuous operator semigroup,*

$$T(\xi) = \lim_{n \rightarrow \infty} \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n \tag{4.26}$$

in the strong sense where $\xi = \zeta\gamma$. Moreover, for $f \in D(A)$,

$$\begin{aligned} & \left\| T(\xi) f - \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n f \right\| \\ & \leq M e^{\omega\xi} \|A f\| \left\{ \frac{1}{\sqrt{n}} \sqrt{\zeta\sigma^2(Y) + \gamma^2\sigma^2(N)} \right. \\ & \quad \left. + \frac{4\omega}{\eta^2 n} \psi_N(\psi_Y^*(\eta)) \exp \left\{ \frac{\omega^2 \psi_N(\psi_Y^*(\eta))}{n(\eta - 2\omega/n)^2} \right\} \right\} \\ & \text{for } n > \max \left(\frac{2\omega}{\eta}, \frac{60}{\psi_N(\psi_Y^*(\eta))} \right), \text{ and } 0 < \eta \leq \delta, \quad (4.27) \end{aligned}$$

and for $f \in D(A^2)$,

$$\begin{aligned} & \left\| T(\xi) f - \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n f \right\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left\{ \frac{1}{n} (\zeta\sigma^2(Y) + \gamma^2\sigma^2(N)) \right. \\ & \quad \left. + \frac{4\omega}{\eta^3 n \sqrt{n}} \{ \psi_N(\psi_Y^*(\eta)) \}^{3/2} \exp \left\{ \frac{2\omega^2 \psi_N(\psi_Y^*(\eta))}{n(\eta - 4\omega/n)^2} \right\} \right\} \\ & \text{for } n > \max \left(\frac{4\omega}{\eta}, \frac{60}{\psi_N(\psi_Y^*(\eta))} \right), \text{ and } 0 < \eta \leq \delta, \quad (4.28) \end{aligned}$$

while for $f \in D(A^3)$,

$$\begin{aligned} & T(\xi) f - \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n f \\ & = -\frac{1}{2n} (\zeta\sigma^2(Y) + \gamma^2\sigma^2(N)) T(\xi) A^2 f + \mathcal{O} \left(\frac{1}{n\sqrt{n}} \right) \quad (n \rightarrow \infty) \end{aligned} \quad (4.29)$$

a possible estimation for the remainder term $R(f)$, say, being

$$\begin{aligned} \|R(f)\| & \leq \frac{M}{6} e^{\omega\xi} \|A^3 f\| \left\{ \frac{3}{n\sqrt{n}} \frac{1}{\eta^3} \{ \psi_N(\psi_Y^*(\eta)) \}^{3/2} \right. \\ & \quad \left. + \frac{8\omega}{\eta^4 n^2} \{ \psi_N(\psi_Y^*(\eta)) \}^2 \exp \left\{ \frac{\omega^2 \psi_N(\psi_Y^*(\eta))}{n(\eta - 3\omega/n)^2} \right\} \right\} \\ & \text{for } n > \max \left(\frac{3\omega}{\eta}, \frac{45}{\psi_N(\psi_Y^*(\eta))} \right), \text{ and } 0 < \eta \leq \delta. \quad (4.30) \end{aligned}$$

Proof. Let $\{Y_n | n \in \mathbb{N}\}$ be an i.i.d. sequence distributed as Y , independent of N , and let

$$X = \sum_{k=1}^N Y_k. \tag{4.31}$$

Let further $\{X_n | n \in \mathbb{N}\}$ be another i.i.d. sequence, distributed as X . The assertions now follow from (2.2), Theorem 3.1 and Theorem 4.1, where in the latter X is to be replaced by \bar{X}_n . ■

Again first and second main theorems are reobtained from Theorem 4.4 if $Y \equiv \gamma$ or Y is exponentially distributed with mean γ ; in the first case, $\psi_Y^*(t) = e^{\gamma t}$, while in the second case, $\psi_Y^*(t) = 1/(1 - \gamma t)$, $t < 1/\gamma$.

Specialising the distribution of N as binomial, geometric, etc., the individual representation theorems mentioned in the Introduction now are also reobtained from Theorem 4.4, all having a rate of convergence of $\mathcal{O}(1/n)$ for $f \in D(A^2)$ with a remainder term of order $\mathcal{O}(1/n\sqrt{n})$ for $f \in D(A^3)$ ($n \rightarrow \infty$). Of course, using the individual moments and generating functions here, improved estimations compared with those of Theorem 4.4 are possible. To shorten matters, we will present some results of this kind for $f \in D(A^2)$, the most important case also for estimations using the second modulus of continuity; for details, see [24].

COROLLARY 4.1. For $f \in D(A^2)$, and $\xi > 0$, we have

$$\begin{aligned} & \left\| T(\xi) f - \left\{ (1 - \xi) I + \xi T\left(\frac{1}{n}\right) \right\}^n f \right\| \\ & \leq \frac{M}{2} e^{\omega \xi} \|A^2 f\| \left[\frac{\xi(1 - \xi)}{n} + 2\omega \sqrt{\frac{2\xi^3(1 - \xi)^3}{n^3}} \exp \left\{ \frac{2\omega^2 \xi}{n} e^{4\omega/n} \right\} \right] \\ & \text{for } n \geq \frac{1}{\xi(1 - \xi)} - 6 \text{ and } 0 < \xi < 1 \text{ (Kendall),} \end{aligned} \tag{4.32}$$

$$\begin{aligned} & \left\| T(\xi) f - \{ (1 - \xi) I + \xi n R(n) \}^n f \right\| \\ & \leq \frac{M}{2} e^{\omega \xi} \|A^2 f\| \left[\frac{\xi(2 - \xi)}{n} + 25\omega \sqrt{\frac{(1 + \xi)^3}{n^3}} \exp \left\{ \frac{4\omega^2 \xi}{n - 4\omega} \right\} \right] \\ & \text{for } n > \max \left(4\omega, \frac{60}{1 + \xi} \right) \text{ and } 0 < \xi < 1 \text{ (Chung),} \end{aligned} \tag{4.33}$$

$$\begin{aligned} & \left\| T(\xi) f - \left\{ (1 + \xi) I - \xi T\left(\frac{1}{n}\right) \right\}^{-n} f \right\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left[\frac{\xi(1 + \xi)}{n} + 2\omega \sqrt{\frac{2\xi^3(1 + \xi)^3}{n^3}} \exp \left\{ \frac{4\omega^2\xi(1 + \xi) e^{4\omega/n}}{n - 4\omega\xi e^{4\omega/n}} \right\} \right] \\ & \quad \text{for } n > \max \left(4\omega(1 + \xi), \frac{1}{\xi(1 + \xi)} + 6 \right) \text{ (Shaw),} \end{aligned} \quad (4.34)$$

$$\begin{aligned} & \left\| T(\xi) f - \{ (1 + \xi) I - \xi n R(n) \}^{-n} f \right\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left[\frac{\xi(2 + \xi)}{n} + 4(2 + \xi)^3 \omega \sqrt{\frac{(1 + \xi)^3}{n^3}} \exp \left\{ \frac{4\omega^2\xi(1 + \xi)}{n - 4(1 + \xi)\omega} \right\} \right] \\ & \quad \text{for } n > \max \left(4\omega(1 + \xi), \frac{60}{1 + \xi} \right) \text{ (Chung),} \end{aligned} \quad (4.35)$$

$$\begin{aligned} & \left\| T(\xi) f - \left\{ 2I - T\left(\frac{\xi}{n}\right) \right\}^{-n} f \right\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left[\frac{2\xi^2}{n} + \frac{8\omega\xi^3}{n\sqrt{n}} \exp \left\{ \frac{8\omega^2\xi^2 e^{4\omega\xi/n}}{n - 4\omega\xi e^{4\omega\xi/n}} \right\} \right] \\ & \quad \text{for } n > \max(7, 6\omega\xi) \text{ (Shaw),} \end{aligned} \quad (4.36)$$

$$\begin{aligned} & \left\| T(\xi) f - \left\{ \frac{n}{\xi} R\left(\frac{n}{\xi}\right) \right\}^n f \right\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left[\frac{\xi^2}{n} + \frac{3\sqrt{3}\xi^3\omega}{n\sqrt{n}} \exp \left\{ \frac{4\omega^2\xi^2}{n - 4\omega\xi} \right\} \right] \\ & \quad \text{for } n > 4\omega\xi \text{ (Post and Widder),} \end{aligned} \quad (4.37)$$

$$\begin{aligned} & \left\| T(\xi) f - \left\{ \frac{n^\xi}{\Gamma(\xi)} \int_0^\infty t^{\xi-1} e^{-nt} T(t)(\cdot) dt \right\}^n f \right\| \\ & \leq \frac{M}{2} e^{\omega\xi} \|A^2 f\| \left[\frac{\xi^2}{n} + 3\sqrt{3}\omega \sqrt{\frac{\xi^3}{n^3}} \exp \left\{ \frac{4\omega^2\xi}{n - 4\omega} \right\} \right] \\ & \quad \text{for } n > \max \left(4\omega, \frac{1}{\xi} \right) \text{ (Butzer and Hahn).} \end{aligned} \quad (4.38)$$

5. OTHER PROBABILISTIC APPROXIMATION THEOREMS

In this section, we shall present some rate of convergence results involving the first and second modulus of continuity. In particular, let

$$\omega_1^*(\delta, t, f) := \sup_{\substack{|s-t| \leq \delta \\ s \geq 0}} \|T(t)f - T(s)f\|, \quad \delta > 0, t \geq 0, f \in \mathcal{X}, \quad (5.1)$$

denote the first modulus of continuity, and

$$\omega_1^b(\delta, f) := \sup\{\|T(t) - T(s)f\| \mid 0 \leq s, t \leq b, |s - t| \leq \delta\}, \\ b > 0, \delta > 0, f \in \mathcal{X}, \quad (5.2)$$

denote the rectified modulus of continuity in $[0, b]$. In this section we will only deal with (5.1) since by the inequality

$$\omega_1^b(\delta, f) \geq \sup_{0 \leq t \leq b - \delta} \omega_1^*(\delta, t, f) \quad (5.3)$$

(cf. Ditzian [7]) corresponding results can be formulated also in terms of the rectified modulus of continuity. Finally, let

$$\omega_2(\xi, f) = \sup_{0 \leq t \leq \xi} \|(T(t) - I)^2 f\|, \quad \xi > 0, f \in \mathcal{X}, \quad (5.4)$$

denote the second modulus of continuity. For simplicity, general results will be given only in the setting of Theorem 4.4 which covers all interesting cases except for the continuous versions of Hille and Phillips' theorems (Theorem 4.3), for which corresponding results will be stated separately. However, for contraction semigroups a general result in the setting of Theorem 4.2 involving the second modulus of continuity will be available, improving corresponding results of Butzer and Hahn [4].

THEOREM 5.1. *Let X be a non-negative real-valued random variable with $\psi_X^*(\delta) < \infty$ for some $\delta > 2\omega$. Further denote $\xi = E(X)$. Then for arbitrary $\varepsilon > 0, t \leq \delta, f \in \mathcal{X}$ there holds the inequality*

$$\|E[T(X)]f - T(\xi)f\| \leq \omega_1^*(\varepsilon, \xi, f) + e^{-\varepsilon t} M \|f\| \{e^{-2t\xi} \psi_X^*(2t) + e^{2t\xi} \psi_X^*(-2t)\}^{1/2} \{\sqrt{\psi_X^*(2\omega)} + e^{\omega\xi}\}. \quad (5.5)$$

Proof. We have

$$\begin{aligned} & \|E[T(X)]f - T(\xi)f\| \\ & \leq \int_{|X-\xi| \leq \varepsilon} \|T(X)f - T(\xi)f\| dP + \int_{|X-\xi| > \varepsilon} \|T(X)f - T(\xi)f\| dP \\ & \leq \omega_1^*(\varepsilon, \xi, f) + \{P(|X-\xi| > \varepsilon)\}^{1/2} \{E(\|T(X)f - T(\xi)f\|^2)\}^{1/2} \\ & \leq \omega_1^*(\varepsilon, \xi, f) + \{P(e^{2t|X-\xi|} > e^{2\varepsilon t})\}^{1/2} \{\sqrt{E(\|T(X)f\|^2)} + \|T(\xi)f\|\} \\ & \leq \omega_1^*(\varepsilon, \xi, f) + e^{-\varepsilon t} \sqrt{E[\exp(2t|X-\xi|)]} \{M \|f\| \sqrt{\psi_X^*(2\omega)} \\ & \quad + M \|f\| e^{\omega\xi}\} \end{aligned}$$

by Hölder’s inequality and Markov’s inequality (Theorem 2.2), applied to the random variables $\exp(2t|X-\xi|)$. Relation (5.5) now follows by means of the inequality $e^{|x|} \leq e^x + e^{-x}$, $x \in \mathbb{R}$. ■

Theorem 5.1 provides an improved estimation of a formula due to Chung [6, Lemma 1].

The following theorem will give the analogue of the general Theorem 4.4 in terms of the first modulus of continuity, from which the individual estimations can be derived by specialisation as in Section 4.

THEOREM 5.2. *Let N be a non-negative integer-valued random variable with $E(N) = \zeta$, and $Y \geq 0$ be a real-valued random variable with $E(Y) = \gamma$ such that $\psi_N(\psi_Y^*(\delta)) < \infty$ for some $\delta > 0$. Then with $\xi = \zeta\gamma$, for $\varepsilon > 0$, $f \in \mathcal{X}$,*

$$\begin{aligned} & \left\| T(\xi)f - \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n f \right\| \\ & \leq \omega_1^*(\varepsilon, \xi, f) + M e^{\omega\xi} \|f\| e^{-\varepsilon\sqrt{n}} \sqrt{2} \exp \left(\frac{\psi_N(\psi_Y^*(\eta))}{(\eta - 2/\sqrt{n})^2} \right) \\ & \quad \times \left\{ 1 + \exp \left(\frac{\omega^2 \psi_N(\psi_Y^*(\eta))}{n(\eta - 2\omega/\sqrt{n})^2} \right) \right\} \\ & \quad \text{for } n > \max \left(\frac{4}{\eta^2}, \frac{2\omega}{\eta} \right) \text{ and } 0 < \eta \leq \delta. \end{aligned} \tag{5.6}$$

Proof. By imitation of the proof of Theorem 4.4 using Theorem 5.1 with $t = \sqrt{n}$. ■

It should be pointed out that the exponential convergence in (5.6) is essentially due to the existence of the moment-generating function, which implies exponential convergence of the deviation probabilities

$P(|\bar{X}_n - \xi| > \varepsilon)$ to zero (cf. also Petrov [18]). Of course, again the choice of Hölder's and other constants in (5.6) and (5.5), resp., is free to a certain degree, hence also other explicit estimations could easily be formulated.

We shall now state a corresponding result for the continuous versions of Hille and Phillips' Theorem 4.3.

THEOREM 5.3. *With the notation of Theorem 4.3, we have for all $f \in \mathcal{X}$, $0 < \gamma < \frac{1}{2}$,*

$$\begin{aligned} & \|T(\xi) f - e^{\xi A_{1/t}} f\| \\ & \leq \omega_1^*(\tau^{-\gamma}, \xi, f) \\ & \quad + e^{-\tau^{(1/2-\gamma)}} M e^{\omega \xi} \|f\| \sqrt{2} \exp(\xi e^{2/\sqrt{\tau}}) \left\{ 1 + \exp\left(\frac{\omega^2 \xi}{\tau} e^{2\omega/\tau}\right) \right\}, \\ & \hspace{20em} \tau > 0, \end{aligned} \tag{5.7}$$

$$\begin{aligned} & \|T(\xi) f - \exp\{\xi \tau^2 R(\tau) - \xi \tau I\} f\| \\ & \leq \omega_1^*(\tau^{-\gamma}, \xi, f) + e^{-\tau^{(1/2-\gamma)}} M e^{\omega \xi} \|f\| \sqrt{2} \exp\left(\frac{2\xi \sqrt{\tau}}{\sqrt{\tau} - 2}\right) \\ & \quad \times \left\{ 1 + \exp\left(\frac{2\omega^2 \xi}{\tau - 2\omega}\right) \right\}, \quad \tau > \max(4, 2\omega). \end{aligned} \tag{5.8}$$

Proof. By imitation of the proof of Theorem 4.3 using Theorem 5.1 with $\varepsilon = \tau^{-\gamma}$ and $t = \sqrt{\tau}$. ■

In [7-9] Ditzian has stated similar rates of convergence for (5.7), (5.8) and Widder's inversion formula (Theorem 5.2 with $N \equiv 1$, Y being exponentially distributed with mean ξ), including the case $\gamma = \frac{1}{2}$ which he proved to be the best possible choice for γ (with a factor larger than 1 for ω_1^*). His methods of proof also use Markov-type inequalities (Theorem 2.2), without, however, stating the probabilistic background of these. Also, his results are due to complicated estimations for the concrete cases treated there, while our general result emerges from a single estimation of this type (Theorem 5.1), giving additionally simple explicit expressions for the constants involved.

A probabilistic argument for $\gamma = \frac{1}{2}$ being the best possible choice could be given as follows. By the Central Limit Theorem 2.3, the sequence $n^{1/2}(\bar{X}_n - \xi)$ tends to a normally distributed random variable in distribution, hence for ε being of order $\mathcal{O}(n^{-\gamma})$ with $\gamma > \frac{1}{2}$, $P(|\bar{X}_n - \xi| > \varepsilon)$ would tend to 1 for $n \rightarrow \infty$, i.e., the second summand in (5.5) and (5.6), resp., would not tend to zero in this case. Note that for $\gamma = \frac{1}{2}$, a strictly positive limit less than one is attained for the sequence $P(|\bar{X}_n - \xi| > \varepsilon)$ in

the non-trivial case, hence stronger estimations than used in the proof of Theorem 5.1 are necessary to obtain a corresponding result for this case.

For the remainder of this section, we shall deal with rates of convergence results for contraction semigroups in terms of the second modulus of continuity.

THEOREM 5.4. *Under the conditions of Theorem 4.2, we have for $f \in \mathcal{X}$*

$$\begin{aligned} & \left\| T(\xi) f - \psi_{N(\tau)} \left(E \left[T \left(\frac{Y}{\tau} \right) \right] \right) f \right\| \\ & \leq K\omega_2 \left(\frac{1}{\sqrt{2\tau}} \left\{ \zeta \sigma^2(Y) + \frac{\gamma^2}{\tau} \sigma^2(N(\tau)) \right\}^{1/2}, f \right), \quad \tau > 0. \end{aligned} \quad (5.9)$$

Alternatively, if N is a non-negative integer-valued random variable with $E(N) = \zeta$, and $Y \geq 0$ is a real-valued random variable with $E(Y) = \gamma$ such that $\sigma^2(N)$ and $\sigma^2(Y)$ exist, then for $f \in \mathcal{X}$

$$\begin{aligned} & \left\| T(\xi) f - \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n f \right\| \\ & \leq K\omega_2 \left(\frac{1}{\sqrt{2n}} \left\{ \zeta \sigma^2(Y) + \gamma^2 \sigma^2(N) \right\}^{1/2}, f \right), \quad n \in \mathbb{N}. \end{aligned} \quad (5.10)$$

Here $\{T(t) \mid t \geq 0\}$ is a contraction semigroup of class (C_0) , and K denotes a generic constant.

Proof. Obvious from the Jackson-type inequalities (4.10) and (4.28) (cf. Butzer and Hahn [4]); note that for equi-bounded semigroups the assumptions of Theorem 4.4 can be weakened similar to those of Theorem 4.2. ■

Theorem 5.4 covers all first and second main theorems treated in Butzer and Hahn [4]; moreover, due to the improved estimations (4.10) and (4.28) the variances of the composed random variables now play the essential role as claimed in [4] only for groups of isometric operators.

It should be pointed out that the corresponding expressions for the individual representation formulas (4.32) to (4.38) as well as (4.12) and (4.15) are immediately available from the estimations given there since the individual variance term always is the leading term in brackets.

6. IMPROVEMENTS OF THE RATE OF CONVERGENCE

As was pointed out in the foregoing sections, the variance of the underlying random variables plays a central role in estimating the rate of con-

vergence in the different representation formulas, giving the rate $\mathcal{O}(1/n)$ for $n \rightarrow \infty$ in representations covered by Theorem 4.4, e.g. It should thus be possible to improve the rate of convergence by suitable elimination of the variance terms in the spirit of Bernstein [1] and Voronovskaja [28], who introduced this approach in the analysis of the convergence behavior of Bernstein polynomials. As will be shown in the following result, their ideas carry over to semigroup theory almost literally.

THEOREM 6.1. *Let N be a non-negative integer-valued random variable with $E(N) = \zeta$, and $Y \geq 0$ be a real-valued random variable with $E(Y) = \gamma$ such that $\psi_N(\psi_N^*(\delta)) < \infty$ for some $\delta > 0$. Then for an arbitrary strongly continuous operator semigroup, with $\xi = \zeta\gamma$,*

$$\begin{aligned} & \left\| T(\xi) f - \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n f \right. \\ & \quad \left. + \frac{1}{2n} (\zeta\sigma^2(Y) + \gamma^2\sigma^2(N)) \left\{ \psi_N \left(E \left[T \left(\frac{Y}{n} \right) \right] \right) \right\}^n A^2 f \right\| \\ & = \mathcal{O} \left(\frac{1}{n^2} \right) \quad \text{for } n \rightarrow \infty, \end{aligned} \tag{6.1}$$

provided that $f \in D(A^4)$. The order $\mathcal{O}(1/n^2)$ for $n \rightarrow \infty$ can in general not be improved.

Proof. According to the proof of Theorem 4.4, it is enough to prove relation (6.1) when $N \equiv 1$, i.e., $\psi_N(E[T(Y/n)])^n = E[T(\bar{X}_n)]$ where $\{X_n | n \in \mathbb{N}\}$ is an i.i.d. sequence identically distributed with Y . A further Taylor expansion in Theorem 4.1 yields

$$\begin{aligned} & E[T(\bar{X}_n)] f - T(\xi) f \\ & = \frac{\sigma^2(Y)}{2n} T(\xi) A^2 f + \frac{1}{6} E(\bar{X}_n - \xi)^3 T(\xi) A^3 f + \mathcal{O} \left(\frac{1}{n^2} \right) \quad (n \rightarrow \infty) \end{aligned} \tag{6.2}$$

which implies that the l.h.s. of (6.1) can be estimated by

$$\frac{1}{6} |E(\bar{X}_n - \xi)^3| M e^{\omega\xi} \|A^3 f\| + \mathcal{O} \left(\frac{1}{n^2} \right). \tag{6.3}$$

Now as a consequence of the Extended Central Limit Theorem 2.3, $E(\bar{X}_n - \xi)^3 = \mathcal{O}(1/n \sqrt{n})$ for $n \rightarrow \infty$; but also $E(\bar{X}_n - \xi)^3 = (1/n^3) \times E(\sum_{k=1}^n X_k - n\xi)^3$, which is a rational function of n , implying that, in fact, $E(\bar{X}_n - \xi)^3 = \mathcal{O}(1/n^2)$ for $n \rightarrow \infty$. This gives the desired estimation (6.1). The rate of convergence cannot be improved except for trivial cases since the remainder term in (6.2) originates from the moment estimation for $E(\bar{X}_n - \xi)^4$ which in this case is exactly $\mathcal{O}(1/n^2)$ by (2.6) and (3.9). ■

As in the concluding remark of Section 5, for the individual representation formulas (4.32) to (4.38) the l.h.s. of (6.1) can again immediately be derived from the estimations given there. Of course, relations analogous to (6.1) can also be formulated in the setting of Theorem 4.2 or Theorem 4.3 which, due to the great similarity, will be omitted here. Further improvements of the rate of convergence could be achieved by successive elimination of remainder terms as indicated in Theorem 6.1. However, things will get quite complicated for orders higher than $\mathcal{O}(1/n^2)$.

7. EXAMPLES

To demonstrate the broadness of possible applications of the results obtained in the preceding sections, we shall deduce some approximation theorems for a class of exponential operators by specialisation on the semigroup of translations, as well as some rate of convergence results for a class of Poisson approximations in probability theory.

For approximation by Bernstein polynomials, we have the following result.

THEOREM 7.1. *For $g \in C[0, 1]$ and $0 < \xi < 1$ we have*

$$\begin{aligned} & \left| g(\xi) - \sum_{k=0}^n \binom{n}{k} \xi^k (1-\xi)^{n-k} g\left(\frac{k}{n}\right) \right| \\ & \leq \|g'\| \sqrt{\frac{\xi(1-\xi)}{n}} \leq \frac{\|g'\|}{2\sqrt{n}}, \quad n \in \mathbb{N}, \\ & \text{if } g' \in C[0, 1], \end{aligned} \tag{7.1}$$

$$\begin{aligned} & \left| g(\xi) - \sum_{k=0}^n \binom{n}{k} \xi^k (1-\xi)^{n-k} g\left(\frac{k}{n}\right) \right| \\ & \leq \|g''\| \frac{\xi(1-\xi)}{2n} \leq \frac{\|g''\|}{8n}, \quad n \in \mathbb{N}, \\ & \text{if } g'' \in C[0, 1], \end{aligned} \tag{7.2}$$

$$\begin{aligned} g(\xi) &= \sum_{k=0}^n \binom{n}{k} \xi^k (1-\xi)^{n-k} g\left(\frac{k}{n}\right) \\ & - g''(\xi) \frac{\xi(1-\xi)}{2n} + \mathcal{O}\left(\frac{1}{n\sqrt{n}}\right), \quad n \rightarrow \infty, \\ & \text{if } g''' \in C[0, 1], \end{aligned} \tag{7.3}$$

$$\begin{aligned}
 & \left| g(\xi) - \sum_{k=0}^n \binom{n}{k} \xi^k (1-\xi)^{n-k} \left\{ g\left(\frac{k}{n}\right) - \frac{\xi(1-\xi)}{2n} g''\left(\frac{k}{n}\right) \right\} \right| \\
 & \leq \frac{\|g'''\|}{6n^2} |\xi(1-\xi)(1-2\xi)| + \frac{3\|g^{(4)}\|}{8n^2} \xi^2(1-\xi)^2 \\
 & \quad + \frac{\|g^{(4)}\|}{24n^3} \xi(1-\xi)(6\xi^2 - 6\xi + 1) \\
 & \leq \frac{\|g'''\|}{60n^2} + \frac{3\|g^{(4)}\|}{128n^2} + \frac{\|g^{(4)}\|}{480n^3}, \quad n \in \mathbb{N}, \\
 & \text{if } g^{(4)} \in C[0, 1]. \tag{7.4}
 \end{aligned}$$

Proof. Let $\mathcal{X} = UCB(\mathbb{R})$, the space of all uniformly continuous and bounded functions on \mathbb{R} , and $\{T(t) \mid t \geq 0\}$ the (contraction) semigroup of left translations. Then A is the differential operator. For $g \in C[0, 1]$ let $g^* \in UCB(\mathbb{R})$ denote a suitable extension of g to \mathbb{R} such that g^* fulfills the same smoothness conditions as g . Then (7.1) to (7.4) follow from Theorem 4.4, (4.32) and a corresponding extension of Theorem 6.1 (with $N \equiv 1$ and Y being binomially distributed) when $T(\xi)g^*(0)$ is considered. ■

Due to similarity of proof, we shall only state the following theorems derived from Hille's and Shaw's representations (4.11) to (4.13) and (4.34), involving Favard (or Szász) and Baskakov operators.

THEOREM 7.2. *For $g \in UCB(\mathbb{R}^+)$ and $\xi > 0$ we have*

$$\begin{aligned}
 & \left| g(\xi) - e^{-\xi\tau} \sum_{k=0}^{\infty} \frac{(\xi\tau)^k}{k!} g\left(\frac{k}{\tau}\right) \right| \leq \|g'\| \sqrt{\frac{\xi}{\tau}}, \quad \tau > 0, \\
 & \text{if } g' \in UCB(\mathbb{R}^+), \tag{7.5}
 \end{aligned}$$

$$\begin{aligned}
 & \left| g(\xi) - e^{-\xi\tau} \sum_{k=0}^{\infty} \frac{(\xi\tau)^k}{k!} g\left(\frac{k}{\tau}\right) \right| \leq \|g''\| \frac{\xi}{2\tau}, \quad \tau > 0, \\
 & \text{if } g'' \in UCB(\mathbb{R}^+), \tag{7.6}
 \end{aligned}$$

$$\begin{aligned}
 & g(\xi) = e^{-\xi\tau} \sum_{k=0}^{\infty} \frac{(\xi\tau)^k}{k!} g\left(\frac{k}{\tau}\right) - g''(\xi) \frac{\xi}{2\tau} + \mathcal{O}\left(\frac{1}{\tau\sqrt{\tau}}\right), \quad \tau \rightarrow \infty, \\
 & \text{if } g''' \in UCB(\mathbb{R}^+), \tag{7.7}
 \end{aligned}$$

$$\begin{aligned}
 & g(\xi) = e^{-\xi\tau} \sum_{k=0}^{\infty} \frac{(\xi\tau)^k}{k!} \left\{ g\left(\frac{k}{\tau}\right) - \frac{\xi}{2\tau} g''\left(\frac{k}{\tau}\right) \right\} + \mathcal{O}\left(\frac{1}{\tau^2}\right), \quad \tau \rightarrow \infty, \\
 & \text{if } g^{(4)} \in UCB(\mathbb{R}^+), \tag{7.8}
 \end{aligned}$$

all estimates holding uniformly in ξ in every bounded interval.

THEOREM 7.3. For $g \in UCB(\mathbb{R}^+)$ and $\xi > 0$ we have

$$\begin{aligned} & \left| g(\xi) - \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\xi^k}{(1+\xi)^{n+k}} g\left(\frac{k}{n}\right) \right| \\ & \leq \|g'\| \sqrt{\frac{\xi(1+\xi)}{n}}, \quad n \in \mathbb{N}, \\ & \text{if } g' \in UCB(\mathbb{R}^+), \end{aligned} \quad (7.9)$$

$$\begin{aligned} & \left| g(\xi) - \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\xi^k}{(1+\xi)^{n+k}} g\left(\frac{k}{n}\right) \right| \\ & \leq \|g''\| \frac{\xi(1+\xi)}{2n}, \quad n \in \mathbb{N}, \\ & \text{if } g'' \in UCB(\mathbb{R}^+), \end{aligned} \quad (7.10)$$

$$\begin{aligned} g(\xi) &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\xi^k}{(1+\xi)^{n+k}} g\left(\frac{k}{n}\right) \\ & - g''(\xi) \frac{\xi(1+\xi)}{2n} + \mathcal{O}\left(\frac{1}{n\sqrt{n}}\right), \quad n \rightarrow \infty, \\ & \text{if } g''' \in UCB(\mathbb{R}^+), \end{aligned} \quad (7.11)$$

$$\begin{aligned} g(\xi) &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\xi^k}{(1+\xi)^{n+k}} \left\{ g\left(\frac{k}{n}\right) \right. \\ & \left. - \frac{\xi(1+\xi)}{2n} g''\left(\frac{k}{n}\right) \right\} + \mathcal{O}\left(\frac{1}{n^2}\right), \quad n \rightarrow \infty, \\ & \text{if } g^{(4)} \in UCB(\mathbb{R}^+), \end{aligned} \quad (7.12)$$

all estimates holding uniformly in ξ in every bounded interval.

We shall conclude with an analysis of some Poisson convergence theorems in probability theory via the Poisson convolution semigroup (cf. [20]) as was first studied by Le Cam [5]. For this purpose, let $\mathcal{X} = l^\infty$ be the Banach space of all bounded sequences $f = (f(0), f(1), \dots)$, and the linear contraction B on \mathcal{X} be defined by

$$Bf = \varepsilon_1 * f, \quad f \in \mathcal{X}, \quad (7.13)$$

where ε_k denotes the unit mass in $k \in \mathbb{Z}^+$, and $*$ means convolution. Then $A = B - I$ is the generator of the Poisson convolution semigroup $\{e^{tA} \mid t \geq 0\}$, i.e.,

$$e^{tA}f = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \varepsilon_k * f = P(t) * f, \quad t \geq 0, \quad (7.14)$$

where $P(t)$ denotes the Poisson distribution over \mathbb{Z}^+ with expectation t . Since Poisson limit theorems usually are expressed in terms of convergence in distribution, we have to provide an adequate metric ρ on the set \mathcal{M} of all probability measures over \mathbb{Z}^+ ; one such is given by

$$\rho(P, Q) = \sup_{S \subseteq \mathbb{Z}^+} |P(S) - Q(S)| = \frac{1}{2} \sum_{k=0}^{\infty} |P(\{k\}) - Q(\{k\})|, \quad P, Q \in \mathcal{M} \tag{7.15}$$

(cf. also Serfling [26]). In fact, convergence in distribution and ρ -convergence as well as norm-convergence (if measures are interpreted as operators) are then equivalent, but only the latter provide useful statements on the rate of convergence. In order to be able to formulate the various Poisson limit theorems we shall need binomial and negative binomial distributions $B(n, p)$ and $\bar{B}(n, p)$, resp., with $n \in \mathbb{N}$, $p \in (0, 1)$, defined by

$$\begin{aligned} B(n, p)(S) &= \sum_{k \in S \cap \{0, 1, \dots, n\}} \binom{n}{k} p^k (1-p)^{n-k}, \\ \bar{B}(n, p)(S) &= \sum_{k \in S} \binom{k+n-1}{k} p^n (1-p)^k, \quad S \subseteq \mathbb{Z}^+. \end{aligned} \tag{7.16}$$

Note that the negative binomial distribution $\bar{B}(n, p)$ can also be extended to a (non-integer) non-negative real parameter n in the same way.

We shall now demonstrate that the known Poisson convergence theorems can be considered as special cases of (an analogue of) Kendall's and Widder's representation theorems (4.32) and (4.37), resp. (cf. also [21]), whereas the Butzer-Hahn representation (4.38) gives a new Poisson convergence theorem. Of course, all of these are given with the exact rates of convergence in the setting of Section 4.

THEOREM 7.4 (Classical Poisson Convergence Theorem). *For $n \geq \xi > 0$ we have*

$$\rho \left(B \left(n, \frac{\xi}{n} \right); P(\xi) \right) \leq \frac{\xi^2}{n} \tag{7.17}$$

with

$$\left\| B \left(n, \frac{\xi}{n} \right) * f - P(\xi) * f \right\| \leq \frac{\xi^2}{2n} \|A^2 f\|, \quad f \in l^\infty, \tag{7.18}$$

and

$$\left\| B\left(n, \frac{\xi}{n}\right) * f - P(\xi) * f \right\| = \frac{\xi^2}{2n} \|e^{\xi A} A^2 f\| + \mathcal{O}\left(\frac{1}{n^2}\right) \quad (n \rightarrow \infty) \quad f \in l^\infty. \tag{7.19}$$

Proof. For (7.16) and (7.17), see [20]; (7.18) follows similarly from an extended Taylor expansion. ■

Note that $B(n, \xi/n) * f = \{(1 - \xi)I + \xi(I + A/n)\}^n * f$, $f \in l^\infty$, hence (7.17) and (7.18) are analogues to Kendall’s representation (4.32) in that the semigroup $T(1/n)$ is replaced by the first two terms of the corresponding Taylor expansion.

THEOREM 7.5. For $\xi > 0$ and all $n \in \mathbb{N}$, we have

$$\rho\left(\bar{B}\left(n, \frac{n}{n + \xi}\right); P(\xi)\right) \leq \frac{\xi^2}{n} \tag{7.20}$$

with

$$\left\| \bar{B}\left(n, \frac{n}{n + \xi}\right) * f - P(\xi) * f \right\| \leq \frac{\xi^2}{2n} \|A^2 f\|, \quad f \in l^\infty, \tag{7.21}$$

and for $n \rightarrow \infty$

$$\left\| \bar{B}\left(n, \frac{n}{n + \xi}\right) * f - P(\xi) * f \right\| = \frac{\xi^2}{2n} \|e^{\xi A} A^2 f\| + \mathcal{O}\left(\frac{1}{n^2}\right), \quad f \in l^\infty. \tag{7.22}$$

Proof. We only need to show that for the resolvent, $\lambda R(\lambda) f = \bar{B}(1, \lambda/(1 + \lambda)) * f$ for arbitrary $\lambda > 0$, $f \in l^\infty$; everything will then follow from Widder’s representation (4.37). But

$$\begin{aligned} \lambda R(\lambda) &= \lambda(\lambda I - A)^{-1} = \lambda((1 + \lambda)I - B)^{-1} \\ &= \frac{\lambda}{1 + \lambda} \left(I - \frac{1}{1 + \lambda} B\right)^{-1} = \frac{\lambda}{1 + \lambda} \sum_{k=0}^{\infty} (1 + \lambda)^{-k} B^k; \end{aligned} \tag{7.23}$$

hence for $f \in l^\infty$

$$\lambda R(\lambda) * f = \frac{\lambda}{1 + \lambda} \sum_{k=0}^{\infty} (1 + \lambda)^{-k} \varepsilon_k * f = \bar{B}\left(1, \frac{\lambda}{1 + \lambda}\right) * f. \quad \blacksquare$$

Using the Butzer–Hahn representation (4.38) in the same way as Widder’s representation in Theorem 7.5, we obtain a third type of Poisson convergence.

THEOREM 7.6. For $\xi > 0$ and $n \in \mathbb{N}$, we have

$$\rho \left(\bar{B} \left(n\xi, \frac{n}{n+1} \right); P(\xi) \right) \leq \frac{\xi^2}{n} \tag{7.24}$$

with

$$\left\| \bar{B} \left(n\xi, \frac{n}{n+1} \right) * f - P(\xi) * f \right\| \leq \frac{\xi^2}{2n} \|A^2 f\|, \quad f \in l^\infty, \tag{7.25}$$

and for $n \rightarrow \infty$

$$\left\| \bar{B} \left(n\xi, \frac{n}{n+1} \right) * f - P(\xi) * f \right\| = \frac{\xi^2}{2n} \|e^{\xi A} A^2 f\| + \mathcal{O} \left(\frac{1}{n^2} \right), \quad f \in l^\infty. \tag{7.26}$$

Proof. As in Theorem 7.5; note that

$$\bar{B} \left(n\xi, \frac{n}{n+1} \right) (S) = \int_0^\infty \frac{n^\xi}{\Gamma(\xi)} t^{\xi-1} e^{-nt} P(t)(S) dt, \quad S \subseteq \mathbb{Z}^+, \tag{7.27}$$

i.e., a negative binomial distribution is obtained by randomization of the Poisson parameter according to a gamma distribution. ■

ACKNOWLEDGMENTS

I want to express my heartiest thanks to Professors Butzer and Rauhut, the former especially for having drawn my attention to probabilistic methods in semigroup theory and for several encouraging remarks, the latter for his permanent interest and helpful comments in the probabilistic part of the work.

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